The Multiple Discrete-Continuous Extreme Value (MDCEV) Model:
Role of Utility Function Parameters, Identification Considerations, and Model Extensions

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ABSTRACT

Many consumer choice situations are characterized by the simultaneous demand for multiple alternatives that are imperfect substitutes for one another. A simple and parsimonious Multiple Discrete-Continuous Extreme Value (MDCEV) econometric approach to handle such multiple discreteness was formulated by Bhat (2005a) within the broader Kuhn-Tucker (KT) multiple discrete-continuous economic consumer demand model of Wales and Woodland (1983). This paper examines several issues associated with the MDCEV model and other extant KT multiple discrete-continuous models. Specifically, the paper proposes a new utility function form that enables clarity in the role of each parameter in the utility specification, presents identification considerations associated with both the utility functional form as well as the stochastic nature of the utility specification, extends the MDCEV model to the case of price variation across goods and to general error covariance structures, discusses the relationship between earlier KT-based multiple discrete-continuous models, and illustrates the many technical nuances and identification considerations of the multiple discrete-continuous model structure through empirical examples.

Keywords: Discrete-continuous system, Multiple discreteness, Kuhn-Tucker demand systems, Mixed discrete choice, Random Utility Maximization.
1. INTRODUCTION

Multiple discreteness (*i.e.*, the choice of multiple, but not necessarily all, alternatives simultaneously) is a rather ubiquitous characteristic of consumer decision-making. For instance, an individual may decide to participate in multiple kinds of maintenance and leisure activities within a given time period (Bhat, 2005a), or a household may own a mix of different kinds of vehicles (such as a sedan and a pick-up truck or a sedan and a minivan; see Bhat and Sen, 2006), or a consumer may purchase multiple flavors of yoghurt in the same shopping purchase (Kim et al., 2002). Such multiple discrete situations may be modeled using the traditional random utility-based (RUM) single discrete choice models by identifying all combinations or bundles of the “elemental” alternatives, and treating each bundle as a “composite” alternative (the term “single discrete choice” is used to refer to the case where a decision-maker chooses only one alternative from a set of alternatives). A problem with this approach, however, is that the number of composite alternatives explodes with the number of elemental alternatives. Another approach is to use the multivariate probit (logit) methods of Manchanda et al. (1999), Baltas (2004), Edwards and Allenby (2003), and Bhat and Srinivasan (2005). But this approach is not based on a rigorous underlying utility-maximizing framework of multiple discreteness; rather, it represents a statistical “stitching” of univariate utility maximizing models. In both the approaches discussed above to handle multiple discreteness, there is also no explicit way to accommodate the diminishing marginal returns (*i.e.*, satiation) in the consumption of an alternative. Additionally, and related to the above point, it is very cumbersome, even if conceptually feasible, to include a continuous dimension of choice or the intensive margin of choice (for example, modeling the durations of participation in the chosen activity purposes, in addition to the choice of activity purpose).

Wales and Woodland (1983) proposed two alternative ways to handle situations of multiple discreteness within a behaviorally-consistent utility maximizing framework. Both approaches assume a direct utility function \( U(x) \) that is assumed to be quasi-concave, increasing, and continuously differentiable with respect to the consumption quantity vector \( x \).\(^1\) Consumers maximize the utility function subject to a linear budget constraint, which is binding in that all the available budget is invested in the consumption of the goods; that is, the budget constraint has an

\[^1\] The assumption of a quasi-concave utility function is simply a manifestation of requiring the indifference curves to be convex to the origin (see Deaton and Muellbauer, 1980, page 30 for a rigorous definition of quasi-concavity). The assumption of an increasing utility function implies that \( U(x^1) > U(x^0) \) if \( x^1 > x^0 \).
equality sign rather than a ‘≤’ sign. This binding nature of the budget constraint is the result of assuming an increasing utility function, and also implies that at least one good will be consumed. The difference in the two alternative approaches proposed by Wales and Woodland (1983) is in how stochasticity, non-negativity of consumption, and corner solutions (i.e., zero consumption of some goods) are accommodated, as briefly discussed below (see Wales and Woodland, 1983 and Phaneuf et al., 2000 for additional details).

The first approach, which Wales and Woodland label as the Amemiya-Tobin approach, is an extension of the classic microeconomic approach of adding normally distributed stochastic terms to the budget-constrained utility-maximizing share equations. In this approach, the direct utility function $U(x)$ itself is assumed to be deterministic by the analyst, and stochasticity is introduced post-utility maximization. The justification for the addition of such normally distributed stochastic terms to the deterministic utility-maximizing allocations is based on the notion that consumers make errors in the utility-maximizing process, or that there are measurement errors in the collection of share data, or that there are unknown factors (from the analyst’s perspective) influencing actual consumed shares. However, the addition of normally distributed error terms to the share equations in no way restricts the shares to be positive and less than 1. The contribution of Wales and Woodland was to devise a stochastic formulation, based on the earlier work of Tobin (1958) and Amemiya (1974), that (a) respects the unit simplex range constraint for the shares, (b) accommodates the restriction that the shares sum to one, and (c) allows corner solutions in which one or more alternatives are not consumed. They achieve this by assuming that the observed shares for the $(K-1)$ of the $K$ alternatives follow a truncated multivariate normal distribution (note that since the shares across alternatives have to sum to one, there is a singularity generated in the $K$-variate covariance matrix of the $K$ shares, which can be accommodated by dropping one alternative).

The second approach proposed by Wales and Woodland, which they label as the Kuhn-Tucker approach, is based on the Kuhn Tucker or KT (1951) first-order conditions for constrained random utility maximization. Unlike the Amemiya-Tobin approach, the KT approach employs a more direct stochastic specification by assuming the utility function $U(x)$ to be random (from the analyst’s perspective) over the population, and then derives the consumption vector for the random utility specification subject to the linear budget constraint by using the KT conditions for constrained optimization. Thus, the stochastic nature of the
consumption vector in the KT approach is based fundamentally on the stochastic nature of the utility function. Consequently, the KT approach immediately satisfies all the restrictions of utility theory, and the stochastic KT first-order conditions provide the basis for deriving the probabilities for each possible combination of corner solutions (zero consumption) for some goods and interior solutions (strictly positive consumption) for other goods. The singularity imposed by the “adding-up” constraint is accommodated in the KT approach by employing the usual differencing approach with respect to one of the goods, so that there are only \((K-1)\) interdependent stochastic first-order conditions.

Among the two approaches discussed above, the KT approach constitutes a more theoretically unified and consistent framework for dealing with multiple discreteness consumption patterns, since stochasticity is introduced directly in the utility function. However, the approach did not receive much attention until relatively recently because the random utility distribution assumptions used by Wales and Woodland lead to a complicated likelihood function that entails multi-dimensional integration. Kim et al. (2002) addressed this issue by using the Geweke-Hajivassiliou-Keane (or GHK) simulator to evaluate the multivariate normal integral appearing in the likelihood function in the KT approach. Also, different from Wales and Woodland, Kim et al. used a generalized variant of the well-known translated constant elasticity of substitution (CES) direct utility function (see Pollak and Wales, 1992; page 28) rather than the quadratic direct utility function used by Wales and Woodland. In any case, the Kim et al. approach, like the Wales and Woodland approach, is not realistic for practical applications and is unnecessarily complicated because of the need to evaluate a multivariate normal integral in the likelihood function. In contrast, Bhat (2005a) introduced a simple and parsimonious econometric approach to handle multiple discreteness, also based on the generalized variant of the translated CES utility function but with a multiplicative log-extreme value error term. Bhat’s model, labeled the multiple discrete-continuous extreme value (MDCEV) model, is analytically tractable in the probability expressions and is practical even for situations with a large number of discrete consumption alternatives. In fact, the MDCEV model represents the multinomial logit (MNL) form-equivalent for multiple discrete-continuous choice analysis and collapses exactly to the MNL in the case that each (and every) decision-maker chooses only one alternative.

Independent of the above works of Kim et al. and Bhat, there has been a stream of research in the environmental economics field (see Phaneuf et al., 2000; von Haefen et al., 2004;
von Haefen, 2003a; von Haefen, 2004; von Haefen and Phaneuf, 2005; Phaneuf and Smith, 2005) that has also used the KT approach to multiple discreteness. These studies use variants of the linear expenditure system (LES) and the translated CES for the utility functions, and use multiplicative log-extreme value errors. However, the error specification in the utility function is different from that in Bhat’s MDCEV model, resulting in a different form for the likelihood function.

Within the context of the KT approach to handling multiple discreteness, the purpose of this research is five-fold. The first objective is to reformulate the utility specification used in earlier studies in a way that explicitly clarifies the role of each parameter in the utility specification. The second objective is to present identification considerations related to both the functional form as well as the stochastic nature of the utility specification. The third objective is to derive the MDCEV model expression for the case when there is price variation across goods and to extend the MDCEV model to accommodate generalized extreme value (GEV)-based and other correlation structures. The fourth objective is to discuss the relationship between the models of Kim et al. (2002), the KT formulations used in Environmental Economics, and the MDCEV formulation. The fifth objective is to illustrate the technical issues related to the properties and identification of the MDCEV model through empirical illustrations.

The rest of the paper is structured as follows. The next section formulates a functional form for the utility specification that enables the isolation of the role of different parameters in the specification. This section also identifies empirical identification considerations in estimating the parameters in the utility specification. Section 3 discusses the stochastic form of the utility specification, the resulting general structure for the probability expressions, and associated identification considerations. Section 4 derives the MDCEV structure for the new utility functional form used in the current paper, and extends this structure to more general error structure specifications. For presentation ease, Sections 2 through 4 consider the case of the absence of an outside good. In section 5, we extend the discussions of the earlier sections to the case when an outside good is present. Section 6 compares the earlier multiple discrete-continuous models used in the literature with the one formulated in the current paper. Section 7 provides empirical illustrations to reinforce the theoretical issues discussed in earlier sections. The final section concludes the paper.
2. FUNCTIONAL FORM OF UTILITY SPECIFICATION

We consider the following functional form for utility in this paper, based on a generalized variant of the translated CES utility function:

\[
U(x) = \sum_{k=1}^{K} \frac{\gamma_k}{\alpha_k} \psi_k \left( \frac{x_k}{\gamma_k + 1} \right)^{\alpha_k} - 1
\]

(1)

where \(U(x)\) is a quasi-concave, increasing, and continuously differentiable function with respect to the consumption quantity \((Kx1)\)-vector \(x\) \((x_k \geq 0 \text{ for all } k)\), and \(\psi_k\), \(\gamma_k\) and \(\alpha_k\) are parameters associated with good \(k\). The function in Equation (1) is a valid utility function if \(\psi_k > 0\) and \(\alpha_k \leq 1\) for all \(k\). Further, for the present, we assume that there is no outside good, so that corner solutions (i.e., zero consumptions) are allowed for all the goods \(k\). Thus, the term \(\gamma_k\), which is a translation parameter, should be greater than zero for all \(k\).\(^2\) The reader will note that there is an assumption of additive separability of preferences in the utility form of Equation (1), which immediately implies that none of the goods are \textit{a priori} inferior and all the goods are strictly Hicksian substitutes (see Deaton and Muellbauer, 1980; page 139). Additionally, additive separability implies that the marginal utility with respect to any good is independent of the levels of all other goods.

