BASIC OPTIMIZATION MODELS FOR WATER AND ENERGY MANAGEMENT

By
Daene C. McKinney
And
Andre G. Savitsky

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Acknowledgements

About the Authors

Daene McKinney is a Professor in the Department of Civil, Architectural and Environmental Engineering at The University of Texas at Austin. During 1998-2000, he was Team Leader and Senior Environmental Policy Advisor for the USAID Environmental Policy and Institutions for Central Asia (EPIC) Program in Almaty, Kazakhastan. During that time the work resulting in the current document was initiated. Dr. McKinney earned his Ph.D. in Civil Engineering with a major in Water Resources Engineering at Cornell University in 1990. After working for the US EPA for one year as an environmental engineer and hydrogeologist, he joined the faculty at the University of Texas at Austin. Dr. McKinney has served as Editor of the Journal of Water Resource Planning and Management and Associate Editor of Operations Research. He has served as the Chair of the ASCE Water Resource Systems Committee, and as a member of the Board of Directors and the International Committee of the Universities Council on Water Resources (UCOWR). Dr. McKinney teaches undergraduate courses in Fluid Mechanics, Numerical Methods, Hydrology, and graduate courses in Water Resources Planning and Management, Transboundary Water Resources and Numerical Methods for Environmental Engineers. Dr. McKinney's research interests include developing and applying numerical methods for simulation, optimization, and uncertainty analysis of water resources management problems, and the development of laboratory and field experimental techniques for the characterization and remediation of aquifer and groundwater contamination.

Andrey Savitsky's has been a Lead Specialist in the Water Resources Department of the Scientific Information Center of the Interstate Coordination Water Commission (SIC ICWC). During 1998 – 2000 he was a Water Resources Advisor for the USAID Environmental Policy and Institutions for Central Asia (EPIC) Program in Tashkent, Uzbekistan. He was responsible for mathematical modeling of optimal water and energy resources management within the EPIC Program. In 1986, he earned his degree of Candidate of Technical Sciences with a major in Hydraulics and Hydrological Engineering on the subject of Water and Salt Regimes of Reservoirs of the Central Asia. At the beginning of his scientific career he worked in the Hydroaerodynamics Laboratory at the Institute of Mechanics and Antiseismic Constructions for the Academy of Sciences of the Uzbek Soviet Socialist Republic. He was involved in studying dynamics of multiphase environments in industry (heat exchange systems in reactors; manufacturing of experimental samples for cumulative, non-stationary, high temperature flows of gas; mathematical modeling of multiphase interpenetrating motions of continuum). Then he joined the Scientific and Research Institute of Irrigation where he worked for a long period of time. The majority of his works and achievements are connected with mathematical simulation of dynamic and cinematic processes in rivers and reservoirs, dams in the process of demolition, in Soil – Plant – Atmosphere systems in agricultural cycles. In SIC ICWC, Savitsky's work is connected with optimization of water resources and development software programs for users. Savitsky's research interests include developing and applying mathematical methods for simulation, optimization of management of natural and artificial phenomena.

PART 1: RESOURCE MANAGEMENT MODELS

1. Introduction

1.1 Systems Approach

Planning, management and design are a critical element of sustainable economic development and expansion. In the process of planning and design there is a need to critically analyze the true economic costs, benefits and environmental consequences of projects. A lack of this analysis can often lead to a level of design quality which falls far short of optimal with respect to the utilization of scarce economic and natural resources and will not improve the ecological balance of systems in general.

More recently, population shifts, industrial and political changes have led political leaders and planners to the conclusion that unlimited expansion and development are no longer the primary objectives in social and economic systems planning and design. The single-objective, single-purpose, single-facility project approach to solving problems that was so common in the past is unacceptable today and has been replaced by multi-objective, multi-purpose, multi-facility solutions at a large scale which must be not only technically feasible but socially, environmentally, economically, and politically feasible as well. In most planning situations it is hard to see how all of these disparate components can be combined into system designs which meet prescribed and sometimes conflicting objectives and constraints imposed on a project.

Systems analysis can aid in identifying those likely situations where a minimum investment of funds and energies will produce maximum gains in terms of resource allocations, economic development and environmental welfare. Generally speaking, systems analysis is the art and science of disassembling complex phenomena into smaller, isolated, more readily understood, subsystems and analyzing the interactions between the subsystems and between the subsystems and the larger environment (Churchman, 1968). This is a natural human process: examining a complex process by directing attention to the component parts and the relationships between the components. Using systems analysis we can focus on the functioning of the components under the various conditions to which the system may be subjected. In many situations, by focusing on the relationships and interactions between the components of complex systems, systems analysis can provide a means of sorting through the myriad of possible solutions to a problem and narrowing the search to a few potentially optimal ones in addition to determining and illustrating the consequences of these alternatives and the tradeoffs between conflicting objectives. The central method used in water resource systems analysis is to couple the descriptions of physical and socioeconomic systems through the use of mathematical models.

A system is a collection of components and their interrelationships forming an entity (e.g., a river basin) which is acted upon by external forces, influences or inputs (precipitation) and produces a specific effect or output (streamflow). That is, a system is a set of objects which transforms an input into an output, the exact output produced depending on certain system properties or parameters (e.g.,

soil types, vegetation, topography). This transformation depends upon the parameters of the system and the design policies imposed on it. Figure 1.1.1 displays a systems representation of a general groundwater system and illustrates the ideas of inputs, outputs, policy variables, and parameters.

Systems analysis involves the construction and linkage of mathematical models of the physical and economic subsystems associated with resource allocation systems. The purpose of constructing these models is to aid engineers, planners and decision makers in identifying and evaluating alternative designs and to determine which ones meet project objectives in an efficient manner. These mathematical models are able to predict a system's response to different design alternatives and conditions. The models are a set of mathematical expressions (partial or ordinary differential or algebraic equations) describing the physical, biological, chemical, and economic processes which take place in the system.

Most systems models are based on statements of basic conservation laws (mass, energy, and momentum), but they can also be empirical or statistical. Systems analysis models are generally broken down into two categories: *simulation* models and *optimization* models.

Policies
Pumping and injection locations, rates and concentrations
Artificial recharge

Groundwater System

Inputs
Subsurface inflows
Natural recharge
Precipitation
Irrigation return flows

Outputs
Subsurfae outflows
Base flow
Evapotranspiration
Phreatophytes

Parameters
Storage coefficient
Porosity
Transmissivity
Leakage factor
Dispersivity
Kinetic reaction rates
Adsorption parameters
Ion exchange parameters

Figure 1.1.1. General diagram of a groundwater system showing inputs, outputs, parameters, and policies. (Adapted from Willis and Yeh, 1987, Fig. 1.5.)

1.2 Simulation and Optimization Models

Simulation models are used to predict a system's response to a given design configuration with great accuracy and detail, and to identify the probable costs, benefits, and impacts of a project. That is,

the simulation model predicts the outcome of a single, specified set of design or policy variables. However, the space of possible design and policy variable values is, in general, infinite. Simulation models, while important tools for managing systems, do not identify or narrow the search for optimal policies or designs for a problem; they provide only localized information regarding the response of the system to one particular design alternative at a time. Separate simulation model runs are required for each design or policy alternative considered. In many situations the number of alternative designs is sufficiently large to preclude simulating each alternative and some other method is normally used to narrow the field of search.

Optimization models provide a means of reducing the number of alternatives which need to be simulated in detail, i.e., screening them. These models search the space of possible design variable values and identify an optimal design and/or operating policy for a given system design objective and set of constraints. The sensitivity of the optimal solution to changes in the model parameters can be readily determined and tradeoffs between several conflicting objectives can also be calculated with most optimization models. These models are usually extensions of simulation models and include as unknowns the design or operating variables (decision variables) of each alternative. These models include relationships which describe the state variables and costs or benefits of each alternative as a function of the decision variables. Constraints are also included in the models to restrict the values of the design or state variables. Optimization models are generally used for preliminary evaluation or screening of alternatives and to identify important data needs prior to extensive data collection and simulation modeling activities.

