A New Closed Form Multiple Discrete-Continuous Extreme Value (MDCEV) Choice Model with Multiple Linear Constraints

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ABSTRACT
Traditional multiple-discrete continuous choice models that have been formulated and applied in recent years consider a single linear resource constraint, which, when combined with consumer preferences, determines the optimal consumption point. However, in reality, consumers may face multiple resource constraints, such as those associated with time, money, and storage capacity. Ignoring such multiple constraints and instead using a single constraint can, and in general will, lead to poor data fit and inconsistent preference estimation, which can then have a serious negative downstream effect on forecasting and welfare/policy analysis. Unlike earlier attempts to address this multiple constraint situation, we formulate a new multiple-constraint (MC) multiple discrete continuous extreme value (MDCEV) model (or the MC-MDCEV model) that retains a closed-form probability structure and is as simple to estimate as the MDCEV model with one constraint. We achieve this by assuming a type-I extreme value distribution for the error term in its minimization form in the baseline utility preference of each good rather than a maximization form as in Bhat's (2005; 2008) original MDCEV formulation. The statistical foundation of the proposed model is based on the fact that the difference between a minimal type-I extreme value random variable with scale σ and the weighted sum of the exponential of standardized minimal type-I extreme value random variables (scaled up by σ) leads to an apparently new multivariate distribution that has an elegant and closed-form survival distribution function. Results from a simulation experiment show that our proposed model substantially outperforms single-constraint models; the results also highlight the serious mis-estimation that is likely to occur if only a subset of active constraints is used. The proposed model is applied to a case of week-long activity participation where individuals are assumed to maximize their utility from time-use subject to time and money budgets. It is hoped that our proposed simple closed-form multi-constraint MDCEV model will contribute to a new direction of application possibilities and to new research into situations where consumers face multiple constraints within a multiple discrete-continuous choice context.

Keywords: Consumer theory, multiple discrete-continuous extreme value model, extreme value distribution, multivariate distributions, multiple constraints, closed-form structure.

1. INTRODUCTION

Single discrete choice models have been widely used in a variety of fields to model the choice of a single alternative from a set of available alternatives. While it is not atypical to derive such single discrete choice models from indirect utility maximization, such models can also be derived from direct utility formulations in which the marginal utility of each good is linear with no satiation effects. In fact, the multiple continuous extreme value (MDCEV) model proposed by Bhat (2005; 2008) is an extension of such a direct utility-based single discrete choice model in which the utility functions of alternatives are considered non-linear. Since its introduction to the literature, the closed-form nature of the MDCEV model and its variants have led to their applications in many different contexts. Some recent applications, to name just a few, include the proportion of annual income spent on different transportation categories (such as vehicle purchase, gas costs, maintenance costs, and air travel; see Ma et al., 2019), the holding and usage level of traditional and alternative fuel vehicles (such as gasoline, diesel, hybrid, electric, and fuel cells; see Shin et al., 2019), participation in different types of activities (such as sleeping, reading, listening to music, playing games, talking with other passengers, and working) that an individual may pursue as part of multi-tasking during travel (Varghese and Jana, 2019), household choice of different types of fuel sources and expenditures for energy (such as kerosene, firewood, Liquid Propane Gas (LPG), and electricity; see Acharya and Marhold, 2019), and children's weekly time spent in different types of out-of-home after-school activities (academic, sports, arts, and other; see Leung et al., 2019).

The essential ingredient of the MDCEV model and related direct utility maximization-based MDC models is the use of a non-linear (but increasing and continuously differentiable) utility structure with decreasing marginal utility (or satiation), which immediately introduces imperfect substitution in the mix and allows the choice of multiple alternatives. Within this context, Bhat (2008) proposed a non-linear utility form that is quite general and subsumes the earlier specifications as special cases. His utility specification also allows a clear interpretation of model parameters. Also, by using log-extreme value error terms in the baseline utility preference of each alternative to introduce stochasticity, and assuming independent and identically distributed (IID) error terms across alternatives, he derived the closed-form MDCEV model. While the MDCEV model can be extended in many ways to include random parameters and/or allow more general non-IID covariance structures for the error terms across alternatives (see, for example, Bhat et al., 2015), doing so can destroy the elegant closed-form nature of the MDCEV. Indeed, it can be argued that the wide applicability of the basic MDCEV model in different fields is because of its attractive (and relatively simple) closed form nature. While the field has evolved substantially in the ability to evaluate multi-dimensional integrals today (see, for example, Bhat, 2018), there is still value in striving to obtain closed-form analytic models. The reasons for this are multi-fold, as also highlighted in a large body of scientific literature across disciplines. First, closed-form solutions are much more computationally efficient than open-form solutions, as highlighted even recently by Krivochiza et al. (2018), Miraldo et al. (2018), and Zaplana and Basañez (2018). This is because open-form solutions are generally solved using numerical or simulation techniques that typically entail the generation of a large number of random sequences and function evaluations. The computational efficiency of closed-form models, on the other hand, can allow a more comprehensive investigation of the many observed factors (and their interaction effects) affecting the choice behavior of interest. As highlighted by Bhat (2000), the end-goal of choice models is to examine the effects of observed variables (and their interaction effects), and so adopting the best systematic specification is paramount in modeling. Second, and related to the first point, closed-form solutions provide a better understanding of how different design variables affect the outputs or the solution. Because closed-form solutions are presented as exact math expressions, they offer a clear view into how variables and interactions between variables affect the result. Moreover, a closed-form structure allows for a better definition of the properties of a solution due to its exactness and also provides better insights about generalization (in some cases) of a solution (see for example, Cong et al., 2020, where the authors underscore the virtues of the exactness of closed-form solutions for interpretations of variables in a machine learning context). Third, closed-form models provide the exact same estimation solution if correctly implemented, regardless of the platform used or the specific numerical technique adopted in estimation. On the other hand, the results from open-form models depend on the numerical technique implemented and the procedure followed for random number generation (also sometimes referred to as the problem of reproducibility; see Lenhard and Küster, 2019). Open-form solutions will also, in many instances, need some hand-holding or specialized code to steer the optimization away from "nether regions" and facilitate convergence. This is because of rounding errors and more basic computational error bounds that can particularly affect gradient computations.

The closed-form MDCEV model (as it stands currently) is based on the notion that consumers maximize utility subject to a single linear binding constraint. But, as discussed in detail in Castro et al. (2012), consumers usually face multiple resource constraints in many MDC situations. For example, consumers' time-use decisions may be based not only on time constraints, but also an income constraint because the expenditure on the chosen activity participations cannot exceed the money budget available. Thus, for example, consider families with children who tend to be time poor (see Bernardo et al., 2015) and who are also budget tight. Such families may not be very responsive to service time reductions at the checkout at upscale retail stores (as a promotion strategy to draw in more patrons) because of the budget constraint they face. Ignoring the budget constraint in this case and using a single time constraint can lead to an under-estimation of time-sensitivity and large model prediction inaccuracies. Similarly, these same families may not be very responsive to a price reduction strategy adopted by upscale retail stores, because of the time constraints they face. In this situation, ignoring the time constraint and using a single budget constraint will lead to an underestimation of price-sensitivity and, again, potentially large model prediction inaccuracies. Another similar example is provided by Satomura et al. (2011) for the purchase of consumer goods, in the presence of both a budget constraint as well as a storage space constraint. In this case, using only a single budget constraint may lead (incorrectly) to purchase predictions of a specific product (say a perishable grocery item that needs to be refrigerated at home) in very large quantities (because of the utility-cost tradeoff); however, consumers may not be able to exercise this preference for large quantities because of say storage space constraints in their refrigerator at home. Thus, ignoring this second storage space constraint will lead to the incorrect estimation of the preference for the good. The fundamental problem is that there is a co-mingling of preference and constraint effects when a constraint is not considered, leading to inconsistent preference estimation. That is, ignoring constraints will, in general, have serious negative repercussions for both model forecasting performance and policy evaluation.

Castro et al. (2012) provide a discussion of earlier studies that have attempted to deal with utility maximization models with more than one constraint, which include Becker (1965), Larson and Shaikh (2001), Hanemann (2006), and Carpio et al. (2008) in the case of a single discrete choice model, and Parizat and Shachar (2010) and Satomura et al. (2011) in the case of multiple discrete continuous models. However, the models of single discrete choice above with multiple constraints are not easily extendable to the MDC case because of the non-linearity of the utility function, while the MDC models with multiple constraints are difficult to estimate and/or consider restrictive utility function forms. For example, Castro et al. (2012) develop a model based on an extension of Bhat's (2008) MDCEV model that is flexible and can accommodate any number of constraints. But it has the drawback that the probabilities do not provide a closed form expression for the probabilities. In fact, the probabilities contain as many integrals (over the real line of univariate extreme value distributions) as the number of constraints in the model. While Castrol et al.'s model can be estimated using simulation techniques, the wide applicability of the basic MDCEV model with a single constraint clearly points to the value of deriving an MDC model that retains the utility structure of the MDCEV model as well as a closed-form expression for the probabilities, regardless of the number of constraints. This is the focus of the current paper. To achieve our objective, we use a stochastic distribution for the error term in the baseline preference for each alternative's utility that, like the traditional MDCEV, employs a log-extreme value error term. However, this error term is introduced in subtractive form in the baseline preference rather than in an additive fashion. Equivalently, rather than using a type-I extreme value error term in additive form based on the maximum of a very large collection of random observations from the same arbitrary distribution, we use the type-I extreme value error term in additive form based on the minimum. In fact, an important supplementary statistical contribution of this paper is that we show that the difference between a minimal type-I extreme value random variable with scale σ and the weighted sum of the exponential of standardized minimal type-I extreme value random variables (scaled up by σ) leads to an apparently new multivariate distribution that has an elegant and closed-form survival distribution function. In addition to this difference in the stochastic distribution of the baseline preference between our proposed model and the MDCEV model, we use a linear form of utility (or the $L\gamma$ MDCEV, as labeled by Bhat, 2018 and also discussed in detail in Bhat et al., 2020) for the baseline preference for the outside goods, rather than the non-linear form of utility (or the $NL\gamma$ MDCEV) adopted in Bhat (2008). Finally, to provide additional model flexibility, we also employ a slightly different version of the utility functional form relative to earlier MDCEV formulations, which nicely integrates with the other two changes indicated above to provide a new closed-form multiple constraint MDCEV model. Interestingly, in the case of a single constraint, our proposed closed-form model collapses to a form that is similar to, though slightly different from, Bhat's original MDCEV structure. Before we move ahead with the model formulation of our proposed methodology, we would like to digress a little and emphasize the importance of having closed form solutions as opposed to open-form solutions.

To summarize, the purpose of this paper is to develop a new MDCEV model that allows for any number of linear constraints, while also retaining a closed-form expression for the probability expressions (we will label the proposed model as the multi-constraint RG $L\gamma$ MDCEV model; that is, the reverse Gumbel $L\gamma$ MDCEV model). The rest of the paper is structured as follows. Section 2 presents the model formulation and estimation procedure. Section 3 discusses the forecasting techniques for our proposed model. A simulation experiment to evaluate our model is presented in Section 4, while Section 5 illustrates an application of the proposed model for analysing time use subject to budget and time constraints. The sixth and final section offers concluding remarks and directions for further research.

2. MODEL FORMULATION

Consider a variant of Bhat's (2018) general functional form for the utility function that is maximized by a consumer subject to two-constraints (we focus on two-constraints for presentation ease, but the closed-form model we derive is immediately extendable to any number of constraints, as we discuss later):

$$U(\boldsymbol{x}) = \psi_1 x_1 + \psi_2 x_2 + \sum_{k=3}^{K} \frac{\gamma_k}{\alpha} \psi_k^{(1-\alpha)} \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha} - 1 \right\}$$

$$\text{s.t.} \quad x_1 + \sum_{k=3}^{K} p_k x_k = E,\ x_1 > 0 \qquad (1)$$

$$x_2 + \sum_{k=3}^{K} g_k x_k = T,\ x_2 > 0$$

where the utility function $U(\boldsymbol{x})$ is quasi-concave, increasing, and continuously differentiable. $\boldsymbol{x} \geq 0$ is the consumption quantity ($\boldsymbol{x}$ is a vector of dimension $(K \times 1)$ with elements x_k), and ψ_k and γ_k are parameters associated with good *k*. The parameter α is a fixed satiation parameter across all the inside goods (however, note that the effective satiation is different across the inside goods because of the presence of the good-specific γ_k parameter). Let the first good be the numeraire good with respect to the first constraint (with a total budget of *E*). This first good does not appear in the second constraint and let its consumption be denoted by x_1 $(x_1 > 0)$. Similarly, let the second good be the numeraire good with respect to the second constraint (with a total budget of *T*), which does not appear in the first constraint. Let its consumption be denoted by x_2 $(x_2 > 0)$ The function $U(\boldsymbol{x})$ in Equation (1) is a valid utility function if $\psi_k > 0$ for all *k*, $\gamma_k > 0$ for all the inside goods ($k = 3, 4, ..., K$) , and $\alpha < 1$. The utility structure for the inside goods is similar to Bhat

(2008), except that the parameter ψ_k is now raised to a power equal to $(1-\alpha)$ (as discussed later, doing so provides a more general model than not doing so, while also retaining a closed form structure for the model). In this way of writing the utility structure for the inside goods, the marginal utility for any inside good may be written as follows:

$$\frac{\partial U(\boldsymbol{x})}{\partial x_k}=\left[\psi_k\left\{\left(\frac{x_k}{\gamma_k}+1\right)^{-1}\right\}\right]^{1-\alpha}=\left[\psi_k\left\{\left(\frac{x_k}{\gamma_k}+1\right)^{-1}\right\}\right]^{\frac{1}{\sigma}}, \text{ where } \frac{1}{\sigma}=1-\alpha,\ k=3,4,...,K. \tag{2}$$

A few things of note here. Since $\alpha<1$, we have $\sigma>0$. Second, if $\alpha\rightarrow 0$, we obtain the γ profile as discussed in Bhat (2008). Additionally, this implies that $\sigma=1$, but then, as discussed in Bhat (2018), one is able to (theoretically) identify the scale of the stochastic parameters ψ_k in the MDCEV model with only one-constraint (however, for our model with multiple constraints, we need to use a standard scale for the stochastic parameters for the outside goods for a closed-form solution, but we make up for that restriction by introducing a fixed satiation parameter α directly in the utility function for the inside goods, and allow it to be estimated; see Bhat et al., 2020 for further discussion, as well as Section 2.3 of this paper for additional details on identification). Third, the baseline marginal utility in this new utility structure is given by $\psi_k^{1/\sigma}$, which is but a simple exponential transformation of the baseline marginal utility ψ_k in Bhat (2008). γ_k is the vehicle to introduce the possibility of corner solutions (that is, zero consumption) for the inside goods k ($k=3,4,...,K$) as well as serves the role of a satiation parameter (higher values of γ_k imply less satiation). Also, the constraint that $\gamma_k>0$ for $k=3,4,...,K$ is maintained by reparametrizing γ_k as $\exp(\delta_k'\boldsymbol{\omega}_k)$, where $\boldsymbol{\omega}_k$ is a vector of decision maker-related characteristics and δ_k is a vector to be estimated.