The form of the utility function in Equation (1) is different from that used in earlier studies. The reason for the specific functional form adopted here is to highlight the role of the various parameters \(\psi_k\), \(\gamma_k\) and \(\alpha_k\), and explicitly indicate the inter-relationships between these parameters that relate to theoretical and empirical identification issues. Also, the utility form returns a value of zero for any good if it is not consumed.\(^3\) Finally, it should be noted that the utility form of Equation (1) collapses to the following linear expenditure system (LES) form when \(\alpha_k \to 0 \forall k\) (see Appendix A).

\[
U(x) = \sum_{k=1}^{K} \gamma_k \psi_k \ln \left( \frac{x_k}{\gamma_k + 1} \right)
\]

(2)

\(^2\) As illustrated in Kim et al. (2002) and Bhat (2005a), the presence of the translation parameters makes the indifference curves strike the consumption axes at an angle (rather than being asymptotic to the consumption axes), thus allowing corner solutions.

\(^3\) As we will show later, however, the utility form we adopt is behaviorally and observationally indistinguishable from those used in Bhat (2005a), Kim et al. (2002), and those in environmental economics. But we have received several queries in response to the form used in the previous studies on the seeming inconsistency of non-zero utility contribution even at zero consumption of a good. The current utility form addresses this issue head-on.
2.1 Role of Parameters in Utility Specification

Role of $\psi_k$

The role of $\psi_k$ can be inferred by computing the marginal utility of consumption with respect to good $k$, which is:

$$\frac{\partial U(x)}{\partial x_k} = \psi_k \left( \frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k-1}$$

(3)

It is obvious from above that $\psi_k$ represents the baseline marginal utility, or the marginal utility at the point of zero consumption. This is the case regardless of the values of $\gamma_k$ and $\alpha_k$ (unlike in earlier studies where the baseline marginal utility, in general, is a function of multiple parameters). For two goods $i$ and $j$ with same unit prices, a higher baseline marginal utility for good $i$ relative to good $j$ implies that an individual will increase overall utility more by consuming good $i$ rather than $j$ at the point of no consumption of any goods. That is, the consumer will be more likely to consume good $i$ than good $j$. Thus, a higher baseline $\psi_k$ implies less likelihood of a corner solution for good $k$.

Role of $\gamma_k$

An important role of the $\gamma_k$ terms is to shift the position of the point at which the indifference curves are asymptotic to the axes from $(0,0,0,...,0)$ to $(-\gamma_1,-\gamma_2,-\gamma_3,...,-\gamma_K)$, so that the indifference curves strike the positive orthant with a finite slope. This, combined with the consumption point corresponding to the location where the budget line is tangential to the indifference curve, results in the possibility of zero consumption of good $k$. To see this, consider two goods 1 and 2 with $\psi_1 = \psi_2 = 1$, $\alpha_1 = \alpha_2 = 0.5$, and $\gamma_2 = 1$. Figure 1 presents the profiles of the indifference curves in this two-dimensional space for various values of $\gamma_1 (\gamma_1 > 0)$. To compare the profiles, the indifference curves are all drawn to go through the point $(0,8)$. The reader will also note that all the indifference curve profiles strike the y-axis with the same slope. As can be observed from the figure, the positive values of $\gamma_1$ and $\gamma_2$ lead to indifference curves that cross the axes of the positive orthant, allowing for corner solutions. The indifference curve
profiles are asymptotic to the $x$-axis at $y = -1$ (corresponding to the constant value of $\gamma_2 = 1$), while they are asymptotic to the $y$-axis at $x = -\gamma_1$.

Figure 1 also points to another role of the $\gamma_k$ term as a satiation parameter. Specifically, the indifference curves get steeper in the positive orthant as the value of $\gamma_1$ increases, which implies a stronger preference (or lower satiation) for good 1 as $\gamma_1$ increases (with steeper indifference curve slopes, the consumer is willing to give up more of good 2 to obtain 1 unit of good 1). This point is particularly clear if we examine the profile of the sub-utility function for alternative $k$. Figure 2 plots the function for alternative $k$ for $\alpha_k \rightarrow 0$ and $\psi_k = 1$, and for different values of $\gamma_k$. All of the curves have the same slope $\psi_k = 1$ at the origin point, because of the functional form used in this paper. However, the marginal utilities vary for the different curves at $x_k > 0$. Specifically, the higher the value of $\gamma_k$, the less is the satiation effect in the consumption of $x_k$. It is important to note that the entire range of satiation effects from immediate and full satiation (flat line) to linear satiation (constant marginal utility) can be accommodated by different values of $\gamma_k$ for any given $\alpha_k$ value.

**Role of $\alpha_k$**

The express role of $\alpha_k$ is to reduce the marginal utility with increasing consumption of good $k$; that is, it represents a satiation parameter. When $\alpha_k = 1$ for all $k$, this represents the case of absence of satiation effects or, equivalently, the case of constant marginal utility. The utility function in Equation (1) in such a situation collapses to $\sum \psi_k x_k$, which represents the perfect substitutes case as proposed by Deaton and Muellbauer (1980) and applied in Hanemann (1984), Chiang (1991), Chintagunta (1993), and Arora *et al.* (1998), among others. Intuitively, when there is no satiation and the unit good prices are all the same, the consumer will invest all expenditure on the single good with the highest baseline (and constant) marginal utility (*i.e.*, the highest $\psi_k$ value). This is the case of single discreteness.\(^4\) As $\alpha_k$ moves downward from the

\(^4\) If there is price variation across goods, one needs to take the derivative of the utility function with respect to expenditures ($e_k$) on the goods. In the case that $\alpha_k = 1$ for all $k$, $U = \sum \psi_k (e_k / p_k)$, where $\psi_k$ is the unit price of good $k$. 

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value of 1, the satiation effect for good $k$ increases. When $\alpha_k \to 0$, the utility function collapses to the form in Equation (2), as discussed earlier. $\alpha_k$ can also take negative values and, when $\alpha_k \to -\infty$, this implies immediate and full satiation. Figure 3 plots the utility function for alternative $k$ for $\gamma_k = 1$ and $\psi_k = 1$, and for different values of $\alpha_k$. Again, all of the curves have the same slope $\psi_k = 1$ at the origin point, and accommodate different levels of satiation through different values of $\alpha_k$ for any given $\gamma_k$ value.

2.2 Empirical Identification Issues Associated with Utility Form

The discussion in the previous section indicates that $\psi_k$ reflects the baseline marginal utility, which controls whether or not a good is selected for positive consumption (or the extensive margin of choice). The role of $\gamma_k$ is to enable corner solutions, though it also governs the level of satiation. The purpose of $\alpha_k$ is solely to allow satiation. Thus, for a given extensive margin of choice of good $k$, $\gamma_k$ and $\alpha_k$ influence the quantity of good $k$ consumed (or the intensive margin of choice) through their impact on satiation effects. The precise functional mechanism through which $\gamma_k$ and $\alpha_k$ impact satiation are, however, different; $\gamma_k$ controls satiation by translating time, while $\alpha_k$ controls satiation by exponentiating time. Clearly, both these effects operate in different ways, and different combinations of their values lead to different satiation profiles. However, empirically speaking, it is very difficult to disentangle the two effects separately, which leads to serious empirical identification problems and estimation breakdowns when one attempts to estimate both $\gamma_k$ and $\alpha_k$ parameters for each good. In fact, for a given $\psi_k$ value, it is possible to closely approximate a sub-utility function profile based on a combination of $\gamma_k$ and $\alpha_k$ values with a sub-utility function based solely on $\gamma_k$ or $\alpha_k$ values. This is illustrated in Figures 4a through 4d for $\psi_k = 1$ and for different satiation levels. In these figures, the subutility functions based solely on $\gamma_k$ assume $\alpha_k = 0$ (i.e., these functions take the form of Equation (2)), while those based solely on $\alpha_k$ for incorporating satiation assume $\gamma_k = 1$ (note that $\gamma_k$, even if fixed, has to be positive to allow corner solutions). In all the figures, the

\[ \frac{\partial U}{\partial e_k} = \frac{\psi_k}{p_k} \]

In this situation, the consumer will invest all expenditures on the single good with the highest price-normalized marginal (and constant) utility $\psi_k/p_k$. 


profile based only on \( \gamma_k \) (the \( \gamma_k \)-profile) or \( \alpha_k \) (the \( \alpha_k \)-profile) tracks the profile based on the combination of values (the combination profile) reasonably well. For moderate satiations (Figures 4a and 4b), one of the two profiles does better than the other, based on how close the \( \gamma_k \) and \( \alpha_k \) values in the combination profile are to the assumed value of \( \alpha_k^* = 0 \) for the \( \gamma_k \)-profile and \( \gamma_k^{**} = 1 \) for the \( \alpha_k \)-profile. For very low and very high satiations, both the \( \alpha_k \)-profile and the \( \gamma_k \)-profile track the combination profile very closely. In actual application, it would behoove the analyst to estimate models based on both the \( \alpha_k \)-profile and the \( \gamma_k \)-profile, and choose a specification that provides a better statistical fit.\(^5\)

In cases where \( \alpha_k \) values are estimated, these values need to be bounded from above at the value of 1. To enforce these conditions, \( \alpha_k \) can be parameterized as \( [1 - \exp(-\delta_k)] \). Further, to allow the satiation parameters to vary across individuals, Bhat (2005a) writes \( \delta_k = \theta_k' y_k \), where \( y_k \) is a vector of individual characteristics impacting satiation for the \( k^{\text{th}} \) alternative, and \( \theta_k \) is a corresponding vector of parameter. In cases where \( \gamma_k \) values are estimated, these values need to be greater than zero, which can be maintained by reparameterizing \( \gamma_k \) as \( \exp(\mu_k) \). Additionally, the translation parameters can be allowed to vary across individuals by writing \( \mu_k = \varphi_k' w_k \), where \( w_k \) is a vector of individual characteristics for the \( k^{\text{th}} \) alternative, and \( \varphi_k \) is a corresponding vector of parameters.