1.3 Model Building Process

The process of developing the mathematical simulation and optimization models which represent the system under investigation consists of several steps. Referring to Figure 2, the first step, problem identification, is to identify the important elements of the system which pertain to the problem at hand and the interactions between the components. That is, a general outline and purpose of the model must be established. The analyst will need to identify the appropriate type of model for the system and the degree of accuracy needed given the time and resources available for modeling. Generally the simplest model with the least number of parameters which will produce reliable results in the time available is preferred. In the next step, conceptualization and development, the mathematical description of the relationships identified previously are established. In this step appropriate computational techniques are also determined and implemented for the problem. Calibration of the mathematical model is then performed to determine reliable estimates of the model parameters. In this procedure the model outputs are compared with actual historical or measured outputs of the system and the parameters are adjusted until the model predicted and the measured values agree. Then a model verification exercise is carried out in which an independent set of input data is used in the model and the predicted results are compared with measured outputs and if they are found to agree the model is considered to be verified and ready for use in simulation or optimization.
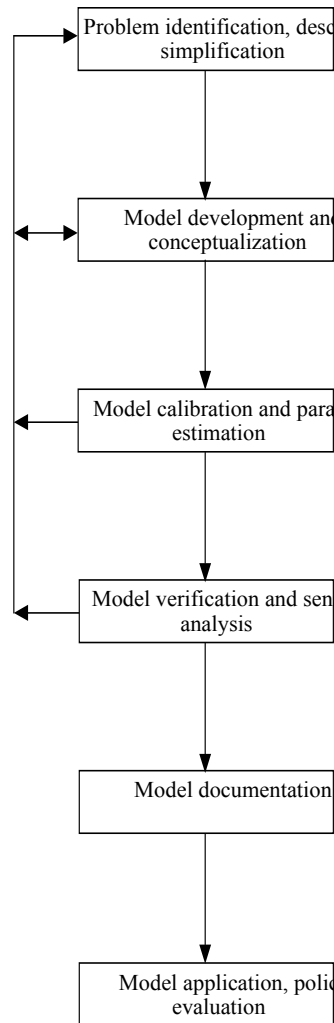
Figure 1.2.1. General diagram of the steps in the model building process. (Adapted from Loucks et al., 1981, Fig. 9.1.)

1.4 Book Organization

This book is intended for use by beginning optimization modelers in science and engineering; and engineers and scientists performing research on the optimal use of resources for water, energy and agricultural problems. Examples of optimization models from several technical areas of interest are covered in Part 1: general equations solving, water resources management, agricultural management, canal design, power system design and management, heat transfer, and fluid flow. These models are presented for the purpose of introducing the reader to the possibilities of modeling these systems using optimization techniques. There are certainly many additional application areas and techniques that could be covered, but the examples selected cover a wide enough range to introduce the reader to the topic. Part 2 of the book covers many of the basics of the modeling language used in this book. That language is the General Algebraic Modeling System (GAMS) a

high-level modeling system for mathematical programming problems (Brooke et al., 1998). This section of the book is intended to provide the reader with sufficient information to construct simple models without having to read through the full language documentation.

In our work we have found that the modeling languages are more useful than the other types of modeling packages such as LINGO (Schrage, 1999) since these products do not allow the easy construction of general modeling structures such as those needed for solving differential equations. There are other high level modeling languages available that can be used for the same purpose, including AIMMS (Bisschop and Roelofs, 2001) and AMPL (Fourer et al., 1999).

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Exercises

1. Discuss the need for benefit – cost analysis in planning and design of water resource projects. What are some consequences of failing to provide adequate analysis in the planning process?
2. What is "systems analysis" and how can it aid in the planning and design of water resources plrjects?
3. What is the difference between "simulation" and "optimization modeling? Give an example when it might be more appropriate to use one rather than the other.
4. In the model building process, why is it important to have independent data sets for the calibration and verification of a model?
5. What is sensitivity analysis and how would you use the results of such an analysis to guide data collection efforts?

2. Economic Analysis of Water Resources

2.1 Cost – Benefit Analysis

2.1.1 Choosing Among Feasible Alternatives

Economic analysis, or the understanding and prediction of decision making under conditions of resource scarcity, plays a major role in the planning, design and management of sustainable water resource systems. Allocation of water among competing uses to obtain an optimum value in terms of market or welfare measures is one of the main problems of water resources planners. Price theory is very relevant where markets are operating efficiently, whereas welfare economics seeks to maximize human welfare in situations where desirable social gains and undesirable social costs are not fully accounted for in a profit maximizing, market economy (North, 1985). Price theory and welfare economics tend to focus on static analyses of projects, whereas, financial analysis considers the time value of investments and decisions. In this section, we will consider some aspects of financial analysis.

Choice is governed by economic and financial feasibility and acceptability with respect to social and environmental impacts. Here we want to consider investment analysis which serves as a guide for allocating resources between present and future consumption. The process consists of:

> Identifying alternatives to be considered;
> Predicting the consequences resulting from these alternatives;
> Converting the consequences into some commensurable units (e.g., $'s); and
> Choosing among the alternatives

One project may produce one type of output, while another project produces another kind of output. In order to compare the projects and make investment decisions, common units must be used to express the outputs of each alternative before any comparison can be made. Monetary units are the most commonly used units.

Some projects will provide outputs in the near future and other projects may delay outputs for an appreciable time or distribute them uniformly over the project lifetime. Outputs today do not have the same value as outputs tomorrow and the following observations are appropriate:

- Investors often prefer early return on investments since it provides them with more flexibility in making future investment decisions;
- Benefits and costs at different times should not be directly compared or combined, since they are not in common units;
- Future benefits and costs must be multiplied by a factor that becomes progressively smaller for times further into the future. This multiplicative factor is called the <u>discount rate</u> and it has a great impact on the alternative selected;

- Future benefits and costs are given more weight with lower discount rates and less weight with higher discount rates; and
- Committing resources to one project may deny the possibility of investing in some other project. This brings up the question of <u>opportunity costs</u>, or what must be foregone in order to undertake some alternative.

One should always keep in mind that different points of view may be adopted in analyzing alternatives, e.g., project sponsors; people in a specific area or region; and an entire nation. Each point of view may value benefits and costs differently and even define items differently (i.e., one person's cost may be another person's benefit.).

2.1.2 Cost-effectiveness analysis

A program is cost-effective if, on the basis of *life cycle cost* analysis of competing alternatives, it is determined to have the lowest costs expressed in present value terms for a given amount of benefits. Cost-effectiveness analysis is appropriate whenever it is unnecessary or impractical to consider the dollar value of the benefits provided by the alternatives under consideration. This is the case whenever (i) each alternative has the same annual benefits expressed in monetary terms; or (ii) each alternative has the same annual affects, but dollar values cannot be assigned to their benefits. Analysis of alternative defense systems often falls in this category (OMB, 1992).

2.1.3 Benefit-cost analysis

Financial benefit-cost analysis evaluates the effect of a project on the water sector or utility by providing projected balance, income, and sources and applications of fund statements (ADB, 2005). This can be distinguished from economic benefit-cost analysis which evaluates the project from the viewpoint of the entire economy. In financial benefit-cost analysis, which we will consider here, the unit of analysis is the *project* and not the entire economy nor the entire water sector or utility. Therefore, it focuses on the *additional* financial benefits and costs to the water sector, attributable to the project.

Benefit – Cost Analysis is a systematic quantitative method of assessing the desirability of government projects or policies when it is important to take a long view of future effects and a broad view of possible side-effects (OMB, 1992).

- Both costs and benefits of a project must be measured and expressed in commensurable units;
- It is the main analytical tool used to evaluate water resource and environmental decisions;
- Benefits of an alternative are estimated and compared with the total costs that society would bear if that action were undertaken; and
- Viewpoint is important - some groups are only concerned with benefits, others are concerned only with costs

In benefit – cost analyses, any costs and benefits that are unaffected by which alternative is selected should be neglected. That is, the differences between alternatives need only be considered.

Estimates of benefits and costs are typically uncertain because of imprecision in both underlying data and modeling assumptions. Because such uncertainty is basic to many analyses, its effects should be analyzed and reported (OMB, 1992). Uncertainty may exist in: objectives, constraints, public response, technological change, or extreme events and recurrence.

2.1.3.1 Interest Rate Calculations

Consider investing $100 at a rate of 5%. At the end of one year the value of the investment would be:

$$F_1 = \$100(1+0.05) = \$105$$

Similarly, at the end of 2 years, the value would be

$$F_2 = \$105(1+0.05) = \$100(1+0.05)^2 = \$110.25$$

or, generalizing this, we have at the end of t years that an initial investment of P would be worth

$$F_t = P(1+i)^t$$

Put another way, a single payment of F_t available t years in the future is worth

$$P = \frac{F_t}{(1+i)^t}$$

and a series of (not necessarily equal) payments F_t available t years in the future is worth

$$P = \sum_{t=1}^{T} \frac{F_t}{(1+i)^t}$$

Example. Assume that an initial investment of $50 disbursement at t=0 is required and that this will result in $200 receipt at t = 1 year and $150 receipt at t = 2 years with an interest rate of 7% annually (see figure).