The first constraint in Equation (1) has a budget of E across all goods k ($k=1,2,...,K$), and $p_k\geq 0$ is the unit price investment of the inside goods k ($k=3,4,...,K$) along this first dimension. The second constraint in Equation (1) has a budget of T, and $g_k\geq 0$ is the unit price investment along this second dimension. Also, $g_1=0$ and $p_2=0$.

2.1. Optimal Allocation

To find the optimal allocation of goods, we construct the Lagrangian and derive the Karush-Kuhn-Tucker (KKT) conditions. The Lagrangian function for the model of Equation (1) is:

$$L=U(\boldsymbol{x})+\lambda\left(E-x_1-\sum_{k=3}^{K}p_kx_k\right)+\mu\left(T-x_2-\sum_{k=3}^{K}g_kx_k\right), \tag{3}$$

where λ and μ are Lagrangian multipliers for the first and second constraints, respectively. The KKT first order conditions for optimal consumption allocations (x_k^*) are as follows, given that $x_1 > 0$ and $x_2 > 0$:

$$\psi_1 - \lambda = 0;\ \psi_2 - \mu = 0$$

$$\left[\psi_k\left(\frac{x_k^*}{\gamma_k}+1\right)^{-1}\right]^{\frac{1}{\sigma}} - \lambda p_k - \mu g_k = 0 \text{ if consumption} = x_k^*\ (x_k^* > 0),\ k = 3,4,\ldots,K, \tag{4}$$

$$[\psi_k]^{\frac{1}{\sigma}} - \lambda p_k - \mu g_k < 0 \text{ if } x_k^* = 0,\ k = 3,4,\ldots,K.$$

As in the case of a single constraint, the positive consumption of an inside good leads to a density contribution (the equality constraint above), while zero consumption of an inside good leads to a mass contribution (the inequality constraint above). Substituting λ and μ from the first equation line into the next two lines, we obtain:

$$\frac{1}{p_k}\left[\psi_k\left(\frac{x_k^*}{\gamma_k}+1\right)^{-1}\right]^{\frac{1}{\sigma}} = \psi_1 + \frac{\psi_2}{(p_k / g_k)} \text{ if consumption} = x_k^*\ (x_k^* > 0),\ k = 3,4,\ldots,K, \tag{5}$$

$$\frac{1}{p_k}[\psi_k]^{\frac{1}{\sigma}} < \psi_1 + \frac{\psi_2}{(p_k / g_k)} \text{ if } x_k^* = 0,\ k = 3,4,\ldots,K.$$

The KKT conditions above have an intuitive interpretation. For example, if p_k refers to the unit price of good *k*, and g_k refers to the unit space needed to store good *k*, the left side of the equations above correspond to the price-normalized marginal utility at the point of optimal consumption x_k^* for the inside goods. If consumption of a good *k* is positive, it will be such that the marginal utility at the consumed point of good *k* is equal to the baseline utility of outside good 1 (with a unit price of one) plus the opportunity cost-normalized baseline utility (where p_k / g_k refers to the opportunity cost of storing one unit of good *k*) of outside good 2. On the other hand, if the price-normalized marginal utility at zero consumption of good *k* is less than the baseline utility of outside good 1 plus the opportunity cost-normalized baseline utility of outside good 2, there will be no consumption of good *k*.

Rearranging terms and taking logarithms, the KKT conditions of Equation (4) may be rewritten as:

$$\begin{aligned} &\ln\psi_k - \sigma\ln(p_k\psi_1 + g_k\psi_2) = \ln\left(\frac{x_k^*}{\gamma_k}+1\right) \text{ if consumption} = x_k^*\ (x_k^* > 0),\ k = 3,4,\ldots,K \\ &\ln\psi_k - \sigma\ln(p_k\psi_1 + g_k\psi_2) < 0 \qquad \text{if } x_k^* = 0,\ k = 3,4,\ldots,K \end{aligned} \tag{6}$$

2.2. Statistical Specification

The baseline random marginal utility for each good is defined as follows:

$$\psi_k = \exp(\boldsymbol{\beta}'\boldsymbol{z_k} + \varepsilon_k),\ k = 1,2,...,K \tag{7}$$

where $\boldsymbol{z_k}$ is a set of attributes that characterize alternative *k* and the decision maker (including a constant), and ε_k captures the idiosyncratic (unobserved) characteristics that impact the baseline utility of good *k*. This parameterization guarantees the positivity of the baseline utility. Substituting this baseline utility form in Equation (6), and defining $V_k = \ln\left(\frac{x_k^*}{\gamma_k}+1\right) - \boldsymbol{\beta}'\boldsymbol{z_k}$, $V_{k0} = -\boldsymbol{\beta}'\boldsymbol{z_k}$, $a_{1k} = p_k e^{\boldsymbol{\beta}' z_1}$, and $a_{2k} = g_k e^{\boldsymbol{\beta}' z_2}$, the KKT conditions, after some algebraic manipulations, are equivalent to:

$$\begin{aligned} \eta_k &= \varepsilon_k - \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}) = V_k \quad \text{if consumption} = x_k^*\ (x_k^* > 0),\ k = 3,4,...,K \\ \eta_k &= \varepsilon_k - \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}) < V_{k0} \quad \text{if } x_k^* = 0,\ k = 3,4,...,K \end{aligned} \tag{8}$$

Note that, by construction, $a_{1k} \geq 0$ and $a_{2k} \geq 0$ ($k = 3,4,...,K$). The challenge now is to assume a distribution for the unobserved error terms ε_k ($k = 1,2,...,K$) that leads to a closed-form expression for both the multivariate cumulative distribution as well as all partial derivatives of the vector $\boldsymbol{\eta} = (\eta_3, \eta_4, ..., \eta_K)'$ (once there are closed-form expressions for these, the probability of any pattern of consumption is easily written down in closed form from the conditions above, as will be discussed later). Unfortunately, assuming the typical type-1 extreme value distribution (based on the limiting distribution of the maximum of random variables) for these error terms does not provide a closed-form solution, as shown in Castro et al. (2012). In this paper, we assume a type-1 distribution based on the limiting distribution of the minimum of random variables. That is, we assume that the error terms ε_1 and ε_2 are independent and identically distributed (IID) with a type-1 standard extreme value (minimum) distribution.

$$\boldsymbol{f}_{\varepsilon_k}(u) = e^{-e^u}.e^u \text{ and } \boldsymbol{F}_{\varepsilon_k}(u) = \text{Prob}(\varepsilon_k < u) = 1 - e^{-e^u} \text{ for } k = 1,2. \tag{9}$$

We also assume that the error terms ε_k ($k = 3,4,...,K$) have a location parameter of zero and a scale parameter of σ. Essentially, we are allowing heteroscedasticity in the error variances, with the error variances of the outside goods normalized to one and the error variances of the inside goods estimated. That is,

$$\boldsymbol{f}_{\varepsilon_k}(u) = e^{-e^{u/\sigma}}.e^{u/\sigma} \text{ and } \boldsymbol{F}_{\varepsilon_k}(u) = \text{Prob}(\varepsilon_k < u) = 1 - e^{-e^{u/\sigma}} \text{ for } k = 3,4,...,K. \tag{10}$$

The key insight is that doing so does lead, very surprisingly, to closed-form expressions. To move forward, we first state the following four properties with proofs.

Property 1

With the above-mentioned distributional assumptions on the error terms ε_k, the multivariate survival distribution function (SDF) of the vector $\boldsymbol{\eta}$ takes a closed-form as follows:

$$\boldsymbol{S}_{\boldsymbol{\eta}}(w_3, w_4, ..., w_K) = \text{Prob}(\eta_3 > w_3, \eta_4 > w_4, ..., \eta_K > w_K) = \frac{1}{\left(1+\sum_{k=3}^{K} a_{1k} e^{w_k/\sigma}\right)\left(1+\sum_{k=3}^{K} a_{2k} e^{w_k/\sigma}\right)} \quad (11)$$

Proof: $\text{Prob}(\eta_3 > w_3, \eta_4 > w_4, ..., \eta_K > w_K) =$

$$= \text{Prob}\left[\varepsilon_3 > w_3 + \sigma \ln(a_{1k} e^{\varepsilon_1} + a_{2k} e^{\varepsilon_2}), \varepsilon_4 > w_4 + \sigma \ln(a_{1k} e^{\varepsilon_1} + a_{2k} e^{\varepsilon_2}), ..., \varepsilon_K > w_K + \sigma \ln(a_{1k} e^{\varepsilon_1} + a_{2k} e^{\varepsilon_2})\right]$$

$$= \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{+\infty} \prod_{k=3}^{K} e^{-e^{-\left[\frac{w_k + \sigma \ln(a_{1k} e^{\varepsilon_1} + a_{2k} e^{\varepsilon_2})}{\sigma}\right]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_1 d\varepsilon_2. \quad (12)$$

Straightforward, even if tedious, integration of the above expression results in the needed multivariate survival distribution function (see Appendix A for the derivation). This is a neat multivariate distribution in statistics that we have not encountered in the literature.

Property 2

The survival distribution of any sub-vector of the $\boldsymbol{\eta}$ vector is readily obtained from the SDF expression above for the entire $\boldsymbol{\eta}$ vector. For example, the SDF of only the first two elements is:

$$\boldsymbol{S}_{\eta_3,\eta_4}(\eta_3 > w_3, \eta_4 > w_4) = \frac{1}{\left(1 + a_{13} e^{w_3/\sigma} + a_{14} e^{w_4/\sigma}\right)\left(1 + a_{23} e^{w_3/\sigma} + a_{24} e^{w_4/\sigma}\right)}. \quad (13)$$

Proof: This is straightforward by putting $w_k = -\infty$ for $k = 5, 6, ..., K$ in Equation (11).

Property 3

The multivariate cumulative distribution function (CDF) of the $\boldsymbol{\eta}$ vector can be written as a function of the SDFs corresponding to the random variates as follows:

$$\boldsymbol{F}_{\boldsymbol{\eta}}(w_3, w_4, ..., w_K) = \text{Prob}(\eta_3 < w_3, \eta_4 < w_4, ..., \eta_K < w_K) = 1 + \sum_{D \subset \{3,...K\}, |D| \geq 1} (-1)^{|D|} \boldsymbol{S}_D(\boldsymbol{w}_D), \quad (14)$$

where $\boldsymbol{S}_D(.)$ is the SDF of dimension *D*, *D* represents a specific combination of the η terms (representing a specific sub-vector of the $\boldsymbol{\eta}$ vector; there are a total of $(K-2) + C(K-2,2) + C(K-2,3) + ...C(K-2,K-2) = 2^{K-2} - 1$ possible combinations, |*D*| is the cardinality of the specific combination *D*, and $\boldsymbol{w}_D$ is a sub-vector of the vector $\boldsymbol{w} = (w_3, w_4, ..., w_K)$ with the appropriate elements corresponding to the combination *D* extracted.

Proof: This is based on the inclusion-exclusion probability law for all Fréchet class of multivariate distribution functions with given univariate margins (Feller, 1960).