### 3. STOCHASTIC FORM OF UTILITY FUNCTION

The KT approach employs a direct stochastic specification by assuming the utility function \( U(\alpha) \) to be random over the population. In all the earlier applications of the KT approach for multiple discreteness, a multiplicative random element is introduced to the baseline marginal utility of each good as follows:

\[
\psi(z_k, \varepsilon_k) = \psi(z_k) \cdot e^{\alpha},
\]

\(^5\) Alternatively, the analyst can stick with one functional form \textit{a priori}, but experiment with various fixed values of \( \alpha_k \) for the \( \gamma_k \)-profile and \( \gamma_k \) for the \( \alpha_k \)-profile.
where $z_k$ is a set of attributes characterizing alternative $k$ and the decision maker, and $\epsilon_k$ captures idiosyncratic (unobserved) characteristics that impact the baseline utility for good $j$. The exponential form for the introduction of the random term guarantees the positivity of the baseline utility as long as $\psi(z_k) > 0$. To ensure this latter condition, $\psi(z_k)$ is further parameterized as $\exp(\beta z_k)$, which then leads to the following form for the baseline random utility associated with good $k$:

$$\psi(z_k, \epsilon_k) = \exp(\beta z_k + \epsilon_k).$$  (5)

The $z_k$ vector in the above equation includes a constant term. The overall random utility function of Equation (1) then takes the following form:

$$\cup(x) = \sum_k \frac{\gamma_k}{\alpha_k} \left[ \exp(\beta z_k + \epsilon_k) \right] \left[ \left( \frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right].$$  (6)

From the analyst’s perspective, the individual is maximizing random utility subject to the binding linear budget constraint that $\sum e_k = E$, where $E$ is total expenditure or income (or some other appropriately defined total budget quantity), $e_k = p_k x_k$, and $p_k$ is the unit price of good $k$.

### 3.1 Optimal Expenditure Allocations

The analyst can solve for the optimal expenditure allocations by forming the Lagrangian and applying the Kuhn-Tucker (KT) conditions. The Lagrangian function for the problem is:

$$\mathcal{L} = \sum_k \frac{\gamma_k}{\alpha_k} \left[ \exp(\beta z_k + \epsilon_k) \right] \left[ \left( \frac{e_k}{\gamma_k p_k} + 1 \right)^{\alpha_k} - 1 \right] - \lambda \left[ \sum_{k=1}^K e_k - E \right],$$  (7)

where $\lambda$ is the Lagrangian multiplier associated with the expenditure constraint (that is, it can be viewed as the marginal utility of total expenditure or income). The KT first-order conditions for the optimal expenditure allocations (the $e_k^*$ values) are given by:

$$\left[ \frac{\exp(\beta z_k + \epsilon_k)}{p_k} \right] \left( \frac{e_k^*}{\gamma_k p_k} + 1 \right)^{\alpha_k - 1} - \lambda = 0, \text{ if } e_k^* > 0, \ k = 1, 2, \ldots, K$$  (8)

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6 For reasons that will become clear later, we solve for the optimal expenditure allocations $e_k$ for each good, not the consumption amounts $x_k$ of each good. This is different from earlier studies that focus on the consumption of goods.
The optimal demand satisfies the conditions in Equation (8) plus the budget constraint \( \sum_{k=1}^{K} e_k^* = E \). The budget constraint implies that only \( K-1 \) of the \( e_k^* \) values need to be estimated, since the quantity consumed of any one good is automatically determined from the quantity consumed of all the other goods. To accommodate this constraint, designate activity purpose 1 as a purpose to which the individual allocates some non-zero amount of consumption (note that the individual should participate in at least one of the \( K \) purposes, given that \( E > 0 \)). For the first good, the KT condition may then be written as:

\[
\lambda = \frac{\exp(\beta'z_k + \varepsilon_k)}{p_k} \left( \frac{e_k^*}{\gamma_k p_k} + 1 \right)^{\alpha_k - 1} - \lambda < 0, \text{ if } e_k^* = 0, \ k = 1, 2, \ldots, K
\]

Substituting for \( \lambda \) from above into Equation (8) for the other activity purposes \( k = 2, \ldots, K \), and taking logarithms, we can rewrite the KT conditions as:

\[
V_k + e_k = V_1 + \varepsilon_1 \text{ if } e_k^* > 0 \ (k = 2, 3, \ldots, K)
\]

\[
V_k + e_k < V_1 + \varepsilon_1 \text{ if } e_k^* = 0 \ (k = 2, 3, \ldots, K), \text{ where}
\]

\[
V_k = \beta'z_k + (\alpha_k - 1)\ln \left( \frac{e_k^*}{\gamma_k p_k} + 1 \right) - \ln p_k \ (k = 1, 2, 3, \ldots, K).
\]

Also, note that, in Equation (10), a constant cannot be identified in the \( \beta'z_k \) term for one of the \( K \) alternatives (because only the difference in the \( V_k \) from \( V_1 \) matters). Similarly, individual-specific variables are introduced in the \( V_k \)'s for \((K-1)\) alternatives, with the remaining alternative serving as the base.\(^7\)

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\(^7\) These identification conditions are similar to those in the standard discrete choice model, though the origin of the conditions is different between standard discrete choice models and the multiple discrete-continuous models. In standard discrete choice models, individuals choose the alternative with highest utility, so that all that matters is relative utility. In multiple discrete-continuous models, the origin of these conditions is the adding up (or budget) constraint associated with the quantity of consumption of each good that leads to the KT first order conditions of Equation (10).
3.2 General Econometric Model Structure and Identification

To complete the model structure, the analyst needs to specify the error structure. In the general case, let the joint probability density function of the $\varepsilon_k$ terms be $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_K)$. Then, the probability that the individual allocates expenditure to the first $M$ of the $K$ goods is:

$$P(e_1^*, e_2^*, e_3^*, \ldots, e_M^*, 0, 0, \ldots, 0) = \left| J \right| \int \int \int \int \int f(\varepsilon_1, V_1 - V_2 + \varepsilon_1, V_1 - V_3 + \varepsilon_1, \ldots, V_1 - V_M + \varepsilon_1, e_{M+1}, e_{M+2}, \ldots, e_{K-1}, e_K) \, \text{d}e_K \, \text{d}e_{K-1} \ldots \text{d}e_{M+2} \text{d}e_{M+1} \text{d}e_1,$$

where $J$ is the Jacobian whose elements are given by (see Bhat, 2005a):

$$J_{ih} = \frac{\partial[V_1 - V_{i+1} + \varepsilon_1]}{\partial e_{h+1}}; \quad i, h = 1, 2, \ldots, M - 1. \quad (12)$$

The probability expression in Equation (11) is a $(K-M+1)$-dimensional integral. The expression for the probability of all goods being consumed is one-dimensional, while the expression for the probability of only the first good being consumed is $K$-dimensional. The dimensionality of the integral can be reduced by one by noticing that the KT conditions can also be written in a differenced form. To do so, define $\tilde{\varepsilon}_{k1} = \varepsilon_k - \varepsilon_1$, and let the implied multivariate distribution of the error differences be $g(\tilde{\varepsilon}_{21}, \tilde{\varepsilon}_{31}, \ldots, \tilde{\varepsilon}_{K1})$. Then, Equation (11) may be written in the equivalent $(K-M)$-integral form shown below:

$$P(e_1^*, e_2^*, e_3^*, \ldots, e_M^*, 0, 0, \ldots, 0) = \left| J \right| \int \int \int \int g(V_1 - V_2, V_1 - V_3, \ldots, V_1 - V_M, \tilde{\varepsilon}_{M+1,1}, \tilde{\varepsilon}_{M+2,1}, \ldots, \tilde{\varepsilon}_{K,1}) \, \text{d}\tilde{\varepsilon}_{K,1} \, \text{d}\tilde{\varepsilon}_{K-1,1} \ldots \text{d}\tilde{\varepsilon}_{M+1,1} \quad (13)$$

The equation above indicates that the probability expression for the observed optimal expenditure pattern of goods is completely characterized by the $(K-1)$ error terms in difference form. Thus, all that is estimable is the $(K-1) \times (K-1)$ covariance matrix of the error differences. In other words, it is not possible to estimate a full covariance matrix for the original error terms $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_K)$ because there are infinite possible densities for $f(.)$ that can map into the same $g(.)$ density for the error differences (see Train, 2003, page 27, for a similar situation in the context of standard discrete choice models). There are many possible ways to normalize $f(.)$ to account for this situation. For example, one can assume an identity covariance matrix for $f(.)$, which
automatically accommodates the normalization that is needed. Alternatively, one can estimate \( g(.) \) without reference to \( f(.) \).

