

Online supplement to

“New Matrix-Based Methods for the Analytic Evaluation of the Multivariate Cumulative Normal Distribution Function”

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Performances in each of four sets of cases corresponding to (a) low correlations, high MVNCD values, (b) low correlations, low MVNCD values, (c) high correlations, high MVNCD values, and (d) high correlations, low MVNCD values.

Table 1. MVNCD evaluations results: low correlation, high MVNCD value

Dim.	Measure	GHK		SSJ		Univariate truncated methods				Bivariate truncated methods		
		500 rep.	10,000 rep.	1 perm.	10 perm.	TGME	ME	OVUS	OVBS	TGBME	BME	TVBS
H=5	MAE	0.00034	0.00004	0.00068	0.00037	0.00937	0.00045	0.00026	0.00014	0.00660	0.00028	0.00018
	MAPE	0.10	0.01	0.22	0.12	2.37	0.12	0.07	0.03	1.69	0.07	0.05
	%MAE>0.005	0.0	0.0	0.0	0.0	66.8	0.4	0.0	0.0	49.2	0.0	0.0
	%MAPE>2	0.0	0.0	0.4	0.0	49.2	0.4	0.0	0.0	29.2	0.0	0.0
	Time (s)	0.018	0.247	0.009	0.056	0.008	0.016	0.018	0.018	0.009	0.013	0.017
H=7	MAE	0.00052	0.00006	0.00085	0.00045	0.01193	0.00036	0.00029	0.00020	0.00940	0.00029	0.00022
	MAPE	0.19	0.02	0.35	0.19	3.75	0.13	0.11	0.07	2.97	0.11	0.08
	%MAE>0.005	0.0	0.0	0.1	0.0	21.2	0.0	0.0	0.0	18.0	0.0	0.0
	%MAPE>2	0.0	0.0	0.3	0.0	20.8	0.0	0.0	0.0	17.6	0.0	0.0
	Time (s)	0.026	0.418	0.014	0.097	0.012	0.028	0.032	0.035	0.014	0.019	0.031
H=10	MAE	0.00091	0.00009	0.00093	0.00045	0.01312	0.00039	0.00032	0.00023	0.01091	0.00032	0.00025
	MAPE	0.39	0.04	0.49	0.26	5.15	0.21	0.17	0.12	4.31	0.17	0.13
	%MAE>0.005	0.1	0.0	0.2	0.0	21.0	0.0	0.0	0.0	19.6	0.0	0.0