Property 4

Define the following matrices for the two-constraint case (the generalization to the case of more than two-constraints will be defined later):

$$\boldsymbol{A} = \begin{bmatrix} a_{13} & a_{14} & a_{15} & \dots & a_{1K} \\ a_{23} & a_{24} & a_{25} & \dots & a_{2K} \end{bmatrix} \text{ [2×(K–2)] matrix,} \tag{15}$$

Let $\boldsymbol{A}_n$ be a sub-matrix of A with all rows included, but only the first n columns (so, the sub-matrix $\boldsymbol{A}_n$ is of dimension $2\times n$). Also define matrix $\boldsymbol{B}_n$ as a matrix of all possible combinations of the elements of $\boldsymbol{A}_n$ of length n (with duplication of elements not allowed within any combination and selection of only one element from each column allowed), with the many combinations stacked vertically (so the matrix $\boldsymbol{B}_n$ is of size $2^n \times n$). Let $\boldsymbol{H}_n$ be another $2^n \times n$ matrix that replaces the entries in matrix $\boldsymbol{B}_n$ with values corresponding to the row of matrix $\boldsymbol{B}_n$ in which each element of $\boldsymbol{B}_n$ lies. Finally, let $\boldsymbol{C}_n$ be a $2^n \times 2$ matrix, with each element in each row of each of the two columns representing a count of the number of elements in each row of matrix $\boldsymbol{B}_n$ appearing in each of the two rows of matrix $\boldsymbol{A}_n$. Thus, if n=3, we will have the following:

$$\boldsymbol{A}_n = \begin{bmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{bmatrix},\ \boldsymbol{B}_n = \begin{bmatrix} a_{13} & a_{14} & a_{15} \\ a_{13} & a_{14} & a_{25} \\ a_{13} & a_{24} & a_{15} \\ a_{13} & a_{24} & a_{25} \\ a_{23} & a_{14} & a_{15} \\ a_{23} & a_{14} & a_{25} \\ a_{23} & a_{24} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{bmatrix},\ \boldsymbol{H}_n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix},\ \text{and } \boldsymbol{C}_n = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} \tag{16}$$

With the above matrix definitions, the general formula for the nth order partial derivative of the multivariate survival distribution with respect to the first n variates $(\eta_3, \eta_4, ..., \eta_{n+2})'$ is:

$$\frac{\partial^n \boldsymbol{S}_\eta(w_3, w_4, \dots, w_K)}{\partial w_3 . \partial w_4 ... \partial w_{n+2}} = (-1)^n \frac{\exp\left(\sum_{i=3}^{n+2} \frac{w_i}{\sigma}\right)}{\sigma^n} \sum_{v=1}^{2^n} \left(\frac{\prod_{r=1}^{2} (\boldsymbol{C}_{n,vr}\,!) \prod_{g=1}^{n} \boldsymbol{B}_{n,vg}}{\prod_{r=1}^{2} \left[1 + \sum_{k=3}^{K} a_{rk} \exp\left(\frac{w_k}{\sigma}\right)\right]^{\boldsymbol{C}_{n,vr}+1}} \right) \tag{17}$$

The full derivative is obtained by putting $n = K–2$ in the above formula, and the multivariate density function of $\boldsymbol{\eta} = (\eta_3, \eta_4, ..., \eta_K)'$ may be obtained as:

$$\boldsymbol{f}_{\eta}(w_3,w_4,\ldots,w_K)=(-1)^{K-2}\frac{\partial^{K-2}\boldsymbol{S}_{\eta}(w_3,w_4,\ldots,w_K)}{\partial w_3.\partial w_4\ldots\partial w_K}=\frac{\exp\left(\sum_{k=3}^{K}\frac{w_k}{\sigma}\right)}{\sigma^{K-2}}\sum_{v=1}^{2^{K-2}}\left(\frac{\prod_{r=1}^{2}(\boldsymbol{C}_{K-2,vr}!)\prod_{g=1}^{K-2}\boldsymbol{B}_{K-2,vg}}{\prod_{r=1}^{2}\left[1+\sum_{k=3}^{K}a_{rk}\exp\left(\frac{w_k}{\sigma}\right)\right]^{\boldsymbol{C}_{K-2,vr}+1}}\right) \quad (18)$$

Proof: Please see Appendix B.

With the properties just discussed, the probability that the first M of the inside goods are consumed ($M\geq 1; M<K-2$) at levels $x_3^*,x_4^*,\ldots,x_{M+2}^*$ (with the implied consumptions of $x_1^*=E-\sum_{k=3}^{K}p_k x_k^*$ and $x_2^*=T-\sum_{k=3}^{K}g_k x_k^*$) may be written from Equation (8) as follows[1]:

$$P\left(x_3^*,\ldots,x_{M+2}^*,0,0,\ldots,0\right)$$

$$=|J|\int_{\eta_{M+3}=-\infty}^{\eta_{M+3}=V_{M+3,0}}\int_{\eta_{M+4}=-\infty}^{\eta_{M+4}=V_{M+4,0}}\cdots\int_{\eta_K=-\infty}^{\eta_K=V_{K,0}}\boldsymbol{f}_{\eta}(V_3,V_4,\ldots,V_{M+2},\eta_{M+3},\eta_{M+4},\ldots,\eta_K)\,d\eta_{M+3}d\eta_{M+4},\ldots,d\eta_K$$

$$=|J|\left.\frac{\partial^M\boldsymbol{F}_{\boldsymbol{\eta}}(\eta_3,\eta_4,\ldots,\eta_{M+2},V_{M+3,0},V_{M+4,0},\ldots,V_{K,0})}{\partial\eta_3\partial\eta_4\ldots\partial\eta_{M+2}}\right|_{\eta_3=V_3,\eta_4=V_4,\ldots,\eta_{M+2}=V_{M+2}}$$

$$=|J|\left[\begin{array}{l}\frac{\exp\left(\sum_{k=3}^{M+2}\frac{V_k}{\sigma}\right)}{\sigma^M}\sum_{v=1}^{2^M}\left(\frac{\prod_{r=1}^{2}(\boldsymbol{C}_{M,vr}!)\prod_{g=1}^{M}\boldsymbol{B}_{M,vg}}{\prod_{r=1}^{2}\left[1+\sum_{k=3}^{M+2}a_{rk}\exp\left(\frac{V_k}{\sigma}\right)\right]^{\boldsymbol{C}_{M,vr}+1}}\right)+\\ \sum_{D\subset\{M+3,M+4,\ldots,K\},|D|\geq 1}(-1)^{|D|}\frac{\exp\left(\sum_{i=3}^{M+2}\frac{V_i}{\sigma}\right)}{\sigma^M}\sum_{v=1}^{2^M}\left(\frac{\prod_{r=1}^{2}(\boldsymbol{C}_{M,vr}!)\prod_{g=1}^{M}\boldsymbol{B}_{M,vg}}{\prod_{r=1}^{2}\left[1+\sum_{k=3}^{M+2}a_{rk}\exp\left(\frac{V_k}{\sigma}\right)+\sum_{k=M+3}^{|D|+M+2}a_{rk}\exp\left(\frac{V_{k0}}{\sigma}\right)\right]^{\boldsymbol{C}_{M,vr}+1}}\right)\end{array}\right] \quad (19)$$

where $|J|=\left[\prod_{i=3}^{M+2}f_i\right]$, $f_i=\left(\frac{1}{x_i^*+\gamma_i}\right)$,

[1] Note, however, that there is no need to have information on the budgets E and T in our framework (because of the linear utility function for the outside goods). Also, important to note is that, simply because there is no need to observe E and T in our formulation, does not mean that the constraints are not at play in consumption decisions. As discussed in the context of Equation (5), consumers still make a trade-off decision among the unit prices of the inside goods along the many constraint dimensions when determining their consumption point. To be specific, the consumption pattern of the inside goods of a consumer, given the unit prices of the inside goods along each dimension (p_k and g_k in the two-constraint case), provides the information needed to extract the trade-offs implied by the constraints for the consumer.

The probability that all the inside goods are consumed at levels $x_3^*, x_4^*, ..., x_K^*$ is:

$$P\left(x_3^*,...,x_K^*\right)$$

$$= |J| \, \boldsymbol{f}_\eta(V_3,V_4,...,V_K) = |J| \frac{\exp\left(\sum_{k=3}^{K}\frac{V_k}{\sigma}\right)}{\sigma^{K-2}} \sum_{v=1}^{2^{K-2}} \left(\frac{\prod_{r=1}^{2}(\boldsymbol{C}_{K-2,vr}\,!)\prod_{g=1}^{K-2}\boldsymbol{B}_{K-2,vg}}{\prod_{r=1}^{2}\left[1+\sum_{k=3}^{K} a_{rk}\exp\left(\frac{V_k}{\sigma}\right)\right]^{\boldsymbol{C}_{K-2,vr}+1}} \right) \tag{20}$$

The probability that none of the inside goods are consumed is:

$$P\left(0,...,0\right) = 1 + \sum_{D\subset\{3,...,K\},|D|\geq 1} (-1)^{|D|} \frac{1}{\left(1+\sum_{k\in D} a_{1k}e^{V_{k0}/\sigma}\right)\left(1+\sum_{k\in D} a_{2k}e^{V_{k0}/\sigma}\right)} \tag{21}$$

Thus, the proposed model is referred to as the multi-constraint MDCEV model, which provides a closed form expression for the probabilities of consumption with any number of constraints.

2.3. Model Identification

A couple of issues related to identification. First, in general, the scale parameter σ will be estimable in the multiple constraint case. This is because the unit prices must vary across the goods for at least one-constraint dimension.[2] As soon as there is price variation in even a single dimension, σ become identifiable (see Bhat, 2018 and Bhat et al., 2020). In fact, from a pure theoretical point, even if there were only a single constraint with no price variation, the scale is identifiable in our proposed model, as it becomes similar to the linear outside good MDCEV model of Bhat et al. (2020). However, empirically speaking (and as discussed in Bhat et al., 2020), allowing a free scale parameter in such a single constraint case without price variation can lead to instability in many situations because of the entangling of two different mechanisms to produce satiation, and so the analyst may have to normalize the scale. Second, as also discussed in Castro et al. (2012), and as can be observed from the KKT conditions in Equation (6), it is not the case in the multiple constraint situation that only differences in the $\boldsymbol{\beta}'\boldsymbol{z}_k$ terms matter. This is because the logarithm functional form operates on a standardized function of the sum of quantities associated with the first two goods. Thus, the KKT conditions themselves (because of their functional form) do not impose any theoretical need for the normalization of constants and consumer-specific variables. However, the KKT conditions in Equation (8), as well as the probability expression in Equation (19), involve only the consumption pattern of $K-2$ goods, and so it would be prudent to set the component of $\boldsymbol{\beta}'\boldsymbol{z}_k$ corresponding to constants and individual-specific variables to zero for one or both of the first two goods.

[2] That is, in the two-constraint case, if the p_k and g_k values are the same across all the inside goods, there is no issue of having two-constraints; if this were the case, the two-constraints effectively collapse to a single constraint.

2.4. More Than Two-Constraints

The applications in the literature of a multi-constraint multiple discrete-continuous (MDC) model have been limited to two constraints, at least in part because adding more constraints poses computation problems in the open-form domain of earlier models. However, our proposed closed-form model should open up possibilities to easily estimate and apply MDC models with more than two constraints. For instance, in the context of household time-use in individual and joint non-work activities, there could be multiple budget constraints of time availability for non-work activity participation corresponding to each individual household member (based on, for example, each individual's employment status and work duration). Another application, especially relevant in today's pandemic world, is the addition of a storage space constraint in the house, along with other time and money budget constraints, in the purchase of grocery and other consumer items (that is, how households prioritize purchases so as to minimize shopping trips and/or home deliveries, while also being cognizant of storage space constraints and the impending possible lack of availability in the marketplace of consumer items).

We now show how our closed-form model can be applied to the general case with R constraints. Each constraint is associated with a limited resource (money, time, space, *etc*.). To estimate the MDCEV model with R constraints, individuals should consume at least R goods from the choice set, and the maximization problem is given by:

$$
\begin{aligned}
& Max\ U(\boldsymbol{x}) = \psi_1 x_1 + \psi_2 x_2 + \ldots\ \psi_R x_R + \sum_{k=R+1}^{K} \frac{\gamma_k}{\alpha} \psi_k^{(1-\alpha)} \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha} - 1 \right\} \\
& s.t.\ x_1 + \sum_{k=R+1}^{K} p_{1k} x_k = E_1,\ x_1 > 0 \\
& \quad x_2 + \sum_{k=R+1}^{K} p_{2k} x_k = E_2,\ x_2 > 0 \\
& \quad \vdots \\
& \quad x_R + \sum_{k=R+1}^{K} p_{Rk} x_k = E_R,\ x_R > 0
\end{aligned}
\qquad (22)
$$

where p_{rk} is the unitary contribution of good k $(k > R)$ to constraint r $(p_{rk} > 0\ \forall k = R+1,\ R+2, ...,\ K)$, and E_r is the total availability of resource r $(\forall r = 1,2,...,R)$. This problem can be solved in the same way as for the case with two-constraints, with $a_{rk} = p_{rk} e^{\beta' z_k}$ $(k = R+1, R+2, \ldots, K)$. The corresponding expressions are:

$$P\left(x_{R+1}^{*},\ ...,\ x_{M+R}^{*},\ 0,\ 0,\ ...,\ 0\right) =$$

$$=|J|\left[\begin{array}{l} \dfrac{\exp\left(\sum_{k=R+1}^{M+R}\dfrac{V_k}{\sigma}\right)}{\sigma^M}\sum_{v=1}^{R^M}\left(\dfrac{\prod_{r=1}^{R}(\boldsymbol{C}_{M,vr}\,!)\prod_{g=1}^{M}\boldsymbol{B}_{M,vg}}{\prod_{r=1}^{R}\left[1+\sum_{k=R+1}^{M+R}a_{rk}\exp\left(\dfrac{V_k}{\sigma}\right)\right]^{\boldsymbol{C}_{M,vr}+1}}\right)+ \\ \sum_{D\subset\{M+R+1,M+R+2,...,K\},|D|\geq 1}(-1)^{|D|}\dfrac{\exp\left(\sum_{i=R+1}^{M+R}\dfrac{V_i}{\sigma}\right)}{\sigma^M}\sum_{v=1}^{R^M}\left(\dfrac{\prod_{r=1}^{R}(\boldsymbol{C}_{M,vr}\,!)\prod_{g=1}^{M}\boldsymbol{B}_{M,vg}}{\prod_{r=1}^{R}\left[1+\sum_{k=R+1}^{M+R}a_{rk}\exp\left(\dfrac{V_k}{\sigma}\right)+\sum_{k=M+R+1}^{|D|+M+R}a_{rk}\exp\left(\dfrac{V_{k0}}{\sigma}\right)\right]^{\boldsymbol{C}_{M,vr}+1}}\right) \end{array}\right], \quad (23)$$

$$|J| = \left[\prod_{i=R+1}^{M+R} f_i\right],\ f_i=\left(\frac{1}{x_i^{*}+\gamma_i}\right)$$

$$P\left(x_{R+1}^{*},...,x_K^{*}\right)=|J|\frac{\exp\left(\sum_{k=R+1}^{K}\dfrac{V_k}{\sigma}\right)}{\sigma^{K-2}}\sum_{v=1}^{R^{K-R}}\left(\frac{\prod_{r=1}^{R}(\boldsymbol{C}_{K-R,vh}\,!)\prod_{g=1}^{K-R}\boldsymbol{B}_{K-R,vg}}{\prod_{r=1}^{R}\left[1+\sum_{k=3}^{K}a_{rk}\exp\left(\dfrac{V_k}{\sigma}\right)\right]^{\boldsymbol{C}_{K-R,vr}+1}}\right) \quad (24)$$

$$P\left(0,...,0\right)=1+\sum_{D\subset\{R+1,...,K\},|D|\geq 1}(-1)^{|D|}\frac{1}{\prod_{r=1}^{R}\left(1+\sum_{k\in D}a_{rk}e^{V_{k0}/\sigma}\right)} \quad (25)$$

Identification considerations in this multiple constraint case are similar to that discussed in the case of the two-constraint case.