$200 $150

0   1   2

$50
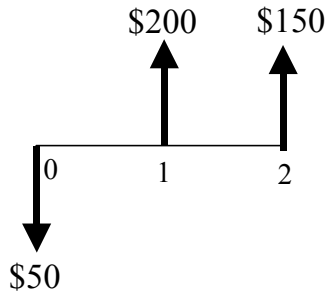
What is the Present Value, P, of the given cash flow?

$$P = C + \sum_t F_t(1+i)^{-t} = C + F_1(1+i)^{-1} + F_2(1+i)^{-2}$$

P = (   ) + (   ) + (   )

P = (   ) $

What is the Future Value of the given cash flow?

$$F = C(1+i)^2 + F_1(1+i)^{+1} + F_2$$

F = (   ) + (   ) + (   )

F = (   ) $

2.1.4 Financial Analysis

Financial analyses use cash flow analysis and discounting techniques in benefit-cost analyses to maximize the rate of return to capital. Capital is the limiting factor here and maximizing profit is not necessarily achieved. The basic principle is that an investment will be made if revenues in the future will repay the cost at a positive rate of interest (North, 1985).

The standard criterion for deciding whether a government program can be justified on economic principles is *net present value* -- the discounted monetized value of expected net benefits (i.e., benefits minus costs). Net present value is computed by assigning monetary values to benefits and costs, discounting future benefits and costs using an appropriate discount rate, and subtracting the sum total of discounted costs from the sum total of discounted benefits. Discounting benefits and costs transforms gains and losses occurring in different time periods to a common unit of measurement. Programs with positive net present value increase social resources and are generally preferred. Programs with negative net present value should generally be avoided (OMB, 1992).

The process proceeds as:

- Define each alternative and predict their consequences
- Place monetary value on consequences
- Select a discount rate
- Convert time streams of benefits and costs
  - Construct cash flow diagrams
  - Convert values of costs and benefits at one date to equivalent values at the present (or another convenient) time.

$$NB_t = B_t - C_t$$

where NB_t = Net benefits at time t

B_t benefits at that time

C_t costs at that time

The Present Value of net benefits is

$$P = \sum_{t=1}^{T} \frac{NB_t}{(1+i)^t}$$

Often it is convenient to convert a present value to an equivalent annual value. For this we can use the Capital Recovery Factor (CRF)

$$CRF_T = \frac{i(1+i)^T}{(i+1)^T - 1}$$

Then

$$A = P \cdot CRF_T$$

2.1.5 Discount Rate

In order to compute net present value, it is necessary to discount future benefits and costs. This discounting reflects the time value of money. Benefits and costs are worth more if they are experienced sooner. All future benefits and costs, including nonmonetized benefits and costs, should be discounted. The higher the discount rate, the lower is the present value of future cash flows. For typical investments, with costs concentrated in early periods and benefits following in later periods, raising the discount rate tends to reduce the net present value (OMB, 1992).

The discount rate measures the rate at which current consumption will be sacrificed to ensure consumption (production) later. Greater sacrifices mean more resources can be devoted to future production. Alternatives discount rates include:

- Zero;

- Interest paid to borrow funds for project financing;
- Internal rate of return;
- Market interest rate for risk free investments
  - Interest on recently issued government bonds having a maturity date approximately equal to the project life
- Most productive investments (opportunity cost of capital)
  - If funds were committed to a project yielding the highest rate of return first, then to subsequent projects in order of rate of return, the IRR of the last project selected before funds run out is the MIRR
- Interest paid on borrowed funds for governments using bond financing
- US federal practice (Senate Doc. 97, 1962)
  - "Average rate of interest payable by the US Treasury on interest-bearing marketable securities outstanding at the end of the fiscal year preceding computation which had terms to maturity of 15 years or more"

  See http://waterhome.tamu.edu/NRCSdata/PriceIndexes/Rates.htm for current and historic rates.

2.1.6 Examples

Example 1. (after Linsley, et. al., 1979) Two alternative plans are considered for a section of an aqueduct. Plan A uses a tunnel, and Plan B uses a lined canal and steel flume. Both plans yield the same revenues over the life of the project.

	Plan A	**Plan B**		
	Tunnel	Canal	Canal lining	Flume
Life	100 yr	100 yr	20 yr	50 yr
First cost	$450,000	$120,000	$50,000	$90,000
Annual O&M cost	$4000		$10,500	

Interest rate = 6 % per year
Study period = 100 years

Compare the equivalent annual costs of the two plans

$$CRF = \frac{i(1+i)^N}{(i+1)^N - 1}$$

where $i = 0.06$ and N = 20, 50, and 100 years

Plan A

Capital recovery cost for the tunnel	$450,000 x 0.060177	$27,080
Annual maintenance cost		$4,000
Total annual cost		$31,080

Plan B

Capital recovery cost for canal	$120,000 x 0.060177	$7,221
Capital recovery cost for canal lining	$50,000 x 0.087184	$4,359
Capital recovery cost for flume	$90,000 x 0.063444	$5,710
Annual maintenance cost		$10,500
Total annual cost		$27,790

Total investment is $450,000 and $260,000, respectively, for the two projects. Even though the annual O&M costs are lower for Plan A, the annual cost comparison tells us that the extra investment is not justified. Thus Plan B should be selected.

Where the capacity of the project is to be determined, we simply determine the project with the maximum net benefits (difference of benefits over costs). That is:

$$Max\ NB(x) = B(x) - C(x)$$

where x is the capacity of the project. Thus

$$\frac{dB(x)}{dx} = \frac{dC(x)}{dx}$$

or

$$\frac{dB(x)}{dC(x)} = 1$$

That is, we increase the capacity up to the point where the marginal (incremental) benefits just exceed the marginal (incremental) costs and then stop.

Example 2. (after North, 1985, Ex. 5-1) A flood control district can construct several alternative control works to alleviate flooding. These alternatives include the construction of dam A, dam B, and a system of levees C. Each of these works can be built to function alone or together with any other or all other projects. Thus we have a possibility of the following combinations: ABC, A, B, C, AB, AC, and BC. The life of each dam is 80 years and the life of the levee system is 60 years. The cost of capital is 4%. Information on total investment, operation and maintainence costs, and average annual flood damages is given in the table. Which flood control undertaking is the most economical?

Table. Flood Control Project data

Project	Total Investment (million $)	Annual Operation and Maintainence (thous. $)	Average Annual Flood Damages (million $)
A	6	90	1.10
B	5	80	1.30
C	6	100	0.70
AB	11	170	0.90
AC	10	190	0.40
BC	9	180	0.50
ABC	15	270	0.25
Do nothing	0	0	2.00

The annual investment costs can be computed for each alternative by multiplying the investiment cost by the appropriate capital recovery factor:

$$CRF_T = \frac{i(1+i)^T}{(i+1)^T - 1}$$

where T = 80 years for dams and 60 for levees, respectively.

Project	Total Investment ($ mln)	CRF	Annual Investment Costs ($ mln)	Annual Operation and Maintainence ($ mln)	Total Annual Cost ($ mln)
A	6	0.04181	0.251	0.090	0.341
B	5	0.04181	0.209	0.080	0.289
C	6	0.04420	0.265	0.100	0.365
AB	11		0.460	0.170	0.630
AC	10		0.516	0.190	0.706
BC	9		0.474	0.180	0.654
ABC	15		0.725	0.270	0.995

The **Incremental Benefit – Cost Ratio Method** compares the additional benefit to the cost of any alternative compared to other alternatives to find the solution. The procedure is:

1. Discard any alternative with B/C < 1
2. Rank order the alternatives from lowest to highest cost
3. Compute the incremental benefit - cost ratio for the contender versus current best alternative. If that ratio is greater than 1, contender becomes current best.
4. Repeat until all alternatives have been tested. Final current best is preferred alternative.