	%MAPE>2	0.1	0.0	0.9	0.2	22.8	0.0	0.0	0.0	22.2	0.0	0.0
	Time (s)	0.043	0.757	0.028	0.198	0.017	0.053	0.058	0.068	0.020	0.031	0.053
H=12	MAE	0.00098	0.00010	0.00094	0.00047	0.01342	0.00036	0.00029	0.00022	0.01129	0.00028	0.00023
	MAPE	0.53	0.05	0.60	0.32	6.45	0.22	0.18	0.14	5.45	0.17	0.13
	%MAE>0.005	0.1	0.0	0.2	0.0	21.4	0.0	0.0	0.0	19.5	0.0	0.0
	%MAPE>2	0.4	0.0	1.1	0.3	23.6	0.1	0.0	0.0	22.7	0.1	0.0
	Time (s)	0.055	1.163	0.038	0.286	0.021	0.074	0.082	0.096	0.024	0.047	0.074
H=15	MAE	0.00132	0.00012	0.00109	0.00047	0.01269	0.00033	0.00029	0.00023	0.01094	0.00026	0.00022
	MAPE	0.86	0.08	0.85	0.36	7.42	0.24	0.22	0.17	6.49	0.20	0.17
	%MAE>0.005	0.3	0.0	0.3	0.0	21.2	0.0	0.0	0.0	20.6	0.0	0.0
	%MAPE>2	2.0	0.0	3.0	0.1	24.2	0.0	0.0	0.0	23.8	0.0	0.0
	Time (s)	0.073	2.205	0.067	0.439	0.029	0.140	0.143	0.146	0.034	0.066	0.106
H=18	MAE	0.00140	0.00015	0.00080	0.00042	0.01195	0.00028	0.00024	0.00020	0.01027	0.00022	0.00020
	MAPE	1.06	0.11	0.75	0.42	7.87	0.27	0.24	0.20	6.96	0.22	0.19
	%MAE>0.005	0.5	0.0	0.0	0.0	19.9	0.0	0.0	0.0	19.2	0.0	0.0
	%MAPE>2	3.7	0.0	2.3	0.4	24.5	0.2	0.1	0.1	24.2	0.1	0.1
	Time (s)	0.100	2.313	0.099	0.663	0.034	0.169	0.173	0.211	0.041	0.092	0.143
H=20	MAE	0.00148	0.00015	0.00080	0.00044	0.01139	0.00026	0.00024	0.00020	0.01000	0.00021	0.00019
	MAPE	1.24	0.13	0.81	0.45	8.86	0.28	0.25	0.21	7.80	0.23	0.20
	%MAE>0.005	1.0	0.0	0.1	0.0	18.7	0.0	0.0	0.0	18.3	0.0	0.0
	%MAPE>2	4.6	0.0	2.2	0.6	24.2	0.0	0.0	0.0	24.0	0.0	0.0
	Time (s)	0.114	2.848	0.106	0.773	0.039	0.200	0.220	0.247	0.049	0.109	0.163