We also present the discrete consumption probability expression for each possible consumption bundle as follows:

$$P(d_{R+1}=1,d_{R+2}=1,\cdots,d_{R+M}=1,d_{R+M+1}=0,\cdots,d_{K-1}=0,d_K=0)$$
$$=\int_{\eta_{R+1}=V_{R+1,0}}^{\eta_{R+1}=\infty}\int_{\eta_{R+2}=V_{R+2,0}}^{\eta_{R+2}=\infty}\cdots\int_{\eta_{R+M}=V_{R+M,0}}^{\eta_{R+M}=\infty}\int_{\eta_{R+M+1}=-\infty}^{\eta_{R+M+1}=V_{R+M+1,0}}\cdots\int_{\eta_{K-1}=-\infty}^{\eta_{K-1}=V_{K-1,0}}\int_{\eta_K=-\infty}^{\eta_K=V_{K,0}}\boldsymbol{f}(\eta_{R+1},\eta_{R+2},...,\eta_K)\,d\eta_K d\eta_{K-1},...,d\eta_{R+1}, \quad (26)$$

where $\boldsymbol{f}(\eta_{R+1},\eta_{R+2},...,\eta_K)$ represents the multivariate density function (pdf) of the random variates $\eta_{R+1},\eta_{R+2},...,\eta_K$. The above expression may be written as:

$$P(d_{R+1}=1,d_{R+2}=1,\cdots d_{R+M}=1,d_{R+M+1}=0,\cdots d_{K-1}=0,d_K=0)$$
$$=\boldsymbol{S}_M(V_{R+1,0},V_{R+2,0},...,V_{R+M,0})+\sum_{D\subset\{R+M+1,...,K-1,K\},|D|\geq 1}(-1)^{|D|}\boldsymbol{S}_{M+|D|}(V_{R+1,0},V_{R+2,0},...,V_{R+M,0},\boldsymbol{V}_{D,0}), \quad (27)$$

where $\boldsymbol{S}_N(.)$ for any dimension N is the multivariate survival distribution function given by Equation (11), D represents a specific combination of the non-consumed goods (there are a total of $2^{K-M-R}-1$ possible combinations of the non-consumed goods), $|D|$ is the cardinality of the specific combination D, and $\boldsymbol{V}_{D,0}$ is a vector which is specific to combination D. The discrete consumption probability for the case of none of the inside goods being consumed is already provided in Equation (21), while the discrete consumption probability for the case of all the inside goods being consumed is given by:

$$\begin{aligned} &P(d_{R+1}=1, d_{R+2}=1, \cdots d_{R+M}=1, d_{R+M+1}=1, \cdots d_{K-1}=1, d_K=1) \\ &= \boldsymbol{S}_{K-R}(V_{R+1,0}, V_{R+2,0}, ..., V_{K,0}) \end{aligned} \tag{28}$$

A final point on model formulation. In the case that one or more of the inside goods is consumed by every individual, the utility formulation of Equation (1) is not applicable because the presence of γ_k requires at least a few individuals to have a corner solution for good k. However, this situation is easily handled through a minor modification of the utility function. In particular, consider that the inside goods $l = R+1, R+2, ..., L$ are consumed by every individual, and the remaining inside goods $t = L+1, L+2, ..., K$ are not. Then, the utility formulation below is the appropriate one:

$$\begin{aligned} &U(\boldsymbol{x}) = \psi_1 x_1 + \psi_2 x_2 + ... \ \psi_R x_R + \sum_{l=R+1}^{L} \frac{1}{\alpha} \psi_l^{(1-\alpha)} x_l^{\alpha} + \sum_{t=L+1}^{K} \frac{\gamma_t}{\alpha} \psi_t^{(1-\alpha)} \left\{ \left(\frac{x_t}{\gamma_t} + 1 \right)^{\alpha} - 1 \right\} \\ &s.t. \ x_1 + \sum_{l=R+1}^{L} p_{1l} x_l + \sum_{t=L+1}^{K} p_{1t} x_t = E_1, \ x_1 > 0 \\ &\quad x_2 + \sum_{l=R+1}^{L} p_{2l} x_l + \sum_{t=L+1}^{K} p_{2t} x_t = E_2, \ x_2 > 0 \\ &\quad \vdots \\ &\quad x_R + \sum_{l=R+1}^{L} p_{Rl} x_l + \sum_{t=L+1}^{K} p_{Rt} x_t = E_R, \ x_R > 0 \end{aligned} \tag{29}$$

There is little change from the earlier formulation, except that $V_l = \ln x_l^* - \boldsymbol{\beta}' \boldsymbol{z}_l \ (l = R+1, R+2, ..., L)$ and some minor adjustments to Equations (23), (24), and (25), as presented in Appendix C.

3. FORECASTING

The KKT conditions of Equation (8) for the inside goods $(k = 3, ..., K)$ translate to the following conditions on the error terms:

$$\begin{aligned} &\varepsilon_k > (-\boldsymbol{\beta}'\boldsymbol{z_k}) + \xi_k \text{ and } x_k^* = \left[\exp(\boldsymbol{\beta}'\boldsymbol{z_k} + \varepsilon_k - \xi_k) - 1\right]\gamma_k \ \text{ if consumption} = x_k^* \ (x_k^* > 0), \ k = 3, 4, ..., K \\ &\varepsilon_k < (-\boldsymbol{\beta}'\boldsymbol{z_k}) + \xi_k \text{ if } x_k^* = 0, k = 3, 4, ..., K, \end{aligned} \tag{30}$$

where $\xi_k = \sigma \ln(a_{1k} e^{\varepsilon_1} + a_{2k} e^{\varepsilon_2})$, $a_{1k} = p_k e^{\boldsymbol{\beta}' z_1}$, and $a_{2k} = g_k e^{\boldsymbol{\beta}' z_2}$.

The simplest forecasting procedure for each observation is as follows:

- Step 1: Draw *K*–*R*+1 independent realizations of ε_k (say μ_k), one for each inside good k $(k = R+1,...,K)$, from the extreme value distribution with location parameter of 0 and the scale parameter equal to the estimated σ value (label this distribution as EV$(0,\hat{\sigma})$). Also, draw R independent realizations of ε_k from EV(0,1) for the outside goods $(k = 1,2,...,R)$.
- Step 2: Given estimated values of $\boldsymbol{\beta}$ and σ, and the input values p_k, and g_k $(k = R+1, R+2,...,K)$ and $\boldsymbol{z_k}$ $(k = 1,2,...,K)$, compute an estimate of ξ_k (say $\hat{\xi}_k$) for each inside good as well as an estimate of $-\boldsymbol{\beta}'\boldsymbol{z_k}$ (say, $\hat{\theta}_k = -\hat{\boldsymbol{\beta}}'\boldsymbol{z_k}$) for each inside good. If $\mu_k > \hat{\theta}_k + \hat{\xi}_k$ $\left(k = R+1, R+2, \ldots, K\right)$, declare the inside good k as being selected for consumption $(d_k = 1)$; otherwise, declare the inside good as not being selected for consumption $(d_k = 0)$.
- Step 3: For the inside goods that are selected $(d_k = 1)$, forecast the continuous value of consumption as follows: $\hat{x}_k^* = \left[\exp\left\{\mu_k - \left(\hat{\theta}_k + \hat{\xi}_k\right)\right\} - 1\right]\hat{\gamma}_k$.

  To conserve on space, two other more advanced forecasting approaches are relegated to an online supplement available at https://www.caee.utexas.edu/prof/bhat/ABSTRACTS/MCMDCEV/OnlineSupp.pdf.

4. SIMULATION EVALUATION

The simulation exercises undertaken in this section examine the effect of employing a single constraint model when there are in fact two budget considerations that consumers are faced with. We do so by estimating the proposed closed-form model that correctly considers both budget constraints as well as the proposed model assuming that only a single constraint is operational. The appropriateness of the models is assessed by the ability to recover parameters from finite samples by generating simulated data sets with known underlying model parameters. As well, we investigate the predictive ability of the different models.

4.1. Experimental Design

In the design, we generate a sample of 3000 observations with four alternatives and two independent variables in the $\boldsymbol{z}_{qk}$ vector in the baseline utility for each alternative (we introduce the subscript q for individuals here; $q = 1,2,...,3000$). For this simulation experiment, we do not consider constants in the baseline preference. Of the two independent variables, the first is a dummy variable, while the other is a continuous variable. That is, consider the following for the $\boldsymbol{z}_{qk}$ vectors (*k*=1 and *k*=2 are the two outside goods):

$$\boldsymbol{z}_{q1} = [0,0],\ \boldsymbol{z}_{q2} = [0,0],\ \boldsymbol{z}_{q3} = \left[y_q, \tilde{z}_{q3}\right], \text{ and } \boldsymbol{z}_{q3} = \left[y_q, \tilde{z}_{q4}\right]. \tag{31}$$

For the dummy variable (y_q) in $\boldsymbol{z}_{qk}$ $(k=3,4)$, we treat this as an individual-specific variable (that does not vary across alternatives). To construct this dummy variable, 3000 independent values are drawn from the standard uniform distribution. If the value drawn is less than 0.5, the value of '0' is assigned to the dummy variable. Otherwise, the value of '1' is assigned. The coefficients on this dummy variable are specified to be 0 for the first two alternatives (the two outside goods) and 1.25 for the second two alternatives (considered as the inside goods). Thus, a single parameter β_1 (=1.25) is to be estimated for the dummy variable. The values for the continuous variable $\tilde{z}_{q3}$ are drawn from a standard univariate normal distribution, while the corresponding values $\tilde{z}_{q4}$ are drawn from a univariate normal distribution with mean 0.5 and standard deviation of 1. The parameter β_2 on this continuous variable is specified to be one. The scale σ is set to one. Furthermore, a constant-only satiation parameter of e^1 is set for both the inside goods (i.e., $\delta_3=1$ and $\delta_4=1$, and the satiation parameters γ_3 and γ_4 are effectively 2.718 for both inside goods). Additionally, we consider two budget considerations. The p_{qk} value for the first outside good is 1, and the p_{qk} value for the second outside good is zero (for all q=1,2,…,Q). The p_{qk} values for the inside goods (k=3,4) for each individual q is drawn from a truncated uniform distribution with a mean of 0.5 for the first inside good (truncated between 0.25 and 0.75) and a mean of 1.5 for the second outside good (truncated between 1 and 2). The g_{qk} value for the first outside good is 0, and the g_{qk} value for the second outside good is one (for all q=1,2,…,Q). The g_{qk} values for the inside goods (k=3,4) for each individual q is drawn similar to that for the first budget consideration, but the values are reversed. That is, g_{qk} is drawn from a truncated uniform distribution with a mean of 1.5 for the first inside good (truncated between 1 and 2) and a mean of 0.5 for the second outside good (truncated between 0.25 and 0.75). Once generated, the independent variable values and the unit prices are held fixed in the entire rest of the simulation exercise.

4.2. Data Generation and Performance Metrics

Using the design presented in the previous sections, we generate the consumption quantity vector $\boldsymbol{x}_q^*$ for each individual using the forecasting algorithm of the previous section. The parameters to be estimated from the data generating process correspond to $\mu=[\beta_1=1.25,\ \beta_2=1, \gamma_3\text{=2.718},\ \gamma_4\text{=2.718}\ \sigma=1]'$. The above data generation process is undertaken 500 times with different realizations of the z_q vector to generate 500 different data sets. For each of the 500 datasets, we estimate three models: (a) the proposed multiple constraint model, (b) the single constraint model with only the first set of unit prices (the p_{qk} values) active and ignoring the second budget constraint, (c) the single constraint model with only the second set of unit prices (the g_{qk} values) active and ignoring the first budget constraint. The performances of the three models in recovering the "true" parameters, their standard errors, as well as predicting the consumption values are evaluated as follows:

(1) Estimate the parameters using each of the three models for each data set *s*. Estimate the standard errors.

(2) Compute the mean estimate for each model parameter across the data sets to obtain **a mean estimate**. Compute the **absolute percentage (finite sample) bias** (APB) of the estimator as:

$$APB(\%) = \left| \frac{\text{mean estimate - true value}}{\text{true value}} \right| \times 100.$$

(3) Compute the standard deviation for each model parameter across the data sets, and label this as the **finite sample standard error** or **FSSE** (essentially, this is the empirical standard error).

(4) Compute the median standard error for each model parameter across the data sets and label this as the **asymptotic standard error** or **ASE** (essentially, this is the standard error of the distribution of the estimator as the sample size increases).

(5) Next, to evaluate the accuracy of the asymptotic standard error formula, compute the APB associated with the ASE of the estimator as:

$$APBASE(\%) = \left| \frac{\text{ASE - FSSE}}{\text{FSSE}} \right| \times 100$$

(6) Examine the data fit at a disaggregate level by comparing the log-likelihood values at convergence of the models. A rigorous statistical test of data fit cannot be undertaken using traditional nested likelihood ratio tests, because the models are not nested forms of each other. But the model with the higher log-likelihood value is to be preferred, because all the models have the same number of estimated parameters. Based on the log-likelihood values for each of the 500 runs (corresponding to the 500 datasets), compute a mean log-likelihood value for each of the two-constraint and one-constraint models. In addition, also compute the average probability of correct prediction for the discrete consumption across the 500 datasets. That is, using Equations (21), (27), and (28), and for each of the 500 datasets, we compute the predicted bivariate probability of the observed discrete choice for each observation (which can be one of four discrete choice combinations based on whether or not good 3 is consumed and whether or not good 4 is consumed), and then compute an average across individuals. This average probability of correct prediction at a dataset-level is then averaged across the 500 datasets to obtain a single average probability of correct prediction. For the continuous consumption level, for each dataset, remove any effects of poor discrete choice predictions on the continuous prediction outcomes by assuming the observed multivariate discrete outcome for each observation, and predict the continuous consumptions for each individual using Step (4) of the forecasting algorithm with 1000 error vector replications per individual observation. Compute an absolute percentage error (APE) for each of goods 3 and 4, and each observation, for these continuous predictions by comparing with the actual consumption values of goods 3 and 4 (ignoring zero consumptions based on the discrete choice), and then compute a mean APE

(MAPE) across all goods and all individuals for each dataset. At the end, compute an MAPE across all the 500 datasets.