Comparison	Project	Benefits ($ mln)	Cost ($ mln)	B/C Ratio	ΔB ($ mln)	ΔC ($ mln)	$\Delta B/\Delta C$	Conclusion
$\varnothing \to B$	B	0.7	0.289	2.42	0.7	0.289	2.4	$\varnothing < B$
$B \to A$	A	0.9	0.341	2.64	0.2	0.052	3.8	$A > B$
$C \to A$	**C**	**1.3**	**0.365**	**3.56**	**0.4**	**0.024**	**17**	**C > A**
$AB \to C$	AB	1.1	0.63	1.75	-0.2	0.265	-0.75	C > AB
$BC \to C$	BC	1.5	0.654	2.29	0.2	0.289	0.69	C > BC
$AC \to C$	AC	1.6	0.706	2.27	0.3	0.341	0.88	C > AC
$ABC \to C$	ABC	1.75	0.995	1.76	0.45	0.63	0.71	C > ABC

Example 3. (after Mays and Tung, Example 2.2.1) Determine the optimal scale of development of a hydroelectric project using benefit – cost analysis. Various alternative size projects and corresponding benefits are shown in the table below.

Scale (MW)	Benefits B ($ mln)	Costs C ($ mln)	Net Benefits B - C ($ mln)
50	18.0	15.0	3.0
60	21.0	17.4	3.6
75	26.7	21.0	5.7
90	29.8	23.4	6.4
100	32.7	26.0	6.7
125	38.5	32.5	6.0
150	42.5	37.5	5.0
200	50.0	50.0	0.0

The following figures show plots of the (1) project benefits and costs versus capacity, and (2) project benefits versus costs. Using a marginal analysis we find that the optimal capacity is 100 MW.
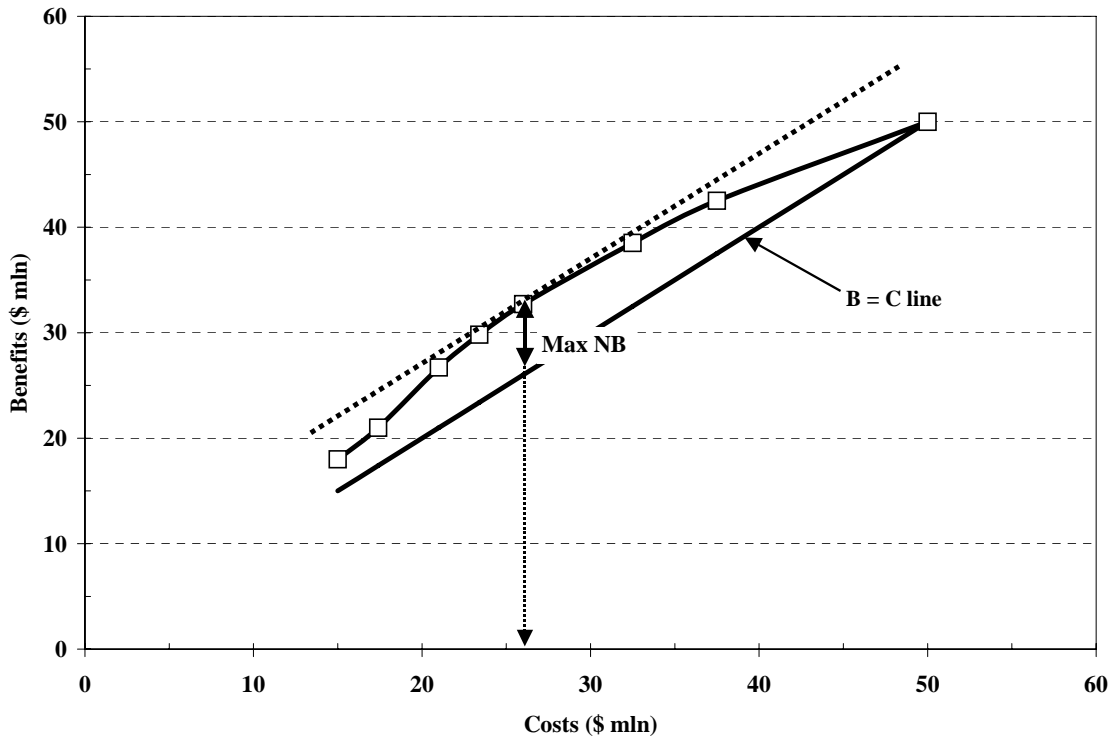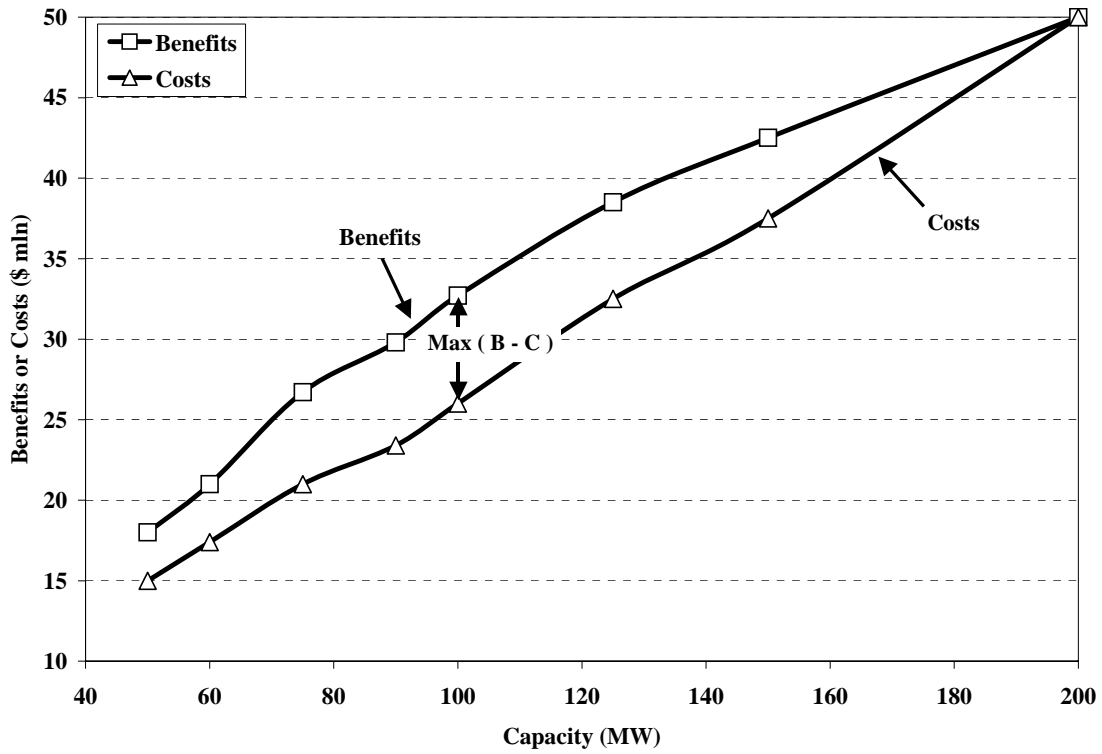
The incremental benefit – cost ratio method to find the same solution as before.

	Comparison	Scale (MW)	B ($ mln)	C ($ mln)	Incremental ΔB ($ mln)	ΔC ($ mln)	ΔB/ΔC	Conclusion
I	$\varnothing \to$ I	50	18.0	15.0	18	15	1.20	$\varnothing <$ I
II	I $\to$ II	60	21.0	17.4	3.0	2.4	1.25	II $>$ I
III	II $\to$ III	75	26.7	21.0	5.7	3.6	1.58	III $>$ II
IV	II $\to$ IV	90	29.8	23.4	3.1	2.4	1.29	IV $>$ III
V	IV $\to$ V	100	32.7	26.0	2.9	2.6	1.11	V $>$ IV
VI	V $\to$ VI	125	38.5	32.5	5.8	6.5	0.89	VI $<$V
VII	V $\to$ VII	150	42.5	37.5	9.8	11.5	0.85	VII $<$ V
VII	V $\to$ VIII	200	50.0	50.0	17.3	24.0	0.72	VIII $<$ V

2.2 Demand for Water

2.2.1 Introduction

Consumers purchase goods produced by firms. They have *preferences* for some goods over others and they choose purchases from a set of feasible options. A *utility function u(x)* is a numerical representation of consumer preferences. If one bundle of goods is preferred to another bundle, then it must have a higher utility. *Indifference curves* are the level sets of a utility function (see Figure 2.2.1.1).
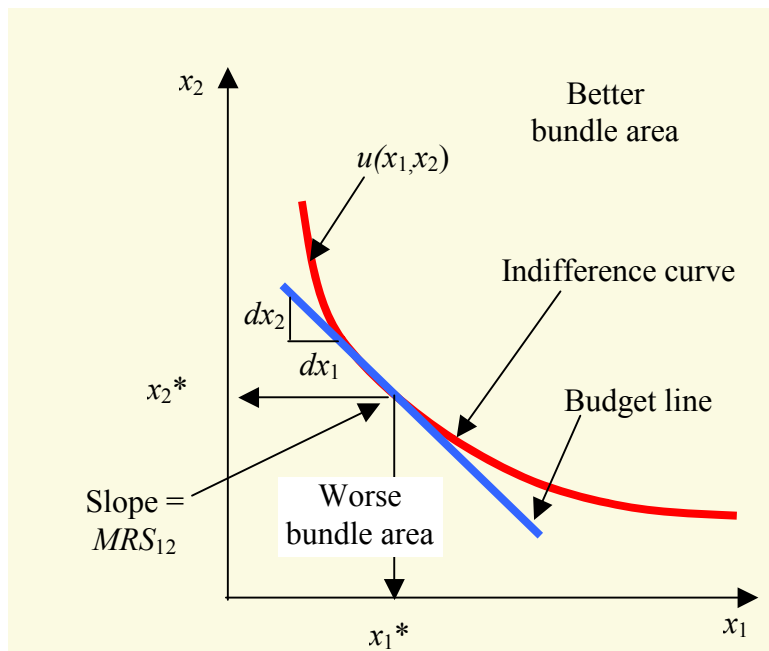
Figure 2.2.1.1. Indifference curve.