In the general case when the unit prices \( p_k \) vary across goods, it is possible to estimate \( K \times (K - 1)/2 \) parameters of the full covariance matrix of the error differences, as just discussed (though the analyst might want to impose constraints on this full covariance matrix for ease in interpretation and stability in estimation). However, when the unit prices are not different among the goods, an additional scaling restriction needs to be imposed. To see this, consider the case of independent and identically distributed error terms for the \( \varepsilon_k \) terms, which leads to a \((K-1)x(K-1)\) covariance matrix for \( \tilde{\varepsilon}_{k_1} \) \( (k = 2, 3, \ldots, K) \) with diagonal elements equal to twice the value of scale parameter of the \( \varepsilon_k \) terms and off-diagonal elements equal to the scale parameter of the \( \varepsilon_k \) terms. Let the unit prices of all goods be the same (see Bhat, 2005a; Bhat and Sen, 2006; Bhat et al., 2006 for examples where the weights or prices on the goods in the budget constraint are equal). Consider the utility function in Equation (6) and another utility function as given below:

\[
\tilde{U} = \sum_k \gamma_k \left[ \exp\{\sigma \times (\beta z_k + \varepsilon_k)\} \right] \cdot \left\{ \frac{x_k}{\gamma_k + 1} \right\} + 1 - 1 \right) \] (14)

The scale of the error terms in the utility function in the above expression is \( \sigma \) times the scale of the error terms in Equation (6). Let \( \alpha_k^* = \sigma (\alpha_k - 1) + 1 \), where \( \alpha_k \) is the satiation parameter in the original Equation (6)\(^8\). The KT conditions for optimal expenditure for this modified utility function can be shown to be:

\[
V_k^* + \sigma \varepsilon_k = V_1^* + \sigma \varepsilon_1 \text{ if } \varepsilon_k^* > 0 \text{ } (k = 2, 3, \ldots, K) \]

\[
V_k^* + \sigma \varepsilon_k < V_1^* + \sigma \varepsilon_1 \text{ if } \varepsilon_k^* = 0 \text{ } (k = 2, 3, \ldots, K), \text{ where} \]

\[
V_k^* = \sigma \beta z_k + (\alpha_k^* - 1) \ln \left( \frac{e_k^*}{\gamma_k p_k} + 1 \right) - \ln p_k \text{ } (k = 1, 2, 3, \ldots, K) \]

\[
= \sigma \beta z_k + \sigma (\alpha_k - 1) \ln \left( \frac{e_k^*}{\gamma_k p_k} + 1 \right) - \ln p_k \text{ } (k = 1, 2, 3, \ldots, K). \]

\(^8\) Note that \( \alpha_k^* \) is less than or equal to 1 by definition, because \( \alpha_k \) is less than or equal to 1 and the scale \( \sigma \) should be non-negative.
If the unit prices are not all the same (i.e., the unit prices of at least two of the $K$ goods are different), the KT conditions above are different from the KT conditions in Equation (10). That is, the utility function in Equation (14) is unique and different from the utility function in Equation (6), which implies that the scale $\sigma$ is identified. However, if the unit prices are all the same ($p_k = p, \forall k$), it is straightforward to note that the KT conditions above collapse exactly to the KT conditions in Equation (10). In this case, the utility function in Equation (14) cannot be uniquely identified from the utility function in Equation (6), which implies that the scale $\sigma$ is not identified theoretically. For convenience, the analyst can set the scale to 1.

In the case that the analyst uses a heteroscedastic specification with no variation in unit prices across alternatives, the scale of one of the alternatives has to be set to unity (similar to the case of the heteroscedastic extreme value or HEV model of Bhat, 1995). With a general error structure and no variation in unit prices, the identification considerations associated with a standard discrete choice model with correlated errors apply (see Train, 2003; Chapter 2).

4. SPECIFIC MODEL STRUCTURES

4.1 The MDCEV Model Structure

Following Bhat (2005a), we specify an extreme value distribution for $\varepsilon_k$ and assume that $\varepsilon_k$ is independent of $z_k$ ($k = 1, 2, \ldots, K$). The $\varepsilon_k$’s are also assumed to be independently distributed across alternatives with a scale parameter of $\sigma$ ($\sigma$ can be normalized to one if there is no variation in unit prices across goods). Let $V_k$ be defined as follows:

$$V_k = \beta'z_k + (\alpha_k - 1)\ln\left(\frac{\varepsilon_k^*}{P_k} + 1\right) - \ln p_k \quad (k = 1, 2, 3, \ldots, K),$$

when the $\alpha$-profile is used, and

$$V_k = \beta'z_k - \ln\left(\frac{\varepsilon_k^*}{\gamma_k P_k} + 1\right) - \ln p_k \quad (k = 1, 2, 3, \ldots, K),$$

when the $\gamma$-profile is used.

As discussed earlier, it is generally not possible to estimate the $V_k$ form in Equation (10), because the $\alpha_k$ terms and $\gamma_k$ terms serve a similar satiation role.

From Equation (11), the probability that the individual allocates expenditure to the first $M$ of the $K$ goods ($M \geq 1$) is:
where $\lambda$ is the standard extreme value density function and $\Lambda$ is the standard extreme value cumulative distribution function. The expression in Equation (17) simplifies to a remarkably simple and elegant closed-form expression. Bhat derived the form of the Jacobian for the case of equal unit prices across goods, which however can be extended in a simple fashion to accommodate the more general case of different unit prices. The resulting form for the determinant of the Jacobian has a compact structure given by:

$$|J| = \left( \prod_{i=1}^{M} c_i \right) \left( \sum_{i=1}^{M} \frac{1}{c_i} \right) \left( 1 - \frac{\alpha_i}{e_i^* + \gamma P_i} \right)^9,$$

(18)

The integration in Equation (17) also collapses to a closed form expression (see Appendix B), providing the following overall expression:

$$P(e_1^*, e_2^*, ..., e_M^*, 0, 0, ..., 0)$$

$$= \frac{1}{\sigma^{M-1}} \left[ \prod_{i=1}^{M} c_i \right] \left[ \sum_{i=1}^{M} \frac{1}{c_i} \right] \left[ \prod_{i=1}^{M} e^{V_i/\sigma} \right] \left[ \frac{1}{\left( \sum_{k=1}^{K} e^{V_k/\sigma} \right)^M} \right] (M - 1)!$$

(19)

In the case when $M = 1$ (i.e., only one alternative is chosen), there are no satiation effects ($\alpha_k = 1$ for all $k$) and the Jacobian term drops out (that is, the continuous component drops out, because all expenditure is allocated to good 1). Then, the model in Equation (19) collapses to the standard MNL model. Thus, the MDCEV model is a multiple discrete-continuous extension of the standard MNL model.\(^{10}\)

The expression for the probability of the consumption pattern of the goods (rather than the expenditure pattern) can be derived to be:

\(^9\) It is important to note that this compact Jacobian form is independent of the assumptions regarding the density and correlation structure of the error terms.

\(^{10}\) Note that when $\alpha_k = 1$ for all $k$, $V_k = \beta_k z_k - \ln p_k$. Even if $M = 1$, when Equation (19) collapses to the MNL form, the scale $\sigma$ is estimable as long as the utility takes the functional form $V_k = \beta_k z_k - \ln p_k$ and there is price variation across goods. This is because the scale is the inverse of the coefficient on the $\ln p_k$ term (see Hanemann, 1984).
\[ P(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0) \]

\[
= \frac{1}{p_1} \cdot \frac{1}{\sigma^{M-1}} \left[ \prod_{i=1}^{M} f_i \right] \left[ \sum_{i=1}^{M} \frac{p_i}{f_i} \right] \left[ \prod_{i=1}^{M} e^{V_i/\sigma} \right] \left( \sum_{k=1}^{K} e^{V_k/\sigma} \right)^M (M-1)!,
\]  

(20)

where \( V_k \) is as defined earlier (see Equation 16) and \( f_i = \left( \frac{1-\alpha_i}{x_i^* + \gamma_i} \right) \). The expression in Equation (20) is, however, not independent of the good that is used as the first one (see the \( 1/p_1 \) term in front). In particular, different probabilities of the same consumption pattern arise depending on the good that is labeled as the first good (note that any good that is consumed may be designated as the first good). In terms of the likelihood function, the \( 1/p_1 \) term can be ignored, since it is simply a constant in each individual’s likelihood function. Thus, the same parameter estimates will result independent of the good designated as the first good for each individual, but it is still awkward to have different probability values for the same consumption pattern. This is particularly the case because different log-likelihood values at convergence will be obtained for different designations of the first good. Thus, the preferred approach is to use the probability expression for expenditure allocations, which will provide the same probability for a given expenditure pattern regardless of the good labeled as the first good. However, in the case that the first good is an outside numeraire good that is always consumed (see Section 5), then \( p_1 = 1 \) and one can use the consumption pattern probability expression or the expenditure allocation probability expression.

4.2 The Multiple Discrete-Continuous Generalized Extreme-Value (MDCGEV) Model Structure

Thus far, we have assumed that the \( \varepsilon_k \) terms are independently and identically extreme value distributed across alternatives \( k \). The analyst can extend the model to allow correlation across alternatives using a generalized extreme value (GEV) error structure. The remarkable advantage of the GEV structure is that it continues to result in closed-form probability expressions for any and all expenditure patterns. However, the derivation is tedious, and the expressions get unwieldy. In this paper, we provide the expressions for a specific nested logit structure with 4 alternatives, two alternatives (labeled 1 and 2) in nest A and the other two alternatives (labeled 3 and 4) in nest B (the derivation is available on request from the author).
The cumulative distribution function for the error terms in the utility expressions take the following form:

\[
\Lambda(e_1 < s_1, e_2 < s_2, e_3 < s_3, e_4 < s_4) = \exp\left[ -\left(e^{-x_1/\sigma_\theta_a} + e^{-x_2/\sigma_\theta_a} + e^{-x_3/\sigma_\theta_b} + e^{-x_4/\sigma_\theta_b}\right) \right] 
\]

Define the following:

\[
A_{12} = \left(e^{V_e / \sigma_{\theta_a}} + e^{V_e / \sigma_{\theta_b}}\right),
\]

\[
B_{34} = \left(e^{V_e / \sigma_{\theta_b}} + e^{V_e / \sigma_{\theta_a}}\right),
\]

\[
H = A_{12}^{\theta_a} + B_{34}^{\theta_b}.
\]

Then,

\[
P(x_1^*, 0, 0, 0) = \frac{A_{12}^{\theta_a-1} \cdot e^{V_e / \sigma_{\theta_a}}}{H}
\]

\[
P(x_1^*, x_2^*, 0, 0) = \frac{1}{\sigma} |J| \cdot e^{V_e / \sigma_{\theta_a}} \cdot e^{V_e / \sigma_{\theta_b}} \cdot \frac{A_{12}^{\theta_a-2}}{H^2} \left( \frac{A_{12}^{\theta_a}}{H} + \frac{1 - \theta_A}{\theta_A} \right)
\]

\[
P(x_1^*, 0, x_3^*, 0) = \frac{1}{\sigma^2} |J| \cdot e^{V_e / \sigma_{\theta_a}} \cdot e^{V_e / \sigma_{\theta_b}} \cdot \frac{B_{34}^{\theta_b-1}}{H^2} \left( \frac{2A_{12}^{\theta_a}}{H} + \frac{1 - \theta_A}{\theta_A} \right)
\]

\[
P(x_1^*, x_2^*, x_3^*, 0) = \frac{1}{\sigma^3} |J| \cdot e^{V_e / \sigma_{\theta_a}} \cdot e^{V_e / \sigma_{\theta_b}} \cdot e^{V_e / \sigma_{\theta_b}} \cdot \frac{A_{12}^{\theta_a-2} B_{34}^{\theta_b-1}}{H^2} \left\{ \frac{6A_{12}^{\theta_a} B_{34}^{\theta_b}}{H^2} + \frac{2(1 - \theta_A)}{\theta_A} \cdot \frac{B_{34}^{\theta_b}}{H} + \frac{2(1 - \theta_B)}{\theta_B} \cdot \frac{A_{12}^{\theta_a}}{H} + \frac{(1 - \theta_A)}{\theta_A} \cdot \frac{(1 - \theta_B)}{\theta_B} \right\}
\]