(7) Finally, at the aggregate level, examine model fit at both the discrete consumption level as well as the continuous consumption level. For the discrete level, for each dataset, predict the aggregate share of individuals participating in each of the four possible discrete outcomes, and compare these predicted shares with the actual percentages of individuals in each combination (using the weighted MAPE statistic, which is the MAPE for each combination weighted by the actual percentage shares of individuals participating in each combination). Next, compute the average of the weighted MAPE statistic across the 500 datasets. For the continuous consumption level, for each dataset, compute an aggregate mean (across observations) of the observation-level continuous consumptions for each of the inside goods (goods 3 and 4), as predicted earlier, and compute an MAPE by comparing the mean of the predicted aggregate consumption of each of the inside goods with the corresponding actual mean value of consumption of the inside good (again, ignoring zero consumptions based on the discrete choice, so this MAPE corresponds to consumption conditional on a positive discrete consumption decision). Then, average the dataset-level MAPE (across the 500 datasets) to obtain an overall MAPE for the continuous consumption quantity.

4.3. Simulation Results

The simulation results for the evaluation of the parameter estimates are provided in Table 1. For each of the five parameters to be estimated, the first row provides the true value, followed by the estimate obtained and the following metrics for each estimate: APB, FSSE, ASE, and APBASE. A number of observations may be made from the table. First, the proposed multi-constraint model (third column of the table) recovers the "true" baseline parameter values (β_1 and β_2) very accurately, with an APB of less than 0.05%; on the other hand, the single-constraint models clearly are nowhere as good as the proposed model in recovering these baseline parameter values. In particular, the single constraint models show a very severe bias of more than 55% for the first parameter, and more moderate biases (but still much larger than the one obtained for the proposed model) for the second parameter. The substantially higher bias for the first parameter is to be expected, because the first variable is a dummy "switch" variable, and so its coefficient is very sensitive to the use of the correct constraint specification. Also, the estimated values of β_1 and β_2 are lower in magnitude than the corresponding "true" parameters in both the single constraint specifications. Again, this is to be expected, because of the KKT conditions. For the two-constraint case, the right side of Equation (5) has the term $\psi_1 + \dfrac{\psi_2}{(p_k / g_k)}$, while in the one-constraint case, this right side would be ψ_1 (if the first constraint is the only one active) or ψ_2 (if the second constraint is the only one active). Thus, the right side is lower in magnitude for the one-constraint cases than for the two-constraint case. At the same time, the magnitude of this right side determines the discrete consumption decision, based on the inequality condition of the KKT. With the same simulated data, therefore, the single constraint cases will underestimate the baseline preference of the inside goods to achieve a reasonable fit for the discrete consumption decisions, which gets translated to the lower estimates of β_1 and β_2. Of course, the recovery of these parameters in the two-constraint and one-constraint cases is quite precise (as can be observed from the entries in the APBASE rows for these parameters in Table 1), because, given individual discrete consumption patterns, the estimation should converge toward the same parameter. Even so, we find that the ASE estimate is closer to the FSSE for the two-constraint estimation relative to the one-constraint estimations, as also evidenced by the higher values of the APBASE metric for the one-constraint estimations.

As far as the satiation parameters γ_3 and γ_4 are concerned, the MC-MDCEV model records an absolute bias of below 0.2%, while the APB of the one-constraint models hover in the 25-55% range. The reason for the biases in the satiation parameters in the single constraint cases may not be that obvious as for the β_1 and β_2 preference parameters, but will become clear in our discussion later in this section. The basic point is that these satiation parameters work in tandem with the preference parameters to determine the continuous consumptions. Thus, given the substantial variations in the estimated preference parameters, it is not surprising that these variations get transmitted to the estimated satiation values. The APBASE values for the satiation parameters are all similar across the three models, even if particularly higher for the γ_4 parameter in the one-constraint cases.

Finally, even in terms of recovering the scale parameter, our proposed model exhibits much smaller APB and APBASE values relative to the single-constraint models. All these findings underscore the superiority of our proposed model in accurately recovering model parameters.

The substantial difference in estimated parameters from the one-constraint models (relative to the "true" parameters) does not necessarily mean that the discrete-continuous predictions will be poor too (after all, as discussed in the context of the KKT conditions, the under-scaling of the preference parameters in the one-constraint cases is but natural, given the quantity that appears on the right side of the KKT conditions). Thus, to obtain a sense of performance, it is imperative that model predictions of the multiple discrete-continuous consumptions also be undertaken. We do so using the approach discussed in the sixth and seventh points in the previous section. Our results show that the mean (across the 500 datasets) log-likelihood at convergence of the two-constraint model is -14262.96, while the log-likelihoods at convergence for the one-constraint models are -14953.20 and -14969.98. The average probability of correct prediction for the discrete consumption is 0.431 from the two-constraint model, and 0.365 and 0.371 from the one-constraint models, while the disaggregate-level MAPE is 20.7% for the two-constraint model and 24.9% and 24.6% for the one-constraint models. All of these disaggregate-level metrics clearly favor the two-constraint estimation, and reveal the pitfalls (from a disaggregate predictive level) of using a one-constraint model when the two-constraint model is the correct formulation.

The aggregate-level fit measures for the three models are shown in Table 2. The top panel of this table corresponds to the discrete choice consumption and the bottom panel correspond to the continuous consumption of the two inside goods. The weighted MAPE value for the proposed model at the discrete choice consumption level comprehensively outperforms the corresponding values for the one-constraint models. It may be illustrative to note that the price value for the first good along the first constraint, by construction in our data generation process, is lower than that of the second good; hence, in the single-constraint model with only the first constraint active, the model, at the discrete consumption level, substantially overpredicts the "first inside good consumption" and underpredicts the "second inside good consumption". A similar but reverse pattern is observed for the one-constraint model with only the second constraint active, where the first good has a higher price than the second good. The results in the lower panel of Table 2 for the continuous choice consumption (conditional on positive discrete choice consumption) once again highlights the superior performance of our proposed model over single-constraint models. The weighted MAPE values for the single-constraint models are in the 50-60% range, nowhere even close to the accuracy level of the two-constraint model with a weighted MAPE of less than 3%. The reason for this is that the first single constraint model leads to an (incorrect) high baseline preference for the first good (as discussed above). However, the continuous consumptions (conditional on consumption) in the data are not large, and so the MDCEV model attempts to compensate/correct for the high baseline preference for the first good by also estimating a high satiation for this first good, even as it fits to the continuous consumption values. This is also clearly evident in the dramatically lower estimate of γ_3 (which implies higher satiation) when only the first constraint is active (see the second row panel and fourth column of Table 1). Thus, the much smaller continuous value (conditional on discrete consumption) predicted by the first constraint-only model in Table 2. The situation, as expected, gets exactly reversed in the second constraint-only model.

In summary, the results from the simulation experiment underscore the utility of the proposed MC-MDCEV model in situations where an individual's multiple discrete-continuous consumption decision is subjected to more than one active constraint. Both from a preference parameter estimation standpoint, as well as a data fit and predictive standpoint, serious mis-specification is likely to occur if only one-constraint is considered in a true multiple constraint scenario. Importantly, our proposed multiple constraint closed-form model is no more difficult to estimate than a single-constraint MDCEV model.

5. APPLICATION

In this section, we demonstrate an application of our proposed model to the case of individuals' weekly activity participation, subject to a time-constraint as well as a money-constraint. We briefly describe the data source and sample, followed by a discussion of the results and data fit.

5.1. Data Description

The data source for the study is the Dutch Longitudinal Internet Studies for the Social Sciences (LISS) panel, which is based on a probability sample of Dutch households drawn from the country's population register. The panel, administered via the internet by CentERdata (www.lissdata.nl), is a standard social monthly survey undertaken in 2009, 2010, and 2012. A few questions related to time use and expenditure in different activities were included in the surveys (Cherchye et al., 2012). In the current paper, we focus on the data from the latest wave (October 2012). Respondents reported (1) the time allocated to multiple activities (including work) during the seven days prior to the survey, and (2) the average monthly expenditure (in euros) in 30 expense categories for each of the 12 months prior to the survey. To achieve consistency between activity durations and expenditures, monthly expenditures were divided by four to obtain weekly expenditures. The resulting weekly time use and consumption data are complemented with socio-demographic information drawn from the LISS panel.

The sample used to estimate our model includes individuals who are the sole workers within their respective households. This recognizes that employed individuals maybe distinctly different in their time use decisions relative to those who are unemployed. Several consistency checks were performed to obtain the estimation sample. Further details are provided in Astroza et al. (2017). The final estimation sample includes the time-use of 1,193 workers.

5.2. Classification of Activities and Unit Price for Expenditures

The focus of our analysis is on time-use subject to the budget constraints of time and money. We consider the following four non-work, non-education, and non-sleep activities as the inside goods:

1) Household chores, including cleaning, shopping, cooking, gardening, washing, dressing, eating, visiting the hairdresser, seeing the doctor, and activities with any own children less than 16 years of age
2) Leisure, including in-home and out-of-home recreational activities, such as watching TV, reading, practicing sports, hobbies, computer as hobby, visiting family or friends, going out, walking the dog, cycling, being physically intimate, etc.
3) Personal business, including family finances and assisting friends/family.
4) Social, including religious activities, civic and volunteer activities, and attending social gatherings.

The MDC dependent variable corresponds to weekly participation and weekly time investment in each of these four inside activity purposes (the procedure to obtain these weekly time-use variables is discussed in detail in Astroza et al., 2017). In addition, we consider two outside goods (or two outside activity purposes, denoted by k=1 and k=2). The first outside activity purpose does not involve any money expenditure (such as, say sleeping), while the second activity purpose does not involve any time expenditure (this may seem tricky, but can be considered as, for example, "clicking to purchase online goods and services", in which the "click" is almost instantaneous; in any case, in our formulation, there is no need to explicitly define the outside purposes, and so these outside purposes can be "imaginary" activities). The time constraint represents time as a limited resource, bounded by the available time after considering activities such as work and sleep. The unit price for time use in each of the inside activities is unity (that is, $p_{qk}=1$ for all q and k=3,4,5,6), since the decision variables themselves represent time investments.

The money constraint represents limited purchasing power, and the unit price for expenditure (that is, g_{qk} for all q and k=3,4,5,6) is computed as the average (across all individuals who participate in the respective activities) of the ratio of expenditure to time duration in activity k. The unit price is in Euros/hour. Among the four inside activities, personal business and social activities did not have expenses associated with them (in the dataset), and so their unit prices of expenditure are zero.

Descriptive statistics for time duration (the dependent variable in our model), and the unit price for expenditure, are presented in Table 3. As can be observed from the table, a large fraction participates in household chores, leisure, and personal business activities during the course of a week. However, less than half of all individuals partake in social activities. Among those who participate in an activity purpose, the average duration of participation is highest for leisure (individuals participate on average for just under 32 hours/week or about 4.6 hours per day) and lowest for personal business (individuals participate on average for 7.5 hours/week or about an hour per day).

5.3. Results

The emphasis of our empirical efforts in this paper is to demonstrate the application of the proposed model rather than necessarily on substantive interpretations and policy implications. But, within the context of the data available, we explored alternative variable specifications to arrive at the best possible specification (including considering alternative functional forms for continuous independent variables such as income and age, including a linear form, piecewise linear forms in the form of spline functions, and dummy variable specifications for different groupings). The final variable specification was based on statistical significance testing as well as intuitive reasoning based on the results of earlier studies.

In this section, we discuss the effects of the variables on activity participation by variable category (see Table 4). The effects relate to the impact of variables on the logarithm of the baseline preference (that is, they correspond to the $\boldsymbol{\beta}$ vector elements in Equation (7)), except when discussing the satiation effects toward the end of this section.

Individual Characteristics: Our results suggest that women are more likely than men to partake in household chores. This gender asymmetry in maintenance-oriented household activities has been well established in the time-use literature (see Bernardo et al., 2015, Bernstein, 2015, and Cerrato and Cifre, 2018). Interestingly, while gender perceptions have changed considerably over the years in terms of universal support for women pursuing professional and political careers, perceptions regarding traditional gender roles dominate on the home front. In particular, while men have picked up a little over time in terms of household chores, there is still a significant gap, with women spending, on average, about an hour more than men (U.S. Bureau of Labor Statistics, 2015). And this gap exists even in the younger generation (18-30 years of age). One possible explanation is that men are pitching in just a little more at home as a means to encourage women to work outside the home and bring in another paycheck, rather than because their gender role perceptions may have shifted. That is, the increasing support for women in the workforce may not necessarily be tied solely to progressive thinking on the part of men, but may be at least as much due to a "money buffering" notion for times of economic challenges (Donner, 2020). Another possible explanation is that society perceives a clean house as a signal of the residence of a "good family" (see Mulder and Lauster, 2010), and this societal pressure for cleanliness appears to rest squarely on the shoulders of women, resulting in the woman being more concerned about how the house looks and investing more time in household chores. Besides, some studies reveal that women think they are more efficient in cleaning and household chores, and do not trust a male partner to do the job "right" (see Vieira et al., 2019).