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Consider the case when there are 2 goods to choose from, x_1 and x_2. If the consumer changes consumption by a small amount (dx_1, dx_2) but keeps utility constant, say at level u_o, then

$$du(x) = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0 \qquad (2.2.1.1)$$

where

$$MU_i = \frac{\partial u}{\partial x_i} \qquad\qquad i = 1,2 \qquad (2.2.1.2)$$

is the *marginal utility* of good i or the change in utility due to a small change in x_i. Then using Eq. 2.2.1.2 in Eq. 2.2.1.1, we can write

$$-\frac{dx_2}{dx_1} = \frac{MU_1}{MU_2} = MRS_{12} \qquad (2.2.1.3)$$

where MRS_{ij} is the *marginal rate of substitution* of good i for good j, that is, the rate at which a consumer can substitute good i for good j.

2.2.2 Consumer's Problem

Consumers attempt to choose the best bundles of goods that they can afford. If there are K goods, whose quantities are represented by the vector $x = (x_1, \cdots, x_K)$, available for consumption with unit prices $p = (p_1, \cdots, p_K)$ and the total amount of money available to the consumer is m, then the consumer must make choices between goods according to a budgetary constraint

$$p^{\mathrm{T}} \cdot x = \sum_{k=1}^{K} p_k x_k \leq m \qquad (2.2.2.1)$$

Consider the case of 2 goods. The budgetary constraint

$$p_1 x_1 + p_2 x_2 \leq m \qquad (2.2.2.2)$$

separates the decision space into two regions: (1) a region containing those combinations of goods whose purchase would exceed the budget; and (2) a region where those combinations that would not exceed the budget (See Figure 2.2.2.1). The slope of the budget line $(-p_i / p_j)$ is the rate at which the market will substitute good i for good j.
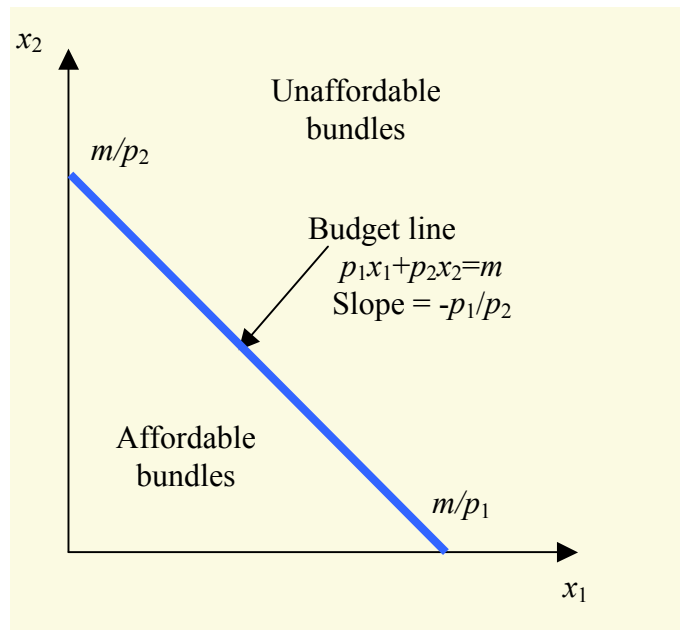
Figure 2.2.2.1. The consumer's budget set

Now, in the general case of K goods, the consumer is faced with the problem

$$\text{Maximize } u(\boldsymbol{x})$$

$$\text{subject to}$$

(2.2.2.3)

$$\boldsymbol{p} \cdot \boldsymbol{x} \le m$$

$$\boldsymbol{x} \ge 0$$

(2.2.2.4)

That is, the consumer tries for maximize utility while satisfying the budget constraint. Now, assume that the budget constraint (Eq. 2.2.2.4) holds as an equality (all funds are expended or one of the goods is actually a savings account) and that the levels of consumption are all positive. Then we have the **classical programming problem** with a **Lagrangian** function

$$L(\boldsymbol{x}, \lambda) = u(\boldsymbol{x}) + \lambda(m - \boldsymbol{p} \cdot \boldsymbol{x})$$

$$= u(\boldsymbol{x}) + \lambda\left(m - \sum_{k=1}^{K} p_k x_k\right)$$

(2.2.2.5)

The **first-order optimality conditions** for this problem are

$$\frac{\partial L}{\partial x_k} = 0 = \frac{\partial u}{\partial x_k} - \lambda p_k, \qquad k = 1,...,K \qquad\qquad (2.2.2.6)$$

$$\frac{\partial L}{\partial \lambda} = 0 = m - \sum_{k=1}^{K} p_k x_k \qquad\qquad (2.2.2.7)$$

The first condition (Eq. 2.2.2.6) says that the ratio of the marginal utility to price is constant for all inputs

$$\frac{MU_k}{p_k} = \frac{\partial u / \partial x_k}{p_k} = \lambda \quad k = 1,...,K \qquad\qquad (2.2.2.8)$$

or

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = ... = \frac{MU_K}{p_K} = \lambda \qquad\qquad (2.2.2.9)$$

That is, a consumer chooses purchases of goods such that the ratio of marginal benefit (marginal utility) to marginal cost (price) is equal among all goods. This ratio, with units of utility/$, is the value of the Lagrange multiplier (λ) which is also the ratio of the change in total utility for a change in income, or

$$\lambda = \frac{\partial u}{\partial m} \qquad\qquad (2.2.2.10)$$

If we write Eq. 2.2.2.9 for two goods, say goods i and j, we have

$$\frac{MU_i}{MU_j} = \frac{p_i}{p_j} = MRS_{ij} \qquad\qquad (2.2.11)$$

which says that the slope of the budget line will equal the slope of the indifference curve. or the ratio of the marginal utilities of any two goods equals the ratio of their prices (See Figure 2.2.2.2).
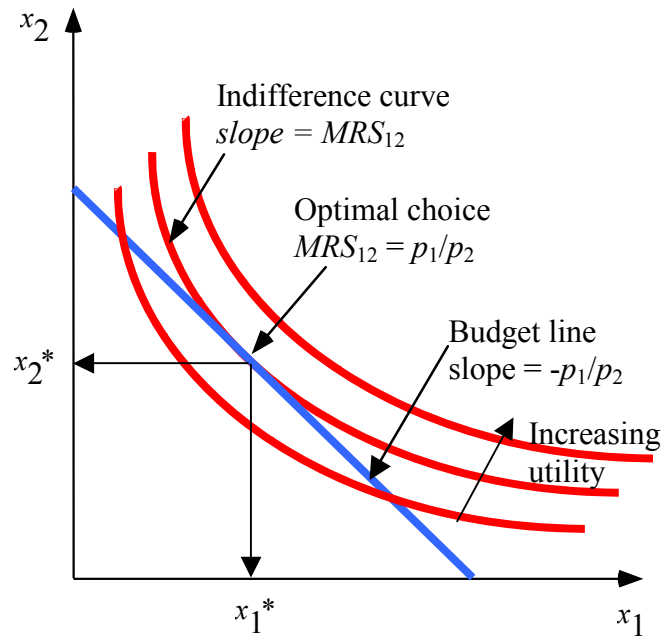
Figure 2.2.2.2. The consumer's problem and solution.