Table 2. MVNCD evaluation results: low correlation, low MVNCD value

Dim.	Measure	GHK		SSJ		Univariate truncated methods				Bivariate truncated methods		
		500 rep.	10,000 rep.	1 perm.	10 perm.	TGME	ME	OVUS	OVBS	TGBME	BME	TVBS
<i>H=5</i>	MAE	0.00013	0.00002	0.00070	0.00035	0.00220	0.00013	0.00009	0.00005	0.00163	0.00010	0.00005
	MAPE	0.18	0.02	2.53	1.36	2.06	0.17	0.12	0.06	1.52	0.12	0.07
	%MAE>0.005	0.0	0.0	0.4	0.0	13.6	0.0	0.0	0.0	8.8	0.0	0.0
	%MAPE>2	0.0	0.0	29.6	13.2	33.6	0.0	0.0	0.0	19.6	0.0	0.0
	Time (s)	0.018	0.247	0.009	0.056	0.008	0.016	0.018	0.018	0.009	0.013	0.017
<i>H=7</i>	MAE	0.00013	0.00002	0.00060	0.00024	0.00198	0.00009	0.00009	0.00005	0.00160	0.00007	0.00006
	MAPE	0.39	0.05	7.63	5.90	3.64	0.40	0.37	0.27	3.01	0.30	0.26
	%MAE>0.005	0.0	0.0	0.0	0.0	10.0	0.0	0.0	0.0	8.0	0.0	0.0
	%MAPE>2	0.8	0.0	55.6	34.4	62.8	2.8	2.0	2.0	50.0	1.6	1.6
	Time (s)	0.026	0.418	0.014	0.097	0.012	0.028	0.032	0.035	0.014	0.019	0.031
<i>H=10</i>	MAE	0.00009	0.00001	0.00031	0.00017	0.00077	0.00002	0.00002	0.00001	0.00062	0.00002	0.00001
	MAPE	1.29	0.16	16.17	10.38	6.43	1.11	1.00	0.72	5.19	0.74	0.62
	%MAE>0.005	0.0	0.0	0.0	0.0	2.8	0.0	0.0	0.0	1.6	0.0	0.0
	%MAPE>2	18.8	0.0	74.8	62.0	76.8	8.4	7.6	4.8	74.0	6.4	5.2
	Time (s)	0.043	0.757	0.028	0.198	0.017	0.053	0.058	0.068	0.020	0.031	0.053
<i>H=12</i>	MAE	0.00006	0.00001	0.00017	0.00008	0.00030	0.00002	0.00001	0.00001	0.00024	0.00001	0.00001
	MAPE	2.43	0.31	26.38	18.04	8.26	1.94	1.86	1.45	7.14	1.82	1.42
	%MAE>0.005	0.0	0.0	0.4	0.0	1.2	0.0	0.0	0.0	0.8	0.0	0.0
	%MAPE>2	37.6	0.8	77.6	72.0	83.2	14.8	14.4	11.2	81.6	10.0	8.8
	Time (s)	0.055	1.163	0.038	0.286	0.021	0.074	0.082	0.096	0.024	0.047	0.074
<i>H=15</i>	MAE	0.00003	0.00000	0.00005	0.00003	0.00011	0.00000	0.00000	0.00000	0.00010	0.00000	0.00000
	MAPE	6.24	1.29	26.65	23.75	11.29	3.24	2.94	2.65	9.90	2.19	1.93
	%MAE>0.005	0.0	0.0	0.0	0.0	0.8	0.0	0.0	0.0	0.8	0.0	0.0
	%MAPE>2	62.0	11.2	83.6	78.0	86.8	22.8	20.8	19.2	84.4	18.0	17.2
	Time (s)	0.073	2.205	0.067	0.439	0.029	0.140	0.143	0.146	0.034	0.066	0.106
<i>H=18</i>	MAE	0.00001	0.00000	0.00002	0.00001	0.00004	0.00000	0.00000	0.00000	0.00004	0.00000	0.00000
	MAPE	10.64	2.94	35.36	24.03	14.07	4.90	4.48	4.25	11.86	3.02	2.73
	%MAE>0.005	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	%MAPE>2	75.6	32.0	86.4	82.0	82.4	31.2	28.8	28.8	86.4	22.8	22.4
	Time (s)	0.100	2.313	0.099	0.663	0.034	0.169	0.173	0.211	0.041	0.092	0.143
<i>H=20</i>	MAE	0.00001	0.00000	0.00001	0.00001	0.00004	0.00000	0.00000	0.00000	0.00004	0.00000	0.00000
	MAPE	15.49	4.52	38.38	32.65	17.04	6.17	5.81	5.36	14.93	4.44	4.83
	%MAE>0.005	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	%MAPE>2	87.2	46.4	87.2	85.2	86.4	33.2	32.8	31.6	84.0	29.6	29.2
	Time (s)	0.114	2.848	0.106	0.773	0.039	0.200	0.220	0.247	0.049	0.109	0.163