Age also impacts time-use. Individuals below the age of 30 years have a lower preference for household chores relative to their older peers. Younger individuals are likely to be either students or young professionals in the formative years of their career, and such individuals generally participate less and allocate relatively limited time to household responsibilities compared to older individuals (Sánchez et al., 2014). Moreover, such individuals are also less likely to invest time in personal and household care activities relative to older individuals (Regitz-Zagrosek, 2012). Also interesting to note is that younger individuals (those below the age of 45 years) have a lower preference for personal business and social activity purposes, implying that they are generally less likely to participate in these two activities relative to their older peers. The lower proclivity for participating in personal business (that is, family finances and assisting family/friends) among younger individuals may be attributed to such individuals being more focused on asset building and not as much on financial planning and asset investment for retirement (Taft et al., 2013). The lower participation in social activities among young individuals may be tied to being more time poor, because of juggling career pressures and family pressures. As observed by Harvey and Mukhopadhyay (2007) and Williams et al. (2016), the convergence of career pressures and life-cycle pressures is a leading cause for time poverty. Further, religious activities, which also serve a social purpose and are included within the social category, are less likely to be engaged in by younger individuals (see Pew Research Center, 2018), because of generally lower religiosity among the younger generation.

Household Attributes: The influence of household attributes indicates that individuals in large households have a higher baseline preference for personal business and social activity purposes. (see Bhat et al., 2016 and Astroza et al., 2017 for similar findings). The latter finding is perhaps a reflection of a simple "numbers" effect; a larger family provides more opportunity to interact and partake in rewarding social activities. The results also point to the higher investment of time in household chores as the number of children (less than 15 years of age) increases, perhaps reflecting a general preference for personally (and in the comfort and privacy of the home) meeting the biological needs of young children (see Farkas et al., 2000 for a similar result). Also, and not surprisingly, respondents in households with many children are less likely (than respondents in households with zero or fewer children) to invest time in leisure activities. This is consistent with the hypothesis that employed parents with children are prone to time poverty for leisure and relaxation activities, reinforcing the high time cost of children found by Ekert-Jaffé (2011) and the time crunch experienced by working parents found by Bernardo et al. (2015) and Craig and Brown (2016). Finally, within the group of household socio-demographics, a higher household income leads to more household chores, and more leisure and less social activities (for income levels of 750 or more euros relative to lower income levels). The elevated participation in leisure among high income individuals is expected, given that leisure activities have an associated financial outlay and higher income households are better positioned to absorb these costs (see Highfill and Franks, 2019, and Parady et al., 2019). In addition, an explicit show of leisure indulgence may be a socio-cultural vehicle to signal wealth, power and status, and privileged access to limited resources. The lower social activity participation among high income individuals (or, alternatively, the higher social activity participation among the low income individuals) is reasonable because social activities typically do not cost money. Besides, this effect may also be proxying for the effect of the closer-knit extended family and community unit of socialization among immigrants in the Netherlands, who generally earn less than domestic-born citizens.

Baseline Preference Constants: The baseline preference constants do not have any clear and substantive interpretations because of the presence of count variables (such as number of household members and number of children in the household). However, the negative sign corresponding to social activities is consistent with the low participation in this activity purpose relative to other activity purposes, as presented earlier in Table 3.

Satiation Effects Through γ_k Parameters: To allow heterogeneity in the parameters across individuals, while also guaranteeing the positivity of the parameters, they are parameterized as $\gamma_k = \exp(\delta_k' \boldsymbol{\omega}_k)$. The estimates in Table 4 for the satiation effects correspond to the elements of the δ_k vector. A positive value for a δ_k element implies that an increase in the corresponding element of the ω_k vector increases γ_k, which has the result of reducing satiation effects and increasing the continuous consumption quantity of alternative *k* (conditional on consumption of alternative *k*). On the other hand, a negative value for a δ_k element implies that an increase in the corresponding element of the $\boldsymbol{\omega}_k$ vector decreases γ_k, which has the result of increasing satiation effects and decreasing the continuous consumption quantity of alternative *k* (conditional on consumption of alternative *k*).

The final specification (see bottom panel of Table 4) indicates that, while there is no statistically significant difference between men and women in participation in leisure, conditional on participation, women engage for less time in leisure. This gender inequality in leisure activity time is well-established in the time poverty literature. Sociologists Hochschild and Machung (1989) coined the term “the second shift” to describe the additional time burdens and responsibilities of working mothers, leaving them with little time for leisure and relaxation pursuits. Hochschild and Machung posit that working women are not only responsible for a daily shift of paid work, but also an additional shift of unpaid work in the home. More recent studies in different countries around the world (see Powell and Craig, 2015, Balish et al., 2016, and Annandale and Hammarstrom, 2015) confirm this gender-based leisure time inequality. This is also supported by the 2018 American Time Use (ATUS) data, which indicates that men, on average, spend 50 more minutes per day in leisure relative to women (U.S. Bureau of Labor Statistics, 2019).

The age effect on duration of participation in social activities is interesting, and highlights the MDCEV model's ability to allow different directions of impact on the discrete choice of participation and the continuous time investment conditional on participation. Specifically, while younger individuals are less likely to partake in one or more social activity episodes during the course of the week (as discussed earlier), they participate for longer periods in social activity if they participate in one or more episodes. This may be reflecting a justification effect or a "fixed cost" effect, wherein once the relatively time-poor young individuals decide to participate in social activities, they decide to invest a good amount of time in it. Finally, as the number of individuals in a household increases, not only does participation in household chores increase, but so does the time invested in household chores.

The satiation effects constants generally reflect the high duration of time investment in leisure activity and the low duration of time investment in personal business activity, subject to participation. Of course, these constants also adjust for the sample range of explanatory variables, as well as the magnitudes of the estimated baseline preferences, to provide the best fit for the continuous consumption values.

Scale Parameter: The estimated value of the scale parameter is presented in the last row of Table 4. The scale parameter is statistically significant and is greater than zero (as it should be). Importantly, it is also statistically different from one (the t-statistic for this test is 45.0). Clearly, an arbitrary normalization of the scale parameter to one will lead to an unnecessary degradation of the model fit.

5.4. Data Fit Measures

To evaluate the fit of our model, we compute the same disaggregate and aggregate metrics as discussed in the simulation experiment section. The log-likelihood at convergence of the two-constraint model is -15,693.2, while the log-likelihood measure at constants is -16425.6. A nested likelihood ratio test clearly rejects the null hypothesis that the estimated model is no better than the constants-only model (the test statistic is 1464.8, which is greater than the chi-squared table statistic with 23 degrees of freedom at any reasonable level of significance. For comparison, the log-likelihoods at convergence for the one-constraint models are -16249.9 and -16715.5, clearly lower than the log-likelihood at convergence of the two-constraint model. Also, the average probability of correct prediction for the discrete consumption is 0.374 from the two-constraint model, and 0.337 and 0.336 from the one-constraint models, while the disaggregate-level MAPE is 31.2% for the two-constraint model and 44.7% and 51.6% for the one-constraint models

The aggregate-level fit measures for the three models are shown in Table 5. For ease of presentation, we provide the pairwise predictions of activity participation at the disaggregate level for the four activities in our application (based on whether or not an individual participates in each of these four activities, there are a total of 2^4 = 16 activity-combinations; however, to make our presentation simple and to avoid clutter, we only provide pairwise predictions of activity participation, which corresponds to 6 possible combinations). For each of the three models, the predicted shares of the pairwise combinations at the discrete level are computed and provided in the top panel of Table 5. Our proposed model with a weighted MAPE value of just over 3% outperforms both the single-constraint models having weighted MAPE values of more than 8%. Even in terms of the number of inside alternatives chosen, the weighted MAPE values of just under 30% for the one-constraint models are 2 times the weighted MAPE value of our proposed model. The aggregate fit measures in the bottom panel of Table 5 correspond to the conditional continuous consumption dimension (that is, to the average predicted continuous values; in our context, these values are the number of hours in a week for which an individual engages in the respective "inside" activity, given that an individual decides to participate in that activity). Once again, the prediction accuracy of the two-constraint model is superior to the one-constraint models. In fact, unlike the case of our simulation exercise, wherein the data generation process itself is embedded in the two-constraints assumption, the data that we use in our application is naturally observed and derived from individuals' weekly activity participation decisions; therefore, the performance of the proposed model at the aggregate as well as disaggregate levels reinforces the notion that consumers indeed consider multiple resource constraints when deciding their multiple discrete-continuous consumption choices. In summary, the closed-form MC-MDCEV proposed here is a clear winner (and by a huge margin) over the single-constraint models.

5.5. Comparison with Earlier Multiple-Constraint Study

In this section, we compare the performance of our closed-form model with Castro et al.'s open-form model (that assumes the usual type-I extreme value based on the maximum distribution for the error terms and proposes an open-form probability structure), in terms of log-likelihood values at convergence and prediction accuracy. To do so, we retain the final empirical specification obtained in our closed-form model. However, the Castro et al. model needs overall budgets for time and expenditure before estimation. We constructed these overall budgets by including all non-work related activities (that included all the four inside activity categories of the current paper as well as the activity categories of "education" and "sleep and relaxing") in the budget computation. The outside Hicksian good along the time duration dimension is considered to be "sleeping and relaxing" (in which there was zero expenditure across all individuals) and the outside Hicksian good along the expenditure dimension is considered to be "education, other weekly cost and savings" (in which there was no or very little time investment among the group of workers, even those who studied).

The performance comparison of our proposed model and the Castro et al. model is presented in Table 6. The top row corresponding to the log-likelihood values at convergence indicate a better fit for our proposed model than Castro et al.'s open-form model for the empirical application. The model predictions in terms of discrete and continuous consumptions are subsequently presented in Table 6. The top panel below the log-likelihood row provides the discrete consumption predictions for joint participation in pairwise activities; our proposed MC-MDCEV model with weighted MAPE of just above 3% is found to outperform the open-form MC-MDCEV which reports a weighted MAPE of close to 13%. The lower panel of the table presents the predictions along the continuous consumption dimension for both the models. The prediction errors for both the models are quite comparable but our closed-form model has a marginally lower weighted MAPE compared to its open-form counterpart. Overall, the performance of our proposed model is slightly better than the model presented by Castro et al. (2012), although both the models perform reasonably well. However, important to emphasize is that our model is computationally much more efficient than the open-form formulation. While the proposed model took a mere 16 minutes to estimate, the open-form simulated method took more than 9 hours to converge. Of course, the convergence time will depend on the capacity of the machine and how one would numerically simulate the open-form probability expression (we used a Halton-sequence method with 100 draws per individual); nevertheless, these numbers give a reasonable sense of the advantage of our closed-form model even for just the two-constraint case. The computational advantage of our model can only be anticipated to be substantially more when going beyond the two-constraint case.

The comparison exercise with the Castro et al. shows promise for our proposed closed-form model with multiple constraints. However, important to keep in mind is that while the Castro et al. model is based on the $NL\gamma$ MDCEV model of Bhat (2008), our proposed model here is based on the RG $L\gamma$ MDCEV with a reverse Gumbel error structure. This itself has some implications that need further careful investigation to assess the relative data fit performances of the different model structures (notwithstanding the computational gains and neat closed-form nature of our proposed model). To keep the presentation manageable here, consider the case of only a single constraint. The linear outside good utility basis of the proposed RG $L\gamma$ MDCEV model (as well as Bhat's (2018) and Bhat et al.'s (2020) $L\gamma$ MDCEV), of course, has the advantage of model estimation when there is no budget information. This is possible because the outside good consumption does not appear in the likelihood function. However, a lingering question may be whether the $L\gamma$-based models perform well regardless of the level of the unobserved budget, because the model formulation itself, while guaranteeing the positivity of all consumed inside goods, does not guarantee the positivity of the outside good consumption. This is in contrast to the traditional $NL\gamma$ MDCEV Model, where the primal feasibility condition of positive consumption of all goods (including the outside good), given a budget, is immediately satisfied based on the complementary slackness KKT first-order stochastic conditions; see footnote eight of Pinjari and Bhat, 2011).

To be sure, the application of the $L\gamma$ MDCEV models, even when budgets are actually observed, does not, by itself, constitute a problem, because the probability expression in the $L\gamma$

MDCEV is based on stochastic KKT conditions (not deterministic KKT conditions, as in most optimization problems). That is, given data, the likelihood expression maximization provides a possible optimal solution (the set of estimated model parameters) that then has to be checked for primal feasibility when translated to consumption predictions to be declared as the true optimal point (primal feasibility here refers to the requirement that the outside good consumption be strictly positive). So, once estimated, the budget condition has to be imposed in the $L\gamma$ MDCEV models during forecasting. This is a case where the model estimation is one step toward optimal consumption determination (but it does not guarantee primal feasibility), which then needs to be vetted through the back-end forecasting process to satisfy primal feasibility and obtain true optimal consumptions. Of course, when the budgets are large (moving toward infinity), there is less need to consider any error truncation operations during forecasting because positivity of the outside good will be near-guaranteed during the estimation step itself. On the other hand, when budgets are tight, there would be more need for truncation operations during forecasting. The empirical effects of different truncation levels (equivalently, different budget tightness levels) on the RG $L\gamma$ MDCEV model (and also the $L\gamma$ MDCEV model) is an important area for future investigation.[3]