2.2.3 Demand

The optimal solution to the consumer's problem depends on income and prices so solving the problem (Eq. 2.2.2.3 and 2.2.2.4) results in an optimal level of consumption $x^* = x^*(p, m)$ which is a function of the prices and the available income. This is the **demand function**. A typical demand function is shown in Figure 2.2.3.1. Often the **inverse demand function, $p = p(x^*, m)$,** is used in analyses; this is simply the inverse of the demand function or price as a function of quantity and income. Market demand is the aggregation of all of the individual consumers' demands. Market demand depends on prices and the distribution of income in the economy.

Price, *p*

Demand curve
x(p,m)

Quantity, *x*
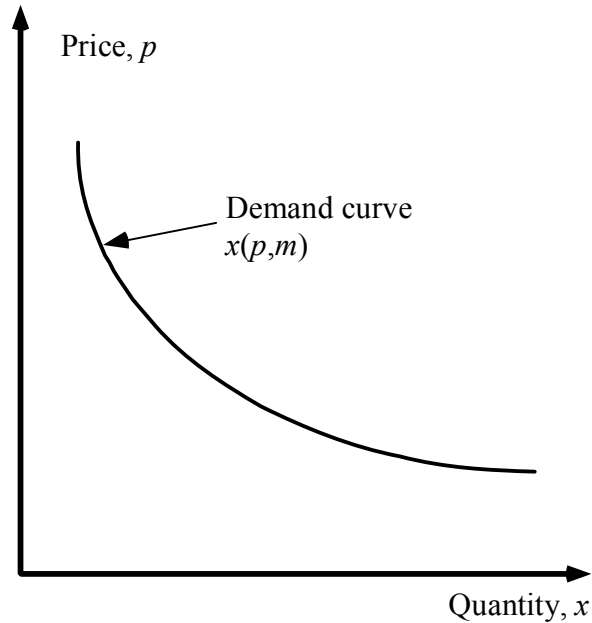
Figure 2.2.3.1. Typical demand curve.

2.2.4 Willingness-to-pay

Demand is only real (or "effective") when it is accompanied by willingness to pay, in cash or kind, for the goods or services offered (Evans, 1992[1]). The value of a good to a person is what that person is willing, and able, to sacrifice for it (willingness-to-pay). How do we measure what a person is willing to pay for a good? Assume that a farmer has no irrigation water for production of a particular crop, but desires to purchase some water. If one unit of water became available, how much would the farmer be willing to pay to obtain that unit of water, rather than have no water at all? Suppose the farmer is willing to pay \$38 for this first unit (see Figure 2.4.1) even though (s)he would prefer to pay less. Now, suppose that the farmer is willing to pay \$26 for a second unit of water. Further, suppose that the farmer is willing to pay \$17 for a third unit. According to the figure, at $p^* = \$10$ per unit, the farmer would purchase 4 units of water for a total cost of \$40, but (s)he would have been *willing to pay* \$93 for that water. Thus, the farmer receives a surplus of \$53 (consumer surplus) when purchasing the 4 units of water.

Evans (1992) suggests three ways of determining willingness-to-pay from direct information in other, similar, situations and from survey information:

- *Indirect method,* involves analyzing what others in similar circumstances to the target population are already paying for services;
- *Direct method* (or contingent valuation method)*,* involves asking people to say what they would be prepared to pay in the future for improved services; and

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1 Evans, Phil, Paying the Piper: An overview of community financing of water and sanitation, Occasional Paper 18, IRC International Water and Sanitation Centre, The Hague, The Netherlands, April 1992

- *Proxy measures*, e.g., use of case studies of water vending to provide indicators of willingness to pay.
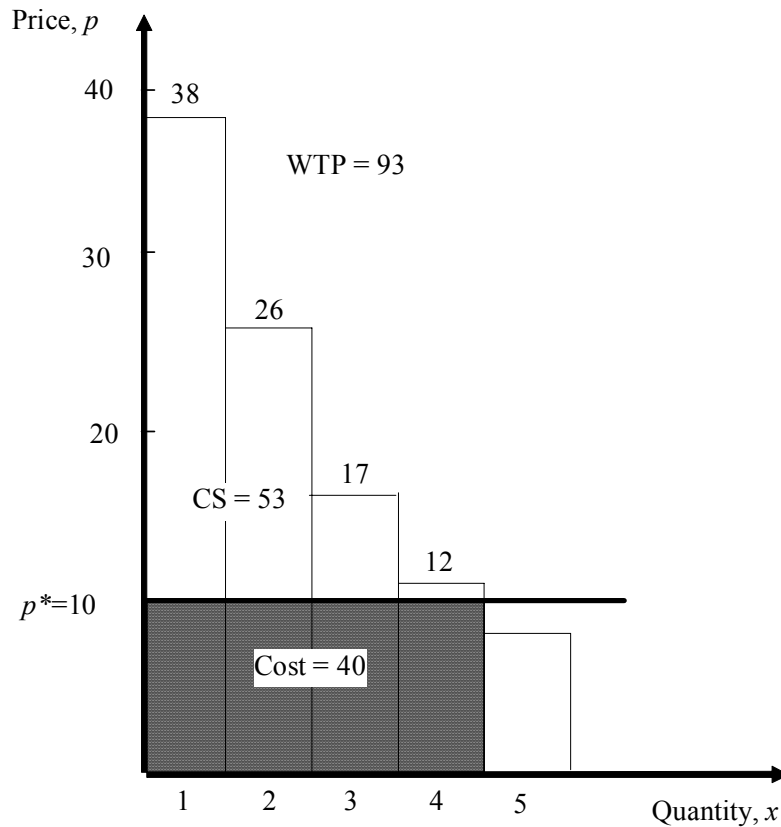
Figure 2.2.4.1. Willingness-to-pay for each additional unit of water.

If we assume that fractional amounts of a unit of a good can be purchased, then we obtain a continuous graph, as in Figure 2.2.4.2. Marginal willingness-to-pay is the height of the curve. Total willingness-to-pay is the sum of the heights of the rectangles between the origin and the particular consumption level, x, of interest. In the case of the continuous curve, willingness-to-pay is the area under the curve from the origin and the particular consumption level of interest and this represents the gross benefit of purchasing this amount of the good. The net-benefit from this purchase is the willingness-to-pay minus the cost or

$$NB = \int_0^{x*} p(\eta, m)d\eta - p * x* \tag{2.2.4.1}$$

which is termed the consumer's surplus.
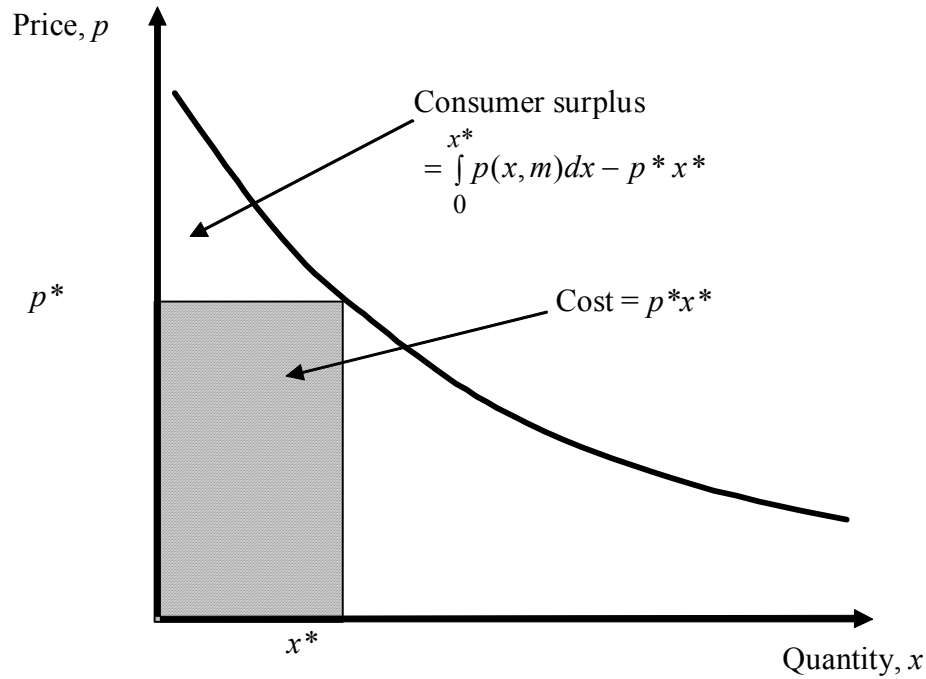
Figure 2.2.4.2. Willingness-to-pay curve.