The probabilities for all other expenditure patterns for the 4 goods can be obtained by interchanging the labels 1, 2, 3, and 4.\(^{11}\)

### 4.3 The Mixed MDCEV Model

The MDCGEV structure is able to accommodate flexible correlation patterns. However, it is unable to accommodate random taste variation, and it imposes the restriction of equal scale of the error terms. Incorporating a more general error structure is straightforward through the use of a mixing distribution, which leads to the Mixed MDCEV (or MMDCEV) model. Specifically,

\(^{11}\) In all the expressions corresponding to the nested structure above, \(\sigma\) is identified only when there is price variation across alternatives (see Section 3.2).
the error term, \( \varepsilon_k \), may be partitioned into two components, \( \zeta_k \) and \( \eta_k \). The first component, \( \zeta_k \), can be assumed to be independently and identically Gumbel distributed across alternatives with a scale parameter of \( \sigma \). The second component, \( \eta_k \), can be allowed to be correlated across alternatives and to have a heteroscedastic scale. Let \( \eta = (\eta_1, \eta_2, \ldots, \eta_K)' \), and assume that \( \eta \) is distributed multivariate normal, \( \eta \sim N(0, \Omega) \).\(^{12}\)

For given values of the vector \( \eta \), one can follow the discussion of the earlier section and obtain the usual MDCEV probability that the first \( M \) of the \( k \) goods are consumed. The unconditional probability can then be computed as:

\[
P(e_1^*, e_2^*, e_3^*, \ldots, e_M^*, 0, 0, \ldots, 0) = \int_{\eta} \frac{1}{\sigma^{M-1}} \left[ \prod_{i=1}^{M} c_i \right] \sum_{i=1}^{M} \frac{1}{c_i} \left[ \prod_{i=1}^{M} e^{(V_i + \eta_i)/\sigma} \right] (M - 1)! dF(\eta).
\]

(22)

where \( F \) is the multivariate cumulative normal distribution (see Bhat, 2005a; Bhat and Sen, 2006; and Bhat et al., 2006).

The model in Equation (22) can be extended in a conceptually straightforward manner to also include random coefficients on the independent variables \( z_k \), and random-effects (or even random coefficients) in the \( \alpha_k \) satiation parameters (if the \( \alpha \) profile is used) or the \( \gamma_k \) parameters (if the \( \gamma \) profile is used).

4.3.1 Heteroscedastic structure within the MMDCEV framework

Consider the case where there is price variation across the alternatives, and the overall errors \( \varepsilon_k \) are heteroscedastic, but not correlated. Assuming a 4-alternative case for ease in presentation, the heteroscedastic structure may be specified in the form of the following covariance matrix for \( \varepsilon = (\varepsilon_{k1}, \varepsilon_{k2}, \varepsilon_{k3}, \varepsilon_{k4}) \):

\(^{12}\) Other distributions may also be used for \( \eta \). Note that the distribution of \( \eta \) can arise from an error components structure or a random coefficients structure or a combination of the two, similar to the case of the usual mixed logit model (see Bhat, 2005b).
where the first component on the right side corresponds to the IID covariance matrix of \( \zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \) and the second component is the heteroscedastic covariance matrix of \( \eta \). The covariance of error differences with respect to the first alternative is:

\[
Cov(\hat{\varepsilon}_1 = \hat{\varepsilon}_{21}, \hat{\varepsilon}_{31}, \hat{\varepsilon}_{41}) = \frac{\pi^2 \sigma^2}{6} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_4^2 \\ \omega_1^2 + \omega_2^2 & \omega_1^2 + \omega_3^2 & \omega_1^2 + \omega_4^2 \\ \omega_1^2 + \omega_2^2 + 2\sigma^2 & \omega_1^2 + \omega_3^2 + 2\sigma^2 & \omega_1^2 + \omega_4^2 + 2\sigma^2 \end{bmatrix} \tag{24}
\]

An inspection of the matrix above shows only four independent equations (the rank condition), implying that at most four parameters are estimable\(^1\). There are two ways to proceed with a normalization, as discussed below.

The first approach is to normalize \( \sigma \) and estimate the heteroscedastic covariance matrix of \( \eta \) (i.e., \( \omega_1, \omega_2, \omega_3, \omega_4 \)). Assume that \( \sigma \) is normalized to \( \tilde{\sigma} \), and let the corresponding values of \( \omega_k \) be \( \tilde{\omega}_k \) (\( k = 1, 2, 3, 4 \)). Then, the following equalities should hold, based on Equation (24), for any normalization of \( \sigma \) to \( \tilde{\sigma} \) (\( q = \pi^2 / 6 \) below):

\[
\begin{align*}
\omega_1^2 + q\sigma^2 &= \tilde{\omega}_1^2 + q\tilde{\sigma}^2 \\
\omega_2^2 + 2\sigma^2 &= \tilde{\omega}_2^2 + 2q\tilde{\sigma}^2 & (k = 2, 3, 4)
\end{align*}
\tag{25}
\]

The above equalities can be rewritten as:

\[
\tilde{\omega}_k^2 = \omega_k^2 + q\sigma^2 - q\tilde{\sigma}^2 & (k = 1, 2, 3, 4) \tag{26}
\]

The normalized variance terms \( \tilde{\omega}_k^2 \) must be greater than or equal to zero, which implies that the following conditions should hold:

\[
\omega_k^2 + q\sigma^2 \geq q\tilde{\sigma}^2 & (k = 1, 2, 3, 4) \tag{27}
\]

Intuitively, the above condition implies that the normalization on \( \tilde{\sigma} \) must be set low enough so that the overall “true” variance of each error term (= \( \omega_k^2 + q\sigma^2 \)) is larger than \( q\tilde{\sigma}^2 \). For example, setting \( \sigma \) to 1 would be inappropriate if the “true” variance of one or more alternatives is less than \( q\tilde{\sigma}^2 \).

\(^1\) Strictly speaking, one can estimate all the five parameters (\( \sigma, \omega_1, \omega_2, \omega_3, \omega_4 \)) because of the difference in the extreme value distributions used for \( \zeta \) and the normal distributions used for \( \eta \) (see Walker, 2002). However, the model will be near singular, and it is important to place the order/rank constraint.
than $\pi^2 / 6$. Since the “true” variance is unknown, the best the analyst can do is to normalize $\sigma$ to progressively smaller values and statistically examine the results.

The second approach is to normalize one of the $\omega_k$ terms instead of the $\sigma$ term. In this case, from Equation (24), we can write:

$$q\tilde{\sigma}_k^2 = \omega_k^2 + q\sigma^2 - \omega_i^2 = \frac{1}{2}\left[\omega_k^2 + \omega_i^2 + 2q\sigma^2 - \omega_i^2 - \tilde{\sigma}_i^2\right]; \ k = 2, 3, 4. \quad (28)$$

After some manipulations, the above equation may be rewritten as:

$$\tilde{\omega}_k^2 = \omega_k^2 + \tilde{\omega}_i^2 - \omega_i^2; \ k = 2, 3, 4. \quad (29)$$

Next, imposing the condition that the normalized terms $\tilde{\omega}_k^2$ must be greater than or equal to zero implies the following:

$$\tilde{\omega}_k^2 \geq \omega_k^2 - \omega_i^2 \ (k = 2, 3, 4). \quad (30)$$

The above condition is automatically satisfied as long as the first alternative is the minimum variance alternative. An associated convenient normalization is $\tilde{\omega}_i^2 = 0$, since the resulting model nests the MDCEV model. The minimum variance alternative can be determined by estimating an unidentified model with all the $k \omega_k$ terms, and identifying the alternative with the minimum variance (see Walker et al., 2004, for an equivalent procedure for a Heteroscedastic specification within the mixed multinomial logit model).

The above discussion assumes there is price variation across goods. In the case of no price variation, the scale $\sigma$ is not identifiable. In this case, the easiest procedure is to normalize $\sigma$ to 1 and the $\omega_k^2$ value for the minimum variance alternative $k$ to zero.

4.3.2 The general error covariance structure within the MMDCEV framework

Appropriate identification normalizations will have to placed on $\sigma$ and the covariance matrix of $\eta$ when the analyst is estimating an error-components structure to allow correlation in unobserved factors influencing the baseline utility of alternatives, since only a $(K-1)x(K-1)$ covariance of error differences is identified. This can be accomplished by imposing a structure based on a priori beliefs or intuitive considerations. However, the analyst must ensure that the elements of the assumed restricted covariance structure can be recovered from the $(K-1)x(K-1)$ covariance of error differences that is actually estimable.
In the most general error covariance structure, and when there is price variation, one way to achieve identification is the following: (1) Normalize the scale parameter $\sigma$ to be a small value such that the variance of the minimum variance alternative exceeds $\pi^2 \sigma^2 / 6$ (since this variance is not known, the analyst will have to experiment with alternative fixed $\sigma$ values), (2) Normalize $\omega_k$ for the minimum variance alternative $k$ to zero, and (3) Normalize all correlations of this minimum variance alternative with other alternatives to zero. Together, these normalizations leave only $K(K-1)/2$ parameters to be estimated, and are adequate for identification. In the case of no price variation, an additional restriction will have to be imposed. One approach would be to set $\sum_{k=2}^{K} \omega_k^2 = 1$ to set the scale in the covariance matrix of $\eta$.