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[3] In a recent paper, Palma and Hess (2020) examine this issue of the performance of linear outside good utility models when budget information is available. However, any results from that study should be viewed with caution and is not immediately applicable to the performance of the $L\gamma$ MDCEV models presented here and in Bhat (2018) and Bhat et al. (2020). This is because the stochastic error basis in that paper is different from the one used in the $L\gamma$ MDCEV models. Specifically, similar to several earlier studies in the environmental economics literature (see, for example, von Haefen and Phaneuf, 2003 and Kuriyama et al., 2010), Palma and Hess assume away any random error term in the outside good utility. The senior author of this paper has already expressed reservations in using such a structure, as articulated in Bhat (2008) (please see Section 6.2 of that paper, especially the text surrounding Equation (41) and (42)). Basically, the environmental economics approach assumes that the analyst knows all consumer-related and market-related factors going into the valuation of the outside good, but not for the inside goods. While it is true that only error differences from the outside good error term matter, considering the presence of an outside good error term engenders a correlation among the differenced error terms, which gets ignored in the absence of an outside good error term. The approach used in the $L\gamma$ MDCEV models, in contrast, is conceptually consistent in considering the utilities of all alternatives as being random. Another way to see this is that, in the Environmental Economics approach, if instead of the outside good's utility, the utility of some other inside good is considered deterministic, we obtain different probability expressions and probability values for the same consumption pattern (again, please see Equations 43 through 48 of Bhat, 2008 for a detailed explanation of this point). There also is an important implication for forecasting if an analyst considers the outside good utility to be linear <u>and</u> deterministic, as in Palma and Hess. In particular, the predicted consumptions of the inside goods is based on draws from a log-extreme value error term (that is, based on exponentiating draws from a type-I (maximum) extreme value distribution). This will have a severe effect on inside good consumption predictions, given the already very fat right tail of the (maximum) extreme value error term (which will need substantial truncation during forecasting). In the case of the $L\gamma$ MDCEV models, on the other hand, as just mentioned, it is the difference in error terms between each inside good and the outside good that are at play and determine consumption quantities. These differenced error terms are, in the univariate margins, symmetric and logistically distributed. The symmetric logistic distribution does not have anywhere close to the problem of the fat right tail as the extreme value, though it still has a fatter tail than the normal distribution. Basically, the stochastic distributions play an important role in $L\gamma$ MDCEV models, and some initial investigations suggest that the budget tightness issue is not that much of a concern in our $L\gamma$ MDCEV models as much as that expressed in Palma and Hess. Also, to be noted is that for both the extreme value distribution based on the Gumbel and the reverse Gumbel, the difference of inside good error terms from the outside good error term, in the univariate margin, takes an identical symmetric logistic distribution. In multivariate space, the discrete and continuous consumptions for the inside goods in the $L\gamma$ MDCEV and RG $L\gamma$ MDCEV models can be determined from a standard multivariate logistic distribution,

Also of importance here is that, for the case when budgets are actually observed, the conventional approach to generate data based on specific model parameters, and then examining whether the resulting data recovers model parameters, is not the right approach for assessing model performance. This is because the data generation process, given parameters, will necessarily consider the truncation process to develop data sets. That is, with the same parameters but different budgets, the data generated will be quite different. Thus, given the data generated, it is meaningless to try to see if the generated data (after truncation based on the budget) recovers the model parameters (except in the case when the budget goes toward infinity, when there will be little truncation needed in generating the data with the parameters). The better measure of the ability of $L\gamma$ MDCEV models to handle cases with budgets would be to examine the closeness of predicted consumptions with actual consumptions in the generated data.

6. CONCLUSION

The MDCEV model, as formulated by Bhat (2005, 2008, 2018), is based on the notion that consumers maximize an additively separable non-linear utility function subject to a single linear binding constraint. However, this may not always be the case. For example, in addition to a time-budget, an individual's activity-participation decisions will also generally be bounded by a money-constraint because certain activities incur expenses. Or, an individual's decisions regarding the purchase of consumer goods may be impacted by both a money-constraint and a storage-constraint. In such cases, an individual's multiple discrete-continuous preferences and the multiple-constraint effects get entangled; thus, ignoring any one of the multiple constraints will, in general, lead to inconsistent and biased model estimations, which, in turn, can have adverse consequences on model forecasting and policy evaluations. A limited number of studies in the past have attempted to consider multiple-constraints in a multiple discrete-continuous framework; however, they either employ restrictive utility function forms or are difficult to estimate because of the non-closed form of the resulting probability expressions.

In this paper, we propose, for the first time, a multiple-constraint MDCEV (MC-MDCEV) model that retains the utility structure of the traditional MDCEV model as well as maintains a closed-form expression for the probabilities, regardless of the number of constraints. To do so, we use a type-I extreme value distribution for the error term in its minimization form in the baseline utility preference of each good rather than a maximization form as in Bhat's original MDCEV formulation. In addition, we use a linear form of utility for the baseline preference for the outside goods as in Bhat (2018), rather than the non-linear form of utility adopted in Bhat (2008). Finally, to provide additional model flexibility, we also employ a slightly different version of the utility functional form relative to earlier MDCEV formulations, which nicely integrates with the other two changes indicated above to provide a new closed-form multiple constraint MDCEV model.

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though using different areas of integration and from different locations of the density space. However, the estimated model parameters (from a given data set) will then simply adjust themselves in the $L\gamma$ MDCEV and RG $L\gamma$ MDCEV models, such that the net effect on the performance of predicted consumptions from both these models should not be much different.

The statistical foundation of the proposed model is based on the fact that the difference between a minimal type-I extreme value random variable with scale σ and the weighted sum of the exponential of standardized minimal type-I extreme value random variables (scaled up by σ) leads to an apparently new multivariate distribution that has an elegant and closed-form survival distribution function. This forms the core of the new model, upon which other model specification and identification issues are developed. The new model provides an easy approach for forecasting, as presented in detail in the paper.

A simulation experiment is conducted to assess the ability of our proposed model to recover true underlying parameters, and also to examine the potential consequences of employing a single-constraint MDCEV model when there are in fact two budget considerations. The appropriateness of the models is assessed by the ability to recover parameters from finite samples by generating simulated data sets. In addition, the predictive ability of the different models is compared. The simulation results clearly indicate the preference estimation bias and the predictive inferiority of the single-constrained models when the two-constrained model is the correct one. Next, to demonstrate an application of our proposed MC-MDCEV model, we use the 2012 LISS (Longitudinal Internet Studies for the Social Sciences) Dutch panel data to investigate the determinants of individuals' week-long activity-participation decisions, subject to both a time as well as money constraint.

The proposed MC-MDCEV should prove to be beneficial in a number of multiple discrete-continuous choice contexts with more than one linear constraint. The closed-form probability structure makes the estimation procedure no more difficult than for traditional MDCEV models. Of course, one can attempt to enhance the model specification by relaxing the IID assumption across the error terms of alternatives or allowing for random coefficients (especially when there are alternative-specific variables available); however, doing so will inevitably dismantle the closed-form probability structure, making the estimation process relatively tedious (especially as the number of constraints increase).

The comparison exercise with the Castro et al. (2012) model shows promise for our proposed closed-form model with multiple constraints. However, based on the discussions in Section 5.5, additional research investigations are needed to gain insights into budgetary contexts when the proposed model can be expected to perform better than in other contexts, as well as to compare the relative data fit performances of different model structures for multiple constraints. In any case, we hope that our proposed simple closed-form multi-constraint MDCEV model will contribute to a new direction of application possibilities and to new research in situations where consumers face multiple constraints within a multiple discrete-continuous choice context.

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Appendix A: Integration to Arrive at the Multivariate Survival Distribution Function

To show that the multivariate survival function collapses to a closed-form expression, we start off with Equation (12) in text, which is,

$$=\text{Prob}\left[\varepsilon_3 > w_3 + \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}), \varepsilon_4 > w_4 + \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}), ..., \varepsilon_K > w_K + \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2})\right]$$

$$= \int_{\varepsilon_1=-\infty}^{+\infty} \int_{\varepsilon_2=-\infty}^{+\infty} \prod_{k=3}^{K} e^{-e^{\left[\frac{w_k + \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2})}{\sigma}\right]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_1 d\varepsilon_2. \qquad \text{(A.1)}$$

The integrand above can be simplified as follows:

$$\prod_{k=3}^{K} e^{-e^{\left[\frac{w_k + \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2})}{\sigma}\right]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2}$$

$$= e^{-\sum_{k=3}^{K} e^{\left[\frac{w_k + \sigma \ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2})}{\sigma}\right]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2}$$

$$= e^{-\sum_{k=3}^{K} (a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}) e^{(w_k/\sigma)}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2}$$

$$= e^{-\sum_{k=3}^{K} a_{1k}e^{\varepsilon_1} . e^{(w_k/\sigma)}} e^{-\sum_{k=3}^{K} a_{2k}e^{\varepsilon_2} . e^{(w_k/\sigma)}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2}$$

$$= e^{-[e^{\varepsilon_1}(1 + \sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})]} e^{-[e^{\varepsilon_2}(1 + \sum_{k=3}^{K} a_{2k}e^{(w_k/\sigma)})]} e^{\varepsilon_1} e^{\varepsilon_2}$$

Given that the random variables ε_1 and ε_2 are independent, the integration in Equation (A.1) can be re-written as,

$$\int_{\varepsilon_1=-\infty}^{+\infty} e^{-[e^{\varepsilon_1}(1 + \sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})]} e^{\varepsilon_1} d\varepsilon_1 \int_{\varepsilon_2=-\infty}^{+\infty} e^{-[e^{\varepsilon_2}(1 + \sum_{k=3}^{K} a_{2k}e^{(w_k/\sigma)})]} e^{\varepsilon_2} d\varepsilon_2 = I_{\varepsilon_1} . I_{\varepsilon_2} .$$

To evaluate the first integration with respect to ε_1, let $s = e^{\varepsilon_1}(1 + \sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})$.

Therefore, $ds = e^{\varepsilon_1}(1 + \sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)}) d\varepsilon_1$.

The first integration then takes the following form (ignoring the limits for the moment),

$$I_{\varepsilon_1} = \int \frac{e^{-s} ds}{(1 + \sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})}.$$

This is a straightforward integration to solve, which results in

$$I_{\varepsilon_1} = -\frac{e^{-s}}{(1 + \sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})}.$$

Now, evaluating the limits, we have,

$$I_{\varepsilon_1} = -\left.\frac{e^{-e^{\varepsilon_1}(1+\sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})}}{(1+\sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})}\right|_{\varepsilon_1=-\infty}^{\varepsilon_1=+\infty} = -\frac{1}{(1+\sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})}[0-1].$$

Therefore,

$$I_{\varepsilon_1} = \frac{1}{(1+\sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})}.$$

Similarly, following the exact same approach,

$$I_{\varepsilon_2} = \frac{1}{(1+\sum_{k=3}^{K} a_{2k}e^{(w_k/\sigma)})}.$$

Therefore, the integration in Equation (A.1) results into the following closed-form expression.

$$=\text{Prob}\left[\varepsilon_3 > w_3 + \sigma\ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}), \varepsilon_4 > w_4 + \sigma\ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2}), \ldots, \varepsilon_K > w_K + \sigma\ln(a_{1k}e^{\varepsilon_1} + a_{2k}e^{\varepsilon_2})\right]$$

$$= \int_{\varepsilon_1=-\infty}^{+\infty}\int_{\varepsilon_2=-\infty}^{+\infty} \prod_{k=3}^{K} e^{-e^{\left[\frac{w_k+\sigma\ln(a_{1k}e^{\varepsilon_1}+a_{2k}e^{\varepsilon_2})}{\sigma}\right]}} e^{-e^{\varepsilon_1}} e^{\varepsilon_1} e^{-e^{\varepsilon_2}} e^{\varepsilon_2} d\varepsilon_1 d\varepsilon_2$$

$$= \frac{1}{(1+\sum_{k=3}^{K} a_{1k}e^{(w_k/\sigma)})(1+\sum_{k=3}^{K} a_{2k}e^{(w_k/\sigma)})}$$

This is exactly Equation (11) in the text.

Appendix B: Proof for the Partial Derivative of the Survival Function Formulation

This proof is best illustrated with the help of an example since the formulation is based on the pattern recognition of the partial derivative terms.

For instance, let us take a case of two-constraints and 4 inside goods (i.e., *k*=3,4,5, and 6, since *k*=1 and 2 are the two outside goods corresponding to the two-constraints).

The survival function is given by, $\boldsymbol{S}_\eta(w_3, w_4, w_5, w_6) = \dfrac{1}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)}$

Now, suppose that we are interested in determining the third order partial derivative of this survival function with respect to w_3, w_4, and w_5. This third order derivative takes the following form.

$$\frac{\partial^3 \boldsymbol{S}_\eta(w_3, w_4, w_5, w_6)}{\partial w_3.\partial w_4.\partial w_5} = -\frac{6a_{13}a_{14}a_{15}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^4\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)\sigma^3} - \frac{2a_{13}a_{14}a_{25}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^3\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^2\sigma^3}$$

$$-\frac{2a_{13}a_{24}a_{15}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^3\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^2\sigma^3} - \frac{2a_{23}a_{14}a_{15}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^3\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^2\sigma^3}$$

$$-\frac{2a_{13}a_{24}a_{25}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^2\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^3\sigma^3} - \frac{2a_{23}a_{14}a_{25}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^2\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^3\sigma^3}$$

$$-\frac{2a_{23}a_{24}a_{15}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)^2\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^3\sigma^3} - \frac{2a_{23}a_{24}a_{25}.e^{\frac{w_1}{\sigma}+\frac{w_2}{\sigma}+\frac{w_3}{\sigma}}}{\left(1+\sum_{k=3}^{6} a_{1k}e^{w_k/\sigma}\right)\left(1+\sum_{k=3}^{6} a_{2k}e^{w_k/\sigma}\right)^4\sigma^3}$$

In our method, we define the following matrices for determining the third (n=3) order derivative.

$$\boldsymbol{A}_3 = \begin{bmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{bmatrix},\ \boldsymbol{B}_3 = \begin{bmatrix} a_{13} & a_{14} & a_{15} \\ a_{13} & a_{14} & a_{25} \\ a_{13} & a_{24} & a_{15} \\ a_{13} & a_{24} & a_{25} \\ a_{23} & a_{14} & a_{15} \\ a_{23} & a_{14} & a_{25} \\ a_{23} & a_{24} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{bmatrix} \text{ and } \boldsymbol{C}_3 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 1 \\ 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

With the above defined matrices for the case in hand, the derivative terms for the survival function can be succinctly written as,

$$\frac{\partial^3 \boldsymbol{S}_\eta(w_3, w_4, w_5, w_6)}{\partial w_3 . \partial w_4 . \partial w_5} = (-1)^3 \frac{\exp\left(\sum_{i=3}^{3+2} \frac{w_i}{\sigma}\right)}{\sigma^3} \sum_{v=1}^{2^3} \left(\frac{\prod_{r=1}^{2} (\boldsymbol{C}_{3,vr}!) \prod_{g=1}^{3} \boldsymbol{B}_{n,vg}}{\prod_{r=1}^{2} \left[1 + \sum_{k=3}^{6} a_{rk} \exp\left(\frac{w_k}{\sigma}\right) \right]^{\boldsymbol{C}_{3,vr}+1}} \right)$$

Expanding this equation will yield all the partial derivative terms obtained earlier.