Measuring benefit of water use in this manner requires that we can derive the demand curve for the water used for a particular purpose. For marketed commodities with available information on prices and quantities we can: (1) derive a demand curve, (2) quantify willingness-to-pay, and (3) use WTP to represent benefits. However, in many cases market prices may not exist, demands may not be revealed, and the change in benefits over time may be extremely uncertain. Examples include (1) the benefits of preserving space for recreation, and (2) the benefits derived from damages prevented due to pollution controls. If the physical damages of pollution can be identified and estimated, then a monetary value may be placed on them (for an example of applying this to the Aral Sea basin, see Anderson, 1997). Sometimes it is possible to survey people to determine their willingness-to-pay for different environmental assets such as environmental preservation, damage reductions, and lower risks. From these survey results we may be able to infer the valuation of the assets. Indeed, we may also be able to infer these values from related markets where values are observable.

The value of municipal water at its source minus any water utility costs is represented by the consumers' surplus. The area under the demand curve for an increment from x_1 to x_2 is (Gibbons, 1986)

$$Area = \frac{px_2^{\chi}}{1-\beta}\left(\frac{x_2}{x_2^{\beta}} - \frac{x_1}{x_1^{\beta}}\right) \qquad \beta = \frac{1}{\varepsilon}$$

(2.2.4.2)

2.2.5 Elasticity (of demand)

The *price elasticity of demand* is a measure of how responsive consumers are to changes in price. The slope of the demand function $x* = x*(p, m)$ is

$$slope = \frac{dx}{dp}$$

(2.2.5.1)

This quantity depends on the units used to describe the inputs and price. If we normalize this function, we obtain the price elasticity of demand

$$elasticity = \varepsilon = \frac{dx/x}{dp/p}$$

(2.2.5.2)

Consider the following example adapted from Merrett (1997). Table 2.2.5.1 shows the quantity of water demanded for different prices along with the price elasticity of water at various increments. Figure 2.2.5.1 plots the demand function for water and illustrates the ranges of elastic and inelastic behavior. Merrett (1997) proposes a cubic form for the demand function

$$p = ax^3 + bx^2 + cx + d$$

(2.2.5.3)

where a < 0, b > 0, c < 0, and d > 0. He points out, at low quantities, higher prices for water have little effect due to the intense need for the water. Similarly, at low quantities, higher prices for water have little effect due to the abundance of water. In the middle quantities, changes in price produce significant changes in the quantity of water demanded.

Price elasticity of municipal water demand was estimated (Gibbons, 1986) and in-house water use was found to be price-inelastic (= -0.23), while sprinkling use was found to be more elastic and differ between the Eastern US (= -1.6) and the Western US (= -0.7).

Table 2.2.5.1. Price Elasticity of Water (adapted from Merrett, 1997)

Quantity (m³/month)	Price (per m³)	ΔQ	ΔP	ε
700	6			
		300	-1	1.8
1000	5			
		500	-1	1.67
1500	4			
		500	-1	1
2000	3			
		500	-1	0.6
2500	2			
		300	-1	0.21
2800	1			
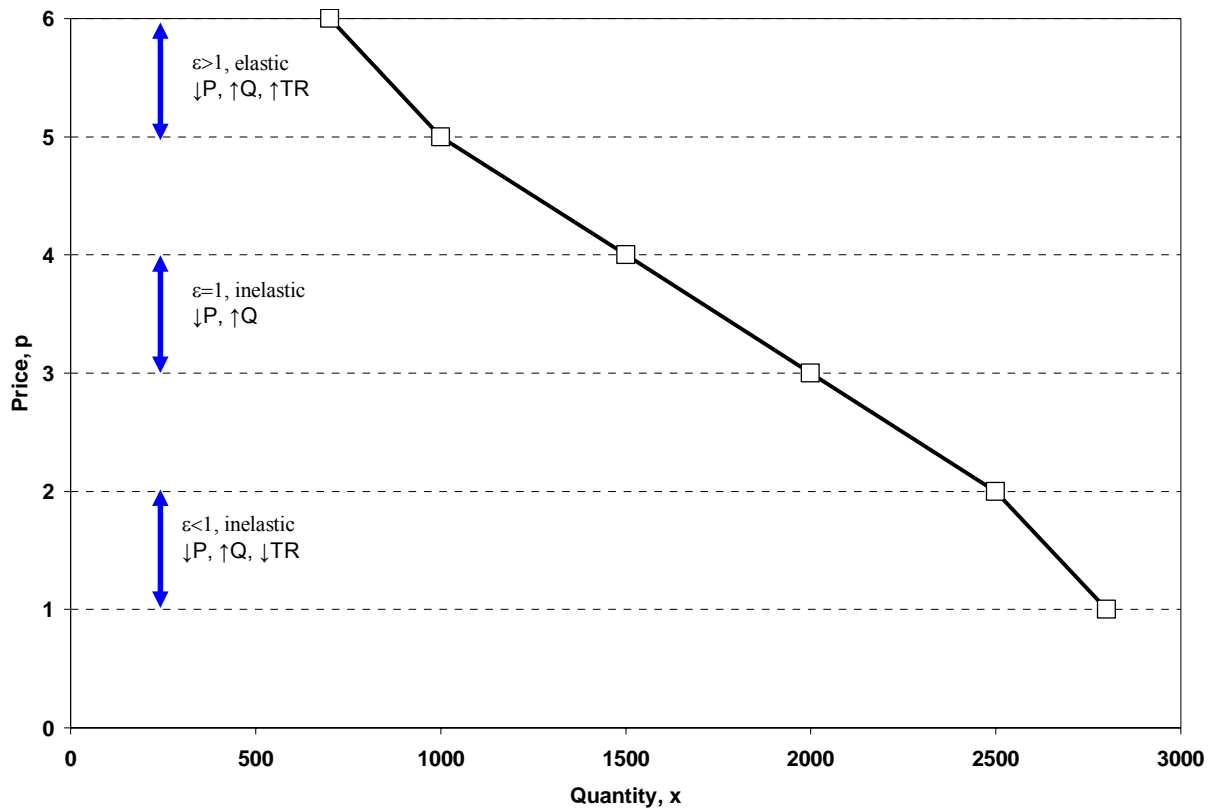
Figure 2.2.5.1. Demand function for water.

2.2.6 Water Values in the US

Table 2.2.6.1 shows data on the value of water for various uses within the United States [Frederrick et al. (1996).

Table 2.2.6.1. National Water Value by Use ($/af) [Frederrick et al. (1996)]

		Average	Median	Min	Max
Instream	Waste Disposal	3	1	0	12
	Recreation/F&W Habitat	48	5	0	2,642
	Navigation	146	10	0	483
	Hydropower	25	21	1	113
Offstream	Irrigation	75	40	0	1228
	Industrial	282	132	28	802
	Thermo Power	34	29	9	63
	Domestic	194	97	37	573

Table 2.2.6.2 shows data on the value of water for recreations and fish & wildlife uses within the United States and Table 2.2.6.3 shows the value of water use in irrigated agriculture.

Table 2.2.6.2. Water Values for Recreation/F&W Habitat ($/af) [Frederrick et al. (1996)]

	Average	Median	Min	Max
Fishing	34	5	0	158
Wildlife Refuge	24	6	1	44
Fishing & Whitewater	1042	1505	6	3
Whitewater	9	9	5	4
Shoreline Recreation	19	19	17	2

	Average	Median
Table 2.2.6.3. Water Values by Crop ($/af) [Frederrick et al. (1996)]		
Alfalfa	51	44
Apples	151	151
Barley	33	39
Beans	58	58
Carrots	550	550
Corn	91	98
Cotton	114	103
Grain Sorgham	57	44
Hay	36	36
Hops	18	18
Lettuce	208	208
Melons	54	54
Onions	40	40
Pears	137	137
Potatoes	710	784
Rice	86	86
Safflower	53	58
Soybeans	121	127
Sugar Beets	121	119
Tomatoes	686	686
Vegetables	206	206
Wheat	51	47