5. THE MODEL WITH AN OUTSIDE GOOD

Thus far, the discussion has assumed that there is no outside numeraire good (i.e., no essential Hicksian composite good). If an outside good is present, arbitrarily label it as the first good which now has a unit price of one. Also, for identification, let $\psi(x_i, \varepsilon_i) = e^{\alpha_i}$. Then, the utility functional form needs to be modified as follows:

$$U(x) = \frac{1}{\alpha_i} \exp(\varepsilon_i) x_i^{\alpha_i} + \sum_{k=2}^{K} \frac{\gamma_k}{\alpha_k} \exp(\beta' z_k + \varepsilon_k) \left( \frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1$$  \hspace{1cm} (31)

Note that there is no translation parameter $\gamma_1$ for the first good, because the first good is always consumed. As in the “no-outside good” case, the analyst will generally not be able to estimate both $\alpha_k$ and $\gamma_k$ for the inside goods 2, 3, …, $K$. The analyst can estimate one of the following three utility forms:

$$U(x) = \frac{1}{\alpha_i} \exp(\varepsilon_i) x_i^{\alpha_i} + \sum_{k=2}^{K} \frac{1}{\alpha_k} \exp(\beta' z_k + \varepsilon_k) \left( x_k + 1 \right)^{\alpha_k} - 1$$

$$U(x) = \frac{1}{\alpha_i} \exp(\varepsilon_i) x_i^{\alpha_i} + \sum_{k=2}^{K} \gamma_k \exp(\beta' z_k + \varepsilon_k) \ln \left( \frac{x_k}{\gamma_k} + 1 \right)$$  \hspace{1cm} (32)

$$U(x) = \frac{1}{\alpha} \exp(\varepsilon_i) x_i^{\alpha} + \sum_{k=2}^{K} \frac{\gamma_k}{\alpha} \exp(\beta' z_k + \varepsilon_k) \left( \frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1$$
The last functional form above is estimable now because the constant \( \alpha \) parameter is obtaining a “pinning effect” from the satiation parameter for the outside good. The analyst can estimate all the three possible functional forms and select the one that fits the data best based on statistical and intuitive considerations. The identification considerations discussed for the “no-outside good” case carries over to the “with outside good” case. The probability expression for the expenditure allocation on the various goods (with the first good being the outside good) is identical to Equation (19), while the probability expression for consumption of the goods (with the first good being the outside good) is

\[
P(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0)
\]

\[
= \frac{1}{\sigma^{M-1}} \left[ \prod_{i=1}^{M} f_i \left[ \sum_{i=1}^{M} \frac{p_i}{f_i} \left( \frac{\prod_{i=1}^{M} e^{V_i / \sigma}}{\sum_{k=1}^{K} e^{V_k / \sigma}} \right)^M \right] (M - 1)!, \right.
\]

where \( f_i = \frac{1 - \alpha_i}{x_i^* + \gamma_i} \).

The expressions for \( V \) in Equation (19) and Equation (33) are as follows for each of the three utility forms in Equation (32):

First form - \( V_k = \beta' z_k + (\alpha_k - 1) \ln(x_k^* + 1) - \ln p_k \) \((k \geq 2)\); \( V_1 = (\alpha_1 - 1) \ln(x_1^*) \)

Second form - \( V_k = \beta' z_k - \ln \left( \frac{x_k^*}{\gamma_k} + 1 \right) - \ln p_k \) \((k \geq 2)\); \( V_1 = (\alpha_1 - 1) \ln(x_1^*) \)

Third form - \( V_k = \beta' z_k + (\alpha - 1) \ln \left( \frac{x_k^*}{\gamma_k} + 1 \right) - \ln p_k \) \((k \geq 2)\); \( V_1 = (\alpha - 1) \ln(x_1^*) \)

6. COMPARISON WITH EARLIER MULTIPLE DISCRETE-CONTINUOUS MODELS

In this section, we discuss how the model developed in this paper differs from the model of Kim et al. (2002), those in Environmental Economics, and the earlier models by Bhat and colleagues. The discussion is in the context of the basic structure with identically and independently distributed error terms across alternatives. Also, for presentation ease, we focus on the situation with no outside good.
6.1 Kim et al.’s Model

Kim et al. (2002) use the following translated constant elasticity of substitution (CES) direct utility form:

\[ U = \sum_{k=1}^{K} \psi_k^1 (x_k + \gamma_k^1)^{\alpha_k^1}, \tag{35} \]

where the superscript ‘1’ is to distinguish the parameters in this functional form from those in the form of Equation (1). In the utility form above, \( \psi_k^1 > 0, \gamma_k^1 > 0, \) and \( 0 \leq \alpha_k^1 \leq 1. \) To empirically identify the utility form, Kim et al. impose the restriction that \( \gamma_k^1 = 1 \) for all \( k \) (i.e., they estimate the \( \alpha \)-profile). The reader will note that Kim et al.’s form leads to a seemingly inconsistent situation of positive utility even at zero consumption of a good. However, this can be easily remedied by revising the utility form above to an empirically indistinguishable alternative form provided below:

\[ U = \sum_{k=1}^{K} \psi_k^1 ((x_k + \gamma_k^1)^{\alpha_k^1} - (\gamma_k^1)^{\alpha_k^1}), \tag{36} \]

The KT conditions and optimal consumptions for both Equations (35) and (36) are identical. But the latter form assigns zero utility to good \( k \) when it is not consumed, while still allowing corner solutions. However, in either of the two forms, the interpretation of \( \psi_k^1 \) is not straightforward. Specifically, the baseline marginal utility of a good (or marginal utility when no quantity of the good is consumed) is \( \psi_k^1 \alpha_k^1, \) which depends on both \( \psi_k^1 \) and \( \alpha_k^1 \) (for \( \gamma_k^1 \) fixed to 1 for all \( k \)). Of course, the satiation rate for good \( k \) with respect to the baseline marginal utility is still determined by \( \alpha_k^1, \) as in the \( \alpha \)-profile based utility form adopted in this paper (i.e., Equation (1)) with \( \gamma_k \) fixed to 1 for all \( k \). In fact, the estimation results and optimal consumptions from using Equation (1) and Equation (35) (with all \( \gamma_k \)'s fixed to 1) will be identical, as long as \( \alpha_k \geq 0 \) for all \( k \) in Equation (1). The only cosmetic difference will be a shift in the constant terms between the \( \psi_k \) and \( \psi_k^1 \) terms. Specifically, if \( \psi_k = \exp(\tau_k + \beta^0 z_k + \varepsilon_k) \) and \( \psi_k^1 = \exp(\tau_k^1 + \beta^1 z_k + \varepsilon_k^1) \), where the constant term is removed out from the \( \beta^0 z_k \) term, the following relationship will hold between the two models as long as the same error distributions are assumed (and assuming that the first alternative is considered as the base):
\[ \tau_k = \tau_k^1 + \ln \left( \frac{\alpha_k^1}{\alpha_1^1} \right) \forall k \]
\[ \alpha_k = \alpha_k^1 \forall k \]
\[ \beta = \beta^1. \]

An important technical nuance is in order here. The form of Equation (35) is restrictive compared to the form of Equation (1) adopted in this paper. Specifically, Equation (1) covers the overall baseline marginal utility satiation space more completely than Equation (35). Kim et al.’s form ties the extensive margin of choice (i.e., whether or not good \( k \) is consumed) and the intensive margin of choice (quantity of good \( k \) consumed) more tightly than the form used in the current paper. To see this, note that the only way to represent high satiation effects (i.e., \( \alpha_k^1 \to 0 \)) in Equation (35) is by having a low baseline marginal utility (i.e., \( \alpha_k^1 \psi_k^1 \to 0 \)). On the other hand, the utility form used in the current paper allows the possibility of high satiation effects (i.e., \( \alpha^1 \to -\infty \)) even at a high baseline marginal utility \( \psi_k \). However, as discussed earlier, the two forms provide the same result as long as \( \alpha_k \geq 0 \) for all \( k \) in Equation (1). If \( \alpha_k < 0 \) for any \( k \) in Equation (1), the resulting marginal utility-satiation space cannot be captured by Kim et al.’s form. But the analyst, at times, may have to impose the constraint \( \alpha_k \geq 0 \) in the functional form of Equation (1) to provide stability in estimation, especially when a scale parameter is being estimated with limited price variation.

Another important difference between Kim et al.’s model and the model here is the distribution of the error terms. Kim et al. assume that the error terms are independent and identically distributed normal. They then use the differencing form of Equation (13) to develop the probabilities, with the \( g(.) \) function being a multivariate normal density. This form requires an appropriate decomposition of the density function for the continuous and discrete components, and multivariate normal integration. The approach is not practical for most realistic applications, and is unnecessarily complicated. As Bhat (2005a) noted, it is much simpler to use the non-differenced probability expression of Equation (11) when assuming an IID form for the error terms. Further, when one consider the error terms to be IID gumbel instead of IID normal, the model structure collapses to the closed-form MDCEV form proposed by Bhat (2005a) and used here.
6.2 Models in Environmental Economics

The studies in Environmental Economics, unlike Kim et al., use the utility function corresponding to the $\gamma_k$-profile. Their utility function takes the LES form (see von Haefen and Phaneuf (2005), von Haefen (2003b), Phaneuf et al. (2000), Phaneuf and Herriges (2000), and Herriges et al. (2004):

$$U = \sum_{k=1}^{K} \psi_k^2 \ln(x_k + \gamma_k^2) ,$$  \hspace{1cm} (38)

The superscript ‘2’ above is to distinguish the parameters from those in Equation (1), and should not be confused with the square power function. In the function above, the utility accrued from zero consumption of a good is positive since $\psi_k^2 > 0$ and $\gamma_k^2 > 0$. However, this can be accommodated by re-writing the utility form in the empirically indistinguishable alternative form shown below:\(^{14}\):

$$U = \sum_{k=1}^{K} \psi_k^2 \ln\left(\frac{x_k}{\gamma_k^2} + 1\right) ,$$  \hspace{1cm} (39)

In particular, it can be readily seen that the KT first order conditions and the optimal consumptions are identical for the utility forms in Equations (38) and (39), as also observed by Herriges et al. (2004). However, in both forms, the interpretation of $\psi_k^2$ is not straightforward, since the baseline marginal utility is $\psi_k^2 / \gamma_k^2$. But, for a given baseline marginal utility, $\gamma_k^2$ serves as a satiation parameter (in addition to allowing corner solutions). In fact, the estimation results and optimal consumptions from using Equation (2) and Equation (39) will be identical, except for a shift in the constant terms between the $\psi_k$ and $\psi_k^2$ terms. Specifically, if $\psi_k = \exp(\tau_k + \beta' z_k + \varepsilon_k)$ and $\psi_k^2 = \exp(\tau_k^2 + \beta'^2 z_k + \varepsilon_k^2)$, the following relationship will hold between the two models as long as the same error distributions are assumed (and assuming that the first alternative is considered as the base):

---

\(^{14}\) von Haefen et al. (2004) and von Haefen and Phaneuf (2003) also recognize this weak complementarity problem in the functional form of Equation (38), where the quality attributes of good $k$ contribute to utility even if the good is not consumed. They address it by interacting the quality attributes with $x_k$, rather than including the quality attributes in $\psi_k$. 