For any n^{th} order partial derivative and with respect to any combination of the inside goods, the above formulation can be generalized as below,

$$\frac{\partial^n \boldsymbol{S}_\eta(w_3, w_4, \ldots, w_K)}{\partial w_3 . \partial w_4 \ldots \partial w_{n+2}} = (-1)^n \frac{\exp\left(\sum_{i=3}^{n+2} \frac{w_i}{\sigma}\right)}{\sigma^n} \sum_{v=1}^{2^n} \left(\frac{\prod_{r=1}^{2} (\boldsymbol{C}_{n,vr}!) \prod_{g=1}^{n} \boldsymbol{B}_{n,vg}}{\prod_{r=1}^{2} \left[1 + \sum_{k=3}^{K} a_{rk} \exp\left(\frac{w_k}{\sigma}\right) \right]^{\boldsymbol{C}_{n,vr}+1}} \right)$$

This is exactly the expression in Equation (17) in the text.

Appendix C: Adjustments to Probability Expressions in the Situation where Some of the Goods are Always Consumed

For all the inside goods 'L' that are always consumed, $V_l = \ln x_l^* - \boldsymbol{\beta}'\boldsymbol{z}_l \;\; (l = R+1, R+2, ..., L)$.

In the situation where 'L' goods are always consumed and some of the other inside goods 'M' are consumed, Equation (23) in the text would take the following form,

$$P\left(x_{R+1}^*, \; ..., \; x_{R+L}^*, x_{R+L+1}^*, ... x_{R+L+M}^*, \; 0, \; 0, \; ..., \; 0 \right) =$$

$$= |J| \left[\begin{array}{l} \dfrac{\exp\left(\sum_{k=R+1}^{R+L+M} \dfrac{V_k}{\sigma} \right)}{\sigma^{M+L}} \sum_{v=1}^{R^{L+M}} \left(\dfrac{\prod_{r=1}^{R} (\boldsymbol{C}_{L+M,vr}\,!) \prod_{g=1}^{L+M} \boldsymbol{B}_{L+M,vg}}{\prod_{r=1}^{R} \left[1 + \sum_{k=R+1}^{R+L+M} a_{rk} \exp\left(\dfrac{V_k}{\sigma} \right) \right]^{\boldsymbol{C}_{L+M,vr}+1}} \right) + \\ \sum_{D \subset \{L+M+R+1, L+M+R+2, ..., K\}, |D| \geq 1} (-1)^{|D|} \dfrac{\exp\left(\sum_{i=R+1}^{R+L+M} \dfrac{V_i}{\sigma} \right)}{\sigma^{M+L}} \sum_{v=1}^{R^{L+M}} \left(\dfrac{\prod_{r=1}^{R} (\boldsymbol{C}_{L+M,vr}\,!) \prod_{g=1}^{L+M} \boldsymbol{B}_{L+M,vg}}{\prod_{r=1}^{R} \left[1 + \sum_{k=R+1}^{R+L+M} a_{rk} \exp\left(\dfrac{V_k}{\sigma} \right) + \sum_{k=R+L+M+1}^{|D|+R+L+M} a_{rk} \exp\left(\dfrac{V_{k0}}{\sigma} \right) \right]^{\boldsymbol{C}_{L+M,vr}+1}} \right) \end{array} \right],$$

$$|J| = \left[\prod_{i=R+1}^{L+M+R} f_i \right], \;\; f_i = \left(\frac{1}{x_i^* + \gamma_i} \right) \qquad \text{(C1)}$$

When all the inside goods are consumed, there is no change in the formulation because the 'L' goods will automatically be accounted for, so the probability expression remains the same as the following.

$$P\left(x_{R+1}^*, \; ..., \; x_K^* \right) = |J| \frac{\exp\left(\sum_{k=R+1}^{K} \dfrac{V_k}{\sigma} \right)}{\sigma^{K-R}} \sum_{v=1}^{R^{K-R}} \left(\frac{\prod_{r=1}^{R} (\boldsymbol{C}_{K-R,vh}\,!) \prod_{g=1}^{K-R} \boldsymbol{B}_{K-R,vg}}{\prod_{r=1}^{R} \left[1 + \sum_{k=3}^{K} a_{rk} \exp\left(\dfrac{V_k}{\sigma} \right) \right]^{\boldsymbol{C}_{K-R,vr}+1}} \right) \qquad \text{(C2)}$$

When no other inside goods other than the 'L' goods that are always consumed are consumed, then the probability expression takes the following form.

$$P\left(x^*_{R+1},\ ...,\ x^*_{R+L},\ 0,\ 0,\ ...,\ 0\right) =$$

$$=|J|\left[\begin{array}{c} \dfrac{\exp\left(\sum_{k=R+1}^{L+R}\dfrac{V_k}{\sigma}\right)}{\sigma^L}\sum_{v=1}^{R^L}\left(\dfrac{\prod_{r=1}^{R}(\boldsymbol{C}_{L,vr}\,!)\prod_{g=1}^{L}\boldsymbol{B}_{L,vg}}{\prod_{r=1}^{R}\left[1+\sum_{k=R+1}^{L+R}a_{rk}\exp\left(\dfrac{V_k}{\sigma}\right)\right]^{\boldsymbol{C}_{L,vr}+1}}\right)+ \\ \sum_{D\subset\{L+R+1,L+R+2,...,L\},|D|\geq 1}(-1)^{|D|}\dfrac{\exp\left(\sum_{i=R+1}^{L+R}\dfrac{V_i}{\sigma}\right)}{\sigma^L}\sum_{v=1}^{R^L}\left(\dfrac{\prod_{r=1}^{R}(\boldsymbol{C}_{L,vr}\,!)\prod_{g=1}^{L}\boldsymbol{B}_{L,vg}}{\prod_{r=1}^{R}\left[1+\sum_{k=R+1}^{L+R}a_{rk}\exp\left(\dfrac{V_k}{\sigma}\right)+\sum_{k=L+R+1}^{|D|+L+R}a_{rk}\exp\left(\dfrac{V_{k0}}{\sigma}\right)\right]^{\boldsymbol{C}_{L,vr}+1}}\right) \end{array}\right], \quad \text{(C3)}$$

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Table 1. Simulation experiment results

Parameter	Metrics	Two-constraints (both constraints active)	Single constraint (only first constraint active)	Single constraint (only second constraint active)
β_1	True value	1.2500	1.2500	1.2500
	Estimate	1.2506	0.4666	0.5505
	APB (%)	0.0448	62.6754	55.9607
	FSSE	0.0436	0.0388	0.0404
	ASE	0.0435	0.0393	0.0411
	APBASE (%)	0.4285	1.3607	1.8936
β_2	True value	1.0000	1.0000	1.0000
	Estimate	0.9996	0.9937	0.8737
	APB (%)	0.0423	0.6295	12.6348
	FSSE	0.0271	0.0265	0.0266
	ASE	0.0276	0.0253	0.0238
	APBASE (%)	1.6402	4.8593	10.4951
γ_3	True value	2.7183	2.7183	2.7183
	Estimate	2.7224	1.3345	3.4059
	APB (%)	0.1523	50.9064	25.2970
	FSSE	0.0591	0.0661	0.0568
	ASE	0.0601	0.0651	0.0558
	APBASE (%)	1.6532	1.4037	1.7262
γ_4	True value	2.7183	2.7183	2.7183
	Estimate	2.7170	3.5689	1.2628
	APB (%)	0.0465	31.2914	53.5430
	FSSE	0.0633	0.0600	0.0717
	ASE	0.0619	0.0569	0.0671
	APBASE (%)	2.2397	5.1256	6.3296
σ	True value	1.0000	1.0000	1.0000
	Estimate	0.9990	0.8919	0.9291
	APB (%)	0.1000	10.8126	7.0908
	FSSE	0.0230	0.0224	0.0216
	ASE	0.0237	0.0233	0.0229
	APBASE (%)	2.6191	3.9395	6.0611

Table 2. Aggregate measures of fit for the simulation experiment

Discrete choice consumption: share of observations with consumption in outside goods and…	**Actual share**	**Two-constraint model prediction**	**One-constraint model prediction: first constraint**	**One-constraint model prediction: second constraint**
No inside good consumption	857	876	1337	1252
First inside good consumption only	479	477	726	205
Second inside good consumption only	788	791	418	1086
First and second inside good consumptions only	876	856	519	457
Weighted Mean Absolute Percentage Error	-	1.47	48.47	46.20
Weighted Mean Absolute Percentage Error for number of inside alternatives picked	-	1.33	32.00	27.93
Continuous consumption (conditional on positive discrete choice consumption)	**Observed**	**Two-constraint model prediction**	**One-constraint model prediction: first constraint**	**One-constraint model prediction: second constraint**
First inside good consumption	20.76	20.89	5.43	13.61
Second inside good consumption	30.90	29.59	19.33	6.44
Weighted Mean Absolute Percentage Error		2.78	52.07	61.89

Table 3. Descriptive statistics for the time-use application (N=1193 observations)

Activity	Participation (%)	Duration (hours/week)				Average unit price (Euros/hour)
		Mean	St. Dev.	Min.	Max.	
Household chores	92.8	25.6	15.7	1.0	108.0	6.05
Leisure	94.3	31.9	16.1	1.0	102.0	1.37
Personal business	93.2	7.5	8.1	0.2	81.3	0.00
Social	42.5	11.7	12.5	0.3	71.0	0.00

Table 4. **Estimation results for the MC-MDCEV model**

Variables	Coefficient estimates (t-stats)			
	Household chores	Leisure	Personal business	Social
Individual characteristics				
Female	0.349 (8.44)	-	-	-
Age (Base: More than 45 years)				
Below 30 years	-0.397 (-7.54)	-	-0.420 (-7.30)	-0.174 (-2.51)
30-45 years	-	-	-0.355 (-9.32)	-0.157 (-3.35)
Household sociodemographic				
Household size	-	-	0.052 (2.81)	0.065 (3.15)
Number of child(ren)	0.074 (1.98)	-0.107 (-4.88)	-	-
Weekly household income (Base: Greater than 1250 Euros)				
Less than 500 Euros	-0.297 (-3.79)	-0.232 (-4.83)	-	0.165 (3.26)
500-749 Euros	-0.213 (-2.90)	-0.163 (-3.77)	-	0.085 (1.89)
750-999 Euros	-0.192 (-1.87)	-	-	-
1000-1250 Euros	-0.162 (-2.01)	-	-	-
Baseline preference constant	2.170 (20.03)	1.882 (20.85)	1.062 (13.09)	-0.201 (-2.97)
Satiation effects				
Female	-	-0.084 (-2.02)	-	-
Age (Base: More than 45 years)				
Below 30 years	-	-	-	0.622 (3.79)
30-45 years	-	-	-	0.418 (3.99)
Household size	0.060 (1.46)	-	-	-
Satiation constant	1.490 (11.31)	2.778 (20.40)	1.199 (12.94)	2.010 (35.82)
Scale parameter				0.439 (35.21)

Table 5. Aggregate measures of fit for the model application

Discrete choice consumption: share of observations with consumption in outside goods and joint participation in…	Actual share	Two-constraint model prediction	One-constraint model prediction: time constraint only	One-constraint model prediction: money constraint only
Household chores and Leisure	1026	1035	1058	1040
Household chores and Personal business	1003	991	1004	943
Household chores and Social	449	493	576	540
Leisure and Personal business	1052	1040	1049	989
Leisure and Social	463	507	590	558
Personal business and social	478	508	574	527
Weighted Mean Absolute Percentage Error	-	3.37%	8.63%	8.32%
Weighted Mean Absolute Percentage Error for number of inside alternatives picked	-	13.75%	29.33%	27.32%
Continuous consumption (conditional on positive discrete choice consumption)	**Observed**	**Two-constraint model prediction**	**One-constraint model prediction: time constraint only**	**One-constraint model prediction: money constraint only**
Household chores	25.60	18.70	12.44	10.89
Leisure	31.90	27.77	18.01	12.22
Personal business	7.50	4.63	4.50	6.35
Social	11.70	15.30	15.17	18.95
Weighted Mean Absolute Percentage Error	-	22.83%	48.70%	55.79%

Table 6. Comparison of new closed-form MC-MDCEV vs. Castro et al.'s (2012) open-form MC-MDCEV

		Two-constraint model (Proposed closed-form model)	Two-constraint model (Castro et al.'s 2012, open-form model)
Log-likelihood values at convergence		**-15693.2**	**-16295.8**
Discrete choice consumption: share of observations with consumption in outside goods and joint participation in…	**Actual share**	**Two-constraint model prediction (Proposed closed-form model)**	**Two-constraint model prediction (Castro's 2012, open-form)**
Household chores and Leisure	1026	1035	819
Household chores and Personal business	1003	991	788
Household chores and Social	449	493	400
Leisure and Personal business	1052	1040	969
Leisure and Social	463	507	446
Personal business and Social	478	508	473
Weighted Mean Absolute Percentage Error	-	**3.37%**	**12.88%**
Continuous consumption (conditional on positive discrete choice consumption)	**Observed**	**Two-constraint model prediction (Proposed closed-form model)**	**Two-constraint model prediction (Castro's 2012, open-form)**
Household chores	25.60	18.70	34.25
Leisure	31.90	27.77	37.65
Personal business	7.50	4.63	8.57
Social	11.70	15.30	14.99
Weighted Mean Absolute Percentage Error	-	**22.83%**	**24.45%**