Table 3. MVNCD evaluation results: high correlation, high MVNCD value

Dim.	Measure	GHK		SSJ		Univariate truncated methods				Bivariate truncated methods		
		500 rep.	10,000 rep.	1 perm.	10 perm.	TGME	ME	OVUS	OVBS	TGBME	BME	TVBS
<i>H=5</i>	MAE	0.00141	0.00024	0.00348	0.00223	0.03741	0.00334	0.00215	0.00122	0.02587	0.00231	0.00144
	MAPE	0.43	0.08	1.34	0.90	10.59	1.21	0.79	0.44	7.27	0.86	0.51
	%MAE>0.005	2.8	0.8	22.4	7.6	93.2	20.8	11.2	3.6	88.8	10.0	4.0
	%MAPE>2	2.8	0.8	17.6	10.8	90.8	17.6	10.8	3.6	88.8	9.6	5.2
	Time (s)	0.018	0.247	0.009	0.056	0.008	0.016	0.018	0.018	0.009	0.013	0.017
<i>H=7</i>	MAE	0.00186	0.00023	0.00271	0.00171	0.03293	0.00239	0.00188	0.00127	0.02575	0.00176	0.00131
	MAPE	0.71	0.09	1.53	1.09	12.24	1.38	1.14	0.78	9.76	0.97	0.69
	%MAE>0.005	5.2	0.4	18.0	4.4	94.0	11.2	6.8	3.2	92.8	6.0	3.2
	%MAPE>2	6.0	0.4	19.2	12.8	94.8	18.0	11.6	6.8	96.0	10.4	6.8
	Time (s)	0.026	0.418	0.014	0.097	0.012	0.028	0.032	0.035	0.014	0.019	0.031
<i>H=10</i>	MAE	0.00259	0.00031	0.00228	0.00133	0.03179	0.00153	0.00128	0.00096	0.02589	0.00116	0.00095
	MAPE	1.13	0.13	1.22	0.73	12.87	0.86	0.71	0.51	10.57	0.64	0.48
	%MAE>0.005	14.0	0.0	8.4	2.0	94.8	4.4	3.2	2.0	94.0	2.4	1.2
	%MAPE>2	16.0	0.0	18.4	8.4	97.2	10.8	8.0	4.4	96.8	6.8	4.4
	Time (s)	0.043	0.757	0.028	0.198	0.017	0.053	0.058	0.068	0.020	0.031	0.053
<i>H=12</i>	MAE	0.00283	0.00034	0.00197	0.00126	0.02854	0.00113	0.00096	0.00077	0.02366	0.00095	0.00076
	MAPE	1.49	0.17	1.37	0.92	13.53	0.82	0.69	0.54	11.17	0.70	0.56
	%MAE>0.005	17.6	0.0	8.4	1.6	95.2	0.8	0.4	0.0	93.2	0.4	0.4
	%MAPE>2	23.2	0.0	17.6	9.2	98.8	9.6	8.0	4.4	98.0	7.6	5.6
	Time (s)	0.055	1.163	0.038	0.286	0.021	0.074	0.082	0.096	0.024	0.047	0.074
<i>H=15</i>	MAE	0.00302	0.00043	0.00179	0.00095	0.02388	0.00082	0.00072	0.00057	0.02029	0.00068	0.00056
	MAPE	1.77	0.23	1.32	0.71	12.63	0.59	0.51	0.40	10.68	0.48	0.39
	%MAE>0.005	16.0	0.0	4.4	0.0	94.4	0.8	0.4	0.4	95.2	0.4	0.4
	%MAPE>2	32.8	0.0	18.8	6.4	98.8	6.8	5.6	3.6	99.2	6.0	2.8
	Time (s)	0.073	2.205	0.067	0.439	0.029	0.140	0.143	0.146	0.034	0.066	0.106
<i>H=18</i>	MAE	0.00310	0.00043	0.00150	0.00092	0.02216	0.00066	0.00058	0.00044	0.01913	0.00054	0.00045
	MAPE	2.24	0.31	1.27	0.80	13.69	0.58	0.50	0.39	11.91	0.45	0.38
	%MAE>0.005	19.6	0.0	1.2	0.0	95.6	0.0	0.0	0.0	92.4	0.0	0.0
	%MAPE>2	41.2	1.2	20.4	8.8	98.8	4.4	4.0	3.6	98.8	3.2	2.4
	Time (s)	0.100	2.313	0.099	0.663	0.034	0.169	0.173	0.211	0.041	0.092	0.143
<i>H=20</i>	MAE	0.00345	0.00049	0.00148	0.00083	0.02012	0.00057	0.00049	0.00041	0.01718	0.00047	0.00042
	MAPE	2.82	0.45	1.70	0.99	15.00	0.65	0.58	0.49	13.07	0.56	0.49
	%MAE>0.005	23.6	0.0	2.8	0.0	86.0	0.0	0.0	0.0	86.4	0.0	0.0
	%MAPE>2	50.8	2.0	24.8	11.6	98.0	6.8	4.4	3.6	99.2	4.0	3.2
	Time (s)	0.114	2.848	0.106	0.773	0.039	0.200	0.220	0.247	0.049	0.109	0.163