25
The stochastic assumptions on the error terms used in the models in the Environmental Economics field are different from the ones used in the current paper. In the Environmental Economics models, one of the alternative’s utility is assumed to be deterministic (say the first alternative). Thus, \( \varepsilon_1 = 0 \), and the remaining error terms are assumed to be independent and identically distributed gumbel. The probability expression then takes the form of the integrand in Equation (17) with \( \varepsilon_i = 0 \) (there is no integration needed because \( \varepsilon_i \) is deterministic).

An issue with the assumption that \( \varepsilon_i = 0 \) is that it is not an innocuous normalization. To see this, note that the probability expression for expenditure on goods can be written in terms of the differences in error terms \( \tilde{\varepsilon}_{k1} = \varepsilon_k - \varepsilon_i \), as shown in Equation (13). It would then be perfectly innocuous to ignore \( \varepsilon_i \) (i.e., assume \( \varepsilon_i = 0 \) and write \( \tilde{\varepsilon}_{k1} = \varepsilon_k \)) if the error terms of the other alternatives \( \tilde{\varepsilon}_{k1} = \varepsilon_k \) were assumed to have a full covariance matrix. But the only way that the remaining error terms can be independent, as assumed by the Environmental Economics studies, is if \( \varepsilon_i \) were truly zero. It is difficult to imagine that the utility of one of the alternatives would be deterministic, while those of all others would not (this is especially so when there is no outside good, though it also applies to the case of the presence of an outside good). Besides, including \( \varepsilon_i \) with an extreme-value distribution does not destroy the closed form expression for the choice probabilities. Also, the MDCEV model collapses to the MNL model for the case when only one alternative is chosen by all decision-makers, while the model used in Environmental Economics does not.

Another point to note is that the Environmental Economics studies (and the Kim et al. study) do not use the compact and simple structure of the Jacobian in the probability expression. Consequently, the Environmental Economics studies use a numerical gradient method in the likelihood estimation, which is less precise than the simple form of the analytic gradient that can

\[
\tau_k = \tau_k^2 + \ln \left( \frac{\gamma_k^2}{\gamma_k^2} \right) \forall k
\]

\[
\gamma_k = \gamma_k^2 \forall k
\]

\[
\beta = \beta^2
\]

15 The studies in environmental economics always assume the presence of an outside good, whose utility is assumed to be deterministic. However, this does not change anything regarding the ensuing discussion.
be obtained by writing the Jacobian in the form of Equation (18). Further, while the numerical gradients provide accurate estimates at acceptable speeds for the simple model form with IID error terms, the approach is extremely slow compared to the analytic gradient approach for the case when random coefficients or richer error correlation patterns are introduced using a mixed approach.

6.3 Bhat’s Earlier Models

The models of Bhat (2005a), Bhat and Sen (2006), and Bhat et al. (2006) use the translated CES direct utility function form as in Kim et al. (2002), rather than the more easy-to-interpret and general utility form used in the current paper. The studies assume independent and identically distributed gumbel error kernel terms that leads to the MDCEV form and its mixed variants, as in the current paper. The studies assume unit “prices” for all the alternatives, and so do not have to deal with the many issues arising from price variation as discussed in this paper. These earlier efforts also do not address identification considerations, nor do they shed light on the role played by the parameters in the utility function.

7. EMPIRICAL ILLUSTRATIONS

7.1 Absence of Outside Good Case

In this section, we supplement the many issues discussed in earlier sections using empirical examples for the case of the absence of an outside good. The data set used for these illustrations is the same as that used by Bhat and Sen (2006), and is drawn from the 2000 San Francisco Bay Area Travel Survey (BATS). The sample includes 3500 households with non-zero vehicle ownership. The vehicles owned by each household are categorized into one of five vehicle types based on their make, model and year. The five vehicle types are (1) Passenger car, (2) Sports Utility Vehicle (SUV), (3) Pickup truck, (4) Minivan, and (5) Van. In this paper, we estimate different MDCEV specifications with these five alternatives. Households may hold a combination of these vehicle types and use different vehicle types in different ways, leading to the discrete-continuous choice of vehicle holding mix and vehicle miles of travel by vehicle type.

In all the results to be discussed, we used the most basic specification with constants in the baseline marginal utility and satiation parameters. This is because the motivation here is to discuss scaling and identification issues, not variable specification considerations. The satiation
parameters are estimated initially in their parameterized form to respect the appropriate bounds. Once estimated, the non-parameterized values are computed and used as the starting values in a final iteration. This final iteration will achieve convergence immediately, and directly provides the necessary standard errors of the satiation parameters. We, however, present the baseline marginal utility parameters in their parameterized form because this allows us to show the equivalence between models in a straightforward manner. That is, we present the $\beta_k$ parameters, where $\psi_k = \exp(\beta_k)$ for all $k$.

7.1.1 Case of no price variation

Consider the situation where the right side of the “budget” constraint is simply the total annual miles of travel across all vehicle types. This is the case of prices not appearing in the budget constraint (or equivalently, the cost of traveling a mile by each vehicle being unity).

Table 1 shows six estimations, the first four being based on the $\alpha$-profile and the last two being based on the $\gamma$-profile (as expected, we were unable to estimate a model with both the $\alpha$ parameters and the $\gamma$ parameters for each vehicle type). The first two estimations fix the scale parameter and estimate all the $\alpha$ satiation parameters. It can be observed from the log-likelihood values at convergence (see last row of the table) that whether one fixes the scale to the value of ‘1’ or ‘2’ does not matter. As discussed in Section 3.2, the $\beta$ parameters in Model 2 are scaled by a factor of 2 and the $\alpha$ parameters in model 2 are related to those in Model 1 by the relationship $\alpha_{\text{model2}} = 2*(\alpha_{\text{model1}} - 1) + 1$. That is, the error term scale is unidentified when one estimates all the $\alpha$ parameters.

The third and fourth models apply a different normalization where one of the $\alpha$ parameters is fixed to the value of ‘0’. In these cases, one can estimate the error term scale. However, these models are no different from Models 1 and 2. Specifically, the $\beta$ parameters in Models 3 and 4 are equal to the estimated scale parameters in these models times the $\beta$ parameters in Model 1. The $\alpha$ parameters in Models 3 and 4 are related to that in Model 1 by $\alpha_{\text{modeld}} = \sigma_d * (\alpha_{\text{model1}} - 1) + 1$ ($d = 3,4$).

The fifth and sixth models correspond to the $\gamma$ profile with a fixed $\alpha$. The fifth model constrains $\alpha$ to 0 and the scale $\sigma$ to 1. The log-likelihood at convergence for this model
indicates that the $\gamma$ profile provides a better fit in this empirical case than does the $\alpha$ profile. The sixth model sets the scale value to 2 rather than 1. This sixth model provides the same results as the fifth model because the $\alpha$ parameter has been set to $\alpha_{\text{model}6} = \sigma_{\text{model}6} \times (\alpha_{\text{model}5} - 1) + 1 = -1$. Thus, in estimating the $\gamma$-profile, the scale can be set to any value as long as the $\alpha$ parameter is fixed appropriately. The normalization used in Model 5 is most convenient in this case.

7.1.2 Case of price variation

The data assembled by Bhat and Sen does not include a composite per mile usage price measure for each vehicle type because of the wide variation in the fixed costs of vehicle purchase and use costs per mile within each broad vehicle type category. For the analysis here, we used synthetic per mile price data by generating uniform random numbers with a mean of 1, 2, 2, 3, and 4 for sedans, sports utility vehicles (SUV), pick-up trucks, minivans, and vans, respectively. The intent here is simply to demonstrate the issues discussed earlier in the context of price variation, rather than to develop an actual model for vehicle type choice and use.

Table 2 shows the results of six model estimations. The first two estimations are $\alpha$-profiles based on the consumption probability expression of Equation (20). In both these estimations, we had to restrict the $\alpha$ parameters to be between 0 and 1 for convergence considerations. The only difference between these first two estimations is that the good labeled as the first good is different (note that any good that is consumed may be designated as the first good). As can be observed from the results, there is effectively no difference in the parameter estimates at convergence (the minor differences are a result of the optimization convergence process). However, the log-likelihood values at convergence are different, because the probability assigned to the same consumption pattern varies based on the vehicle type designated as the first good.

The third model is also an $\alpha$-profile specification (with the $\alpha$’s constrained to between 0 and 1), but based on the expenditure probability expression of Equation (19). The results of this third model are identical to those in the first two models in terms of parameter estimates. However, this model, by structure, is independent of the alternative that is designated as the first good.
The remaining models in Table 2 are all based on the expenditure probability expression. The fourth model, also an $\alpha$-profile specification, restricts the scale parameter to one. A comparison of this model with the third model in terms of a nested likelihood ratio test yields a test statistic of 603.74, which rejects the null hypothesis of a unit scale parameter at any reasonable level of statistical significance. This is a clear indication that the scale parameter is estimable when there is price variation.

The last two models in Table 2 correspond to the $\gamma$-profile, with the sixth model restricting the scale to one. A comparison of these last two models with one another using a nested likelihood ratio test again clearly rejects the null hypothesis that the scale value is immaterial. We also estimated another model that constrains the $\alpha$ value to $(1/\sigma^{\text{model5}}) \times (\alpha^{\text{model5}} - 1) + 1 = 0.418$. If the scale did not matter, this model should provide identical results to Model 5, as in the case of no price variation. The log-likelihood at convergence of this model was $-9046.3$, which is substantially worse than the convergence value for Model 5. Taken together, the results clearly indicate that the scale is identifiable when there is price variation.