Table 4. MVNCD evaluation results: high correlation, low MVNCD value

Dim.	Measure	GHK		SSJ		Univariate truncated methods				Bivariate truncated methods		
		500 rep.	10,000 rep.	1 perm.	10 perm.	TGME	ME	OVUS	OVBS	TGBME	BME	TVBS
<i>H=5</i>	MAE	0.00055	0.00006	0.00243	0.00168	0.01159	0.00105	0.00064	0.00040	0.00849	0.00064	0.00038
	MAPE	2.97	0.46	11.37	9.40	21.28	5.73	5.15	3.38	16.40	4.24	2.65
	%MAE>0.005	0.8	0.0	13.2	7.6	50.4	4.0	0.4	0.4	42.4	0.8	0.4
	%MAPE>2	14.4	2.8	62.8	55.2	86.4	35.6	26.4	16.8	80.4	22.4	14.4
	Time (s)	0.018	0.247	0.009	0.056	0.008	0.016	0.018	0.018	0.009	0.013	0.017
<i>H=7</i>	MAE	0.00050	0.00005	0.00176	0.00100	0.00522	0.00040	0.00029	0.00022	0.00406	0.00029	0.00022
	MAPE	5.81	1.57	21.41	15.02	25.79	6.05	5.89	6.32	19.40	5.19	4.87
	%MAE>0.005	0.4	0.0	8.4	1.6	26.8	1.2	0.4	0.4	20.8	0.8	0.4
	%MAPE>2	38.8	8.4	76.4	68.4	93.6	36.4	34.0	26.8	86.8	28.8	23.2
	Time (s)	0.026	0.418	0.014	0.097	0.012	0.028	0.032	0.035	0.014	0.019	0.031
<i>H=10</i>	MAE	0.00024	0.00004	0.00056	0.00033	0.00193	0.00008	0.00007	0.00006	0.00160	0.00008	0.00007
	MAPE	18.50	7.40	26.20	24.07	34.76	9.91	10.28	10.05	31.16	11.74	12.20
	%MAE>0.005	0.0	0.0	1.6	0.4	9.6	0.0	0.0	0.0	8.8	0.0	0.0
	%MAPE>2	69.2	34.4	84.4	82.0	93.6	47.6	45.2	41.2	92.0	42.4	41.2
	Time (s)	0.043	0.757	0.028	0.198	0.017	0.053	0.058	0.068	0.020	0.031	0.053
<i>H=12</i>	MAE	0.00016	0.00003	0.00030	0.00015	0.00088	0.00003	0.00003	0.00002	0.00073	0.00002	0.00002
	MAPE	23.69	8.17	38.90	30.90	30.26	10.60	9.50	8.97	26.34	8.28	7.72
	%MAE>0.005	0.0	0.0	0.4	0.0	5.2	0.0	0.0	0.0	4.4	0.0	0.0
	%MAPE>2	79.6	42.8	91.2	87.2	94.4	48.4	45.6	42.8	95.2	42.4	39.6
	Time (s)	0.055	1.163	0.038	0.286	0.021	0.074	0.082	0.096	0.024	0.047	0.074
<i>H=15</i>	MAE	0.00008	0.00002	0.00013	0.00006	0.00041	0.00001	0.00001	0.00001	0.00037	0.00001	0.00001
	MAPE	34.86	17.84	36.00	28.07	33.52	13.10	11.98	12.01	31.41	12.29	12.68
	%MAE>0.005	0.0	0.0	0.0	0.0	1.6	0.0	0.0	0.0	1.6	0.0	0.0
	%MAPE>2	85.6	61.6	91.6	90.0	90.8	55.6	54.4	48.8	93.6	46.8	46.8
	Time (s)	0.073	2.205	0.067	0.439	0.029	0.140	0.143	0.146	0.034	0.066	0.106
<i>H=18</i>	MAE	0.00002	0.00000	0.00002	0.00001	0.00003	0.00000	0.00000	0.00000	0.00003	0.00000	0.00000
	MAPE	48.76	24.69	39.31	33.11	36.07	16.47	14.63	13.75	33.07	13.04	11.76
	%MAE>0.005	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	%MAPE>2	95.2	78.8	94.8	89.2	94.0	62.0	61.2	59.6	93.2	57.2	56.4
	Time (s)	0.100	2.313	0.099	0.663	0.034	0.169	0.173	0.211	0.041	0.092	0.143
<i>H=20</i>	MAE	0.00001	0.00000	0.00001	0.00001	0.00001	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000
	MAPE	55.30	38.15	40.19	33.88	39.39	15.18	14.52	15.06	35.83	14.91	15.39
	%MAE>0.005	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	%MAPE>2	96.8	88.0	94.0	90.0	97.6	64.0	62.4	63.6	96.4	61.6	63.6
	Time (s)	0.114	2.848	0.106	0.773	0.039	0.200	0.220	0.247	0.049	0.109	0.163