### 7.2 Presence of Outside Good

The data set used for the case with an outside good is the same as that used in Bhat et al. (2006), and is also drawn from the 2000 San Francisco Bay Area Travel Survey (BATS). The sample comprises the time use characteristics (participation and duration of participation) of 2000 adult individuals in ten different activity purposes over a weekend day: (1) maintenance activities (in-home meals, in-home and out-of-home personal household chores and personal care, in-home and out-of-home personal business, out-of-home maintenance shopping, and out-of-home medical appointments), (2) in-home relaxation, (3) in-home recreation (hobbies, TV, etc.), (4) non-work internet use, (5) social activities (in-home and out-of-home), (6) out-of-home meals, (7) out-of-home non-maintenance shopping, (8) out-of-home volunteer activities (including civic and religious activities), (9) out-of-home recreation ((hobbies, exercise, etc.), and (10) pure recreation (travel episodes that began and ended at home without any stops in-between, such as walking or bicycling around the neighborhood). The reader is referred to Bhat et al. for further details of the activity typology and definitions. The first activity purpose, maintenance activities, is an “outside” good in which every individual participates.
To demonstrate the issues discussed earlier in the paper in the context of price variation, we synthetically generated unit prices (i.e., cost per minute of participation in each type of activity) for each of the 10 activity types. The unit price for the maintenance activity category (the outside good) is set to unity, while those for the other activity types are generated using draws from a uniform distribution with preset mean values. We only discuss the results for the case of price variation here for brevity (and do not include the case of no price variation).

Table 3 shows the estimation results of four different specifications. The first specification is based on an $\alpha$-profile corresponding to the expenditure formulation of Equation (19), while the second is based on an $\alpha$-profile corresponding to the consumption formulation of Equation (33). Both these specifications use the first $V_k$ form in Equation (34) and provide the same parameter estimates, but with different log-likelihood values. The $\alpha$ values had to be bounded between 0 and 1 for convergence. As can be observed, the satiation parameter for the first “good” is zero (however, note that we did not explicitly restrict this parameter to zero). The third specification is the same as Model 2, but constrains the scale parameter to 1. A nested likelihood ratio test between Models 2 and 3 clearly indicates the statistically superior fit of Model 2, showing that the scale is identifiable.

The remaining two model specifications in Table 2 are also based on the consumption formulation. The fourth model corresponds to the $\gamma$-profile for the internal goods (that is, the second utility form of Equation (32)). Interestingly, the $\alpha$ parameter that is estimable for the outside good was estimated to be zero (we had to bound this $\alpha$ value to be between 0 and 1 for obtaining convergence). This is equivalent to a log-form for both the outside and inside goods in the second utility form in Equation (32). The log-likelihood value for this $\gamma$ profile is much superior to the $\alpha$-profile specification of the second model. The fifth specification estimates a common $\alpha$ parameter across the outside and inside alternatives (third utility form in Equation (32)). This specification can be compared to the fourth specification, because the $\alpha$ parameter for the outside good in the fourth model turned out to be zero (in the fourth model, the $\alpha$ parameters for the inside goods are constrained to be zero). The common $\alpha$ parameter in the fifth model is statistically significantly different from zero, indicating that this model is statistically superior to the fourth model. Overall, this fifth specification turns out to be the preferred one in the current empirical context.
8. CONCLUSIONS

Classical discrete and discrete-continuous models deal with situations where only one alternative is chosen from a set of mutually exclusive alternatives. Such models assume that the alternatives are perfectly substitutable for each other. On the other hand, many consumer choice situations are characterized by the simultaneous demand for multiple alternatives that are imperfect substitutes for one another.

A simple and parsimonious econometric approach to handle multiple discreteness was formulated by Bhat (2005a) based on a specific satiation-based formulation within the broader Kuhn-Tucker (KT) multiple discrete-continuous economic model of consumer demand. Bhat’s model, labeled the MDCEV model, is analytically tractable in the probability expressions and is practical even for situations with a large number of discrete consumption alternatives.

This paper examines several issues associated with extant KT multiple discrete-continuous models. Specifically, the paper proposes a new utility function form that enables clarity in the role of each parameter in the utility specification. The paper also presents identification considerations associated with the utility specification, extends the MDCEV model to the case of price variation across goods and to general error covariance structures, discusses the relationship between earlier KT-based multiple discrete-continuous models, and illustrates the many technical nuances and identification considerations of the multiple discrete-continuous model structure through empirical examples.

Overall, the paper contributes toward the modeling of multiple discrete-continuous choice situations, a field of research that is at an exciting and challenging stage. There have been important contributions to the area from marketing, transportation, and environmental economics, especially within the past five years. At the same time, several challenges lie ahead, including (1) Accommodating more than one constraint in the utility maximization problem (for example, recognizing both time and money constraints in activity type choice and duration models; see Anas, 2006 for a recent theoretical effort to accommodate such multiple constraints), (2) Incorporating latent consideration sets in a theoretically appropriate way within the MDCEV structure, and (3) Using more flexible utility structures that can handle both complementarity as well as substitution among goods, and that do not impose the constraints of additive separability.
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REFERENCES


Appendix A: Form of Utility Function as $\alpha_k \to 0$ for all Goods $k$

From Equation (1),

$$U(x) = \sum_{k=1}^{K} \gamma_k x^\gamma_k - \frac{(x_k + 1)^{\alpha_k}}{\alpha_k} + 1$$

(A.1)

Consider the expression in parenthesis and write it in the form shown below when $\alpha_k \to 0$:

$$\lim_{\alpha_k \to 0} \left[ \frac{(x_k + 1)^{\alpha_k}}{\alpha_k} - 1 \right] = \lim_{\alpha_k \to 0} \frac{e^{\alpha_k \ln\left(\frac{x_k}{\gamma_k} + 1\right)} - 1}{\alpha_k}$$

(A.2)

Using L'Hospital’s rule, we can write the above expression as:

$$\lim_{\alpha_k \to 0} \left[ e^{\alpha_k \ln\left(\frac{x_k}{\gamma_k} + 1\right)} \cdot \ln\left(\frac{x_k}{\gamma_k} + 1\right) \right] = \ln\left(\frac{x_k}{\gamma_k} + 1\right).$$

(A.3)

Thus, $U(x)$ collapses to Equation (2) when $\alpha_k \to 0$ for all $k$. 


Appendix B: Derivation of the Structure of the Multiple Discrete-Continuous Extreme Value Model with Error Scale Parameter

From Equation (17) of the text:

\[
P(e_1^*, e_2^*, \ldots, e_m^*, 0, 0, \ldots, 0)
\]

\[
= |J| \int_{\varepsilon_1=-\infty}^{\varepsilon_1=+\infty} \left\{ \prod_{i=2}^{M} \frac{1}{\sigma} \left( \frac{V_i - V_1 + \varepsilon_1}{\sigma} \right) \right\} \times \left\{ \prod_{i=M+1}^{K} \Lambda \left( \frac{V_1 - V_i + \varepsilon_1}{\sigma} \right) \right\} \frac{1}{\sigma} \left( \frac{\varepsilon_1}{\sigma} \right) d\varepsilon_1
\]

\[
= |J| \frac{1}{\sigma^{M-1}} \left[ \prod_{i=2}^{M} e^{\left( \frac{-V_i - V_1}{\sigma} \right)} \right] \left[ \int_{\varepsilon_{m+1}=-\infty}^{\varepsilon_{m+1}=+\infty} \left( \frac{\varepsilon_1}{\sigma} \right)^{M-1} e^{-\sum_{k=1}^{K} \left( \frac{-1}{\sigma} \right) \left( \frac{V_i - V_1}{\sigma} \right)^{M-1}} \right] \frac{1}{\sigma} e^{\frac{-\varepsilon_1}{\sigma}} d\varepsilon_1
\]

Now, consider the last term within the integral in the expression above, and let \( t = e^{\frac{-\varepsilon_1}{\sigma}} \). Then

\[
dt = -e^{\frac{-\varepsilon_1}{\sigma}} \cdot \frac{1}{\sigma} d\varepsilon_1
\]

\[
\int_{t=+\infty}^{0} t^{M-1} e^{\frac{-t}{\sigma}} \cdot dt = \frac{(M-1)!}{\sum_{k=1}^{K} \left( \frac{V_i - V_1}{\sigma} \right)^{M}}
\]

Putting this back in (B.1), we get

\[
P(e_1^*, e_2^*, \ldots, e_m^*, 0, 0, \ldots, 0)
\]

\[
= \frac{1}{\sigma^{M-1}} |J| \left[ \prod_{i=2}^{M} e^{\left( \frac{-V_i - V_1}{\sigma} \right)} \right] \left[ \frac{1}{\sum_{k=1}^{K} e^{\left( \frac{-V_i - V_1}{\sigma} \right)^{M}}} \right] (M-1)!
\]

\[
= \frac{1}{\sigma^{M-1}} \left[ \prod_{i=1}^{M} \gamma_i \gamma_j \frac{1}{c_j} \right] \left[ \prod_{i=1}^{M} \gamma_i^{\alpha_i} \right] (M-1)!, \text{ where } c_j = \left( \frac{1-\alpha_i}{e_i^* + \gamma_j \beta_i} \right).
\]
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<td>-9648.48</td>
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### Table 2. Specifications for the “No Outside Good” Case with Price Variables

<table>
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<tr>
<th>Parameters</th>
<th>Model 1 (Consumption-based)</th>
<th>Model 2 (Consumption-based)</th>
<th>Model 3 (Expenditure-based)</th>
<th>Model 4 (Expenditure-based)</th>
<th>Model 5 (Expenditure-based)</th>
<th>Model 6 (Expenditure-based)</th>
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<td>Estimate</td>
<td>t-stat</td>
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<tr>
<td>(\alpha_{\text{passenger car}})</td>
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<td>0.00</td>
<td>0.000(^{17})</td>
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<td>(\alpha_{\text{Van}})</td>
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\(^{16}\) The value of 0.000 was the estimated value at convergence (the standard error was 0.0595)

\(^{17}\) The value of 0.000 was the estimated value at convergence (the standard error was 0.0600)

\(^{18}\) The value of 0.000 was the estimated value at convergence (the standard error was 0.0597)
Table 3. Specifications for Case with Outside Good and with Price Variables

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model 1 (Expenditure-based)</th>
<th>Model 2 (Consumption-based)</th>
<th>Model 3 (Consumption-based)</th>
<th>Model 4 (Consumption-based)</th>
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<tr>
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<td>Est.</td>
<td>t-stat</td>
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<td>Maintenance activity</td>
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<td>-6.795</td>
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<td>Pure recreation</td>
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$^{19}$ The value of 0.000 was the estimated value at convergence (the standard error was 0.0372)

$^{20}$ The value of 0.000 was the estimated value at convergence (the standard error was 0.0363)

$^{21}$ The value of 0.000 was the estimated value at convergence (the standard error was 0.0311)
Table 3. Specifications for Case with Outside Good and with Price Variables (Continued)

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<th>Parameters</th>
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