2.3 Supply of Water

2.3.1 Introduction

Firms produce outputs from various combinations of inputs. The objective of a firm is to maximize profit subject to constraints imposed by technological capabilities. As long as inputs are costly, we can limit our consideration to those combinations of inputs that will produce the *maximum* output for a given level of inputs. This represents the boundary of the so-called production possibilities set and it is called the *production function* (see Figure 2.3.1.1)

$$y = f(x) \tag{2.3.1.1}$$

Level curves of the production function are called isoquants where $f(x) = y_0$ and y_0 is a fized level of output. For a firm producing a single output (y) from two inputs (x_1 and x_2), the isoquants may look like those in Figure 2.3.1.2.
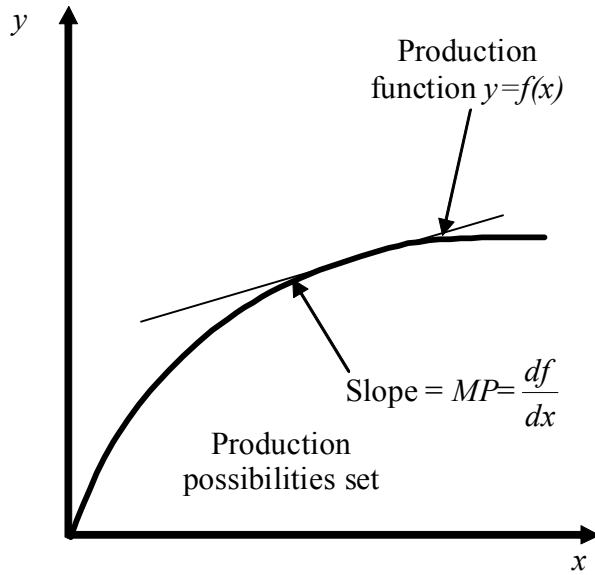
Figure 2.3.1.1. Production function.
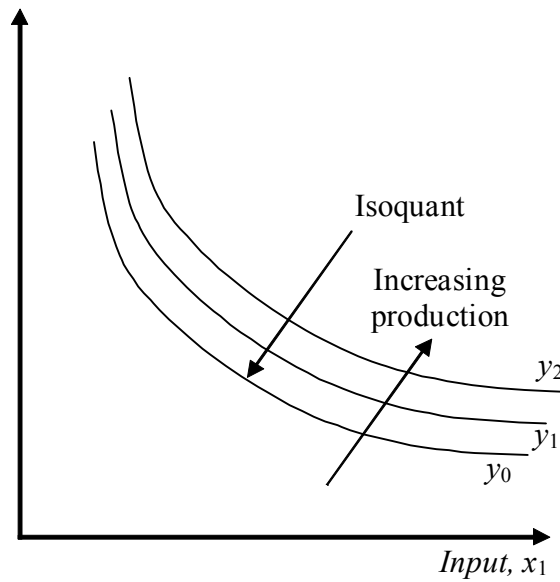
Figure 2.3.1.2. Isoquants

Suppose that a firm wants to increase the amount of one input and decrease the amount of another while maintaining production at a constant level of output $f(x) = y_0$. So if production is to remain constant, we can write

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0 \qquad (2.3.1.2)$$

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where

$$\frac{\partial f}{\partial x_i} = \frac{\partial y}{\partial x_i} = MP_i, \qquad i = 1,2 \tag{2.3.1.2}$$

is the *marginal product* or the additional output available from using an additional unit of input x_i in the production process. Now, from Equations 2.3.1.1 and 2.3.1.2, we have

$$\frac{dx_2}{dx_1} = -\frac{MP_1}{MP_2} = TRS_{12} \tag{2.3.1.3}$$

where TRS_{12} is the *technical rate of substitution*. That is, the rate at which x_1 can replace x_2 (see Figure 2.3.1.3). TRS_{12} is the slope of the isoquant $f(x_1, x_2) = y_0$.
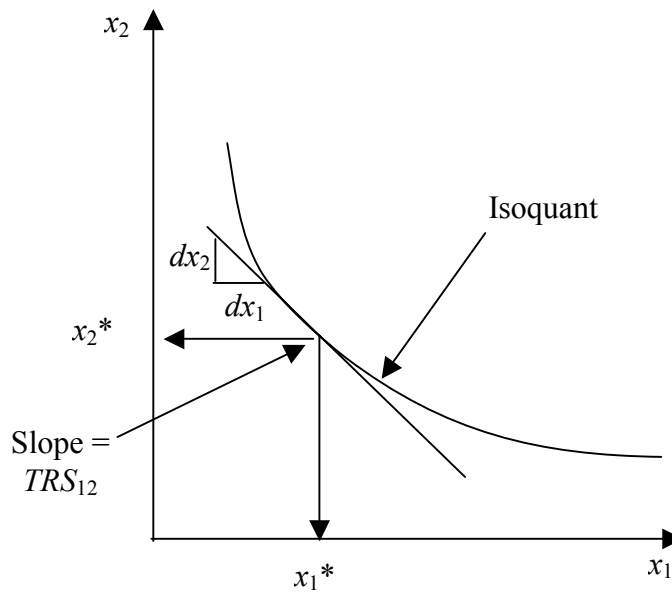
Figure 2.3.1.3. Technical rate of substitution.

Revenue R is the amount of money that a firm receives for selling an amount y of a product for a particular price p:

$$R = py \tag{2.3.1.4}$$

Marginal revenue is the change in revenue for a change in the output or the quantity sold

$$\frac{dR}{dy} = \frac{\partial R}{\partial y} + \frac{\partial R}{\partial p}\frac{dp}{dy} = p + y\frac{dp}{dy} \tag{2.3.1.5}$$

Example: Consider the case of a linear inverse demand function:

$$p(y) = a - by \qquad\qquad (2.3.1.6)$$

Then, revenue is given by the quadratic function

$$R = py = ay - by^2 \qquad\qquad (2.3.1.7)$$

and marginal revenue, the derivative of revenue with respect to output, is

$$\frac{dR}{dy} = a - 2by \qquad\qquad (2.3.1.8)$$

Thus, the slope of the marginal revenue curve is twice as steep as that of the demand curve (see Figure 2.3.1.4).
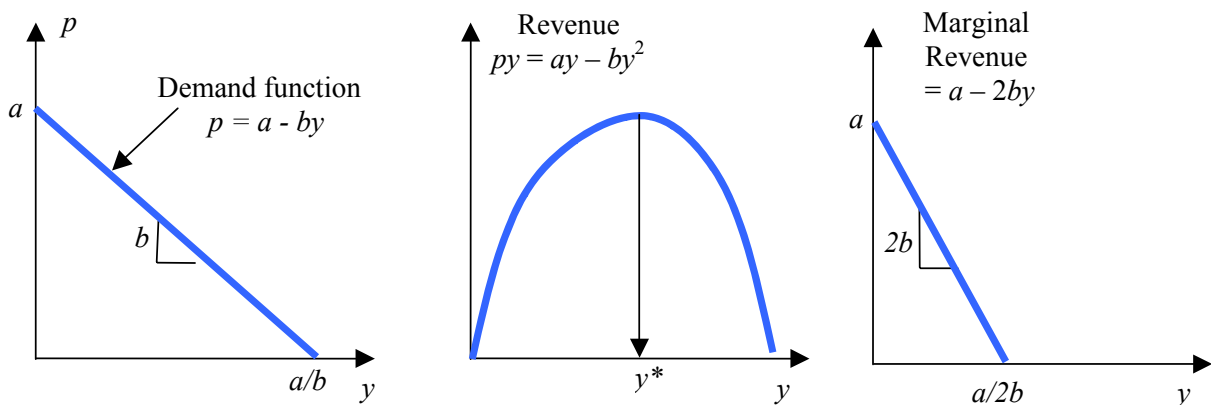
Figure 2.3.1.4. Marginal revenue and demand curves for a linear demand function.

Profit is the difference between the revenue a firm receives and the cost that it incurs

$$\pi(p,w) = py - \sum_{n=1}^{N} w_n x_n = pf(x_1,...,x_n) - \sum_{n=1}^{N} w_n x_n \qquad\qquad (2.3.1.9)$$

2.3.2 The Firm's Problem

The firm's problem is to maximize profit

$$\textit{maximize } \pi(p,w) = pf(x_1,...,x_n) - \sum_{n=1}^{N} w_n x_n \qquad\qquad (2.3.2.1)$$

The **first-order optimality conditions** for this problem are

$$\frac{\partial \pi}{\partial x_n} = 0 \qquad \Rightarrow \qquad p \frac{\partial f}{\partial x_n} = w_n, \text{ or } \frac{\partial f}{\partial x_n} = \frac{w_n}{p} \quad n = 1, \dots, N \qquad (2.3.2.2)$$

This condition says that the value of the marginal product (price times marginal product $p\, \partial f / \partial x_n$) for input n must equal the price of that input (w_n). This tangency condition is illustrated in Figure 2.3.2.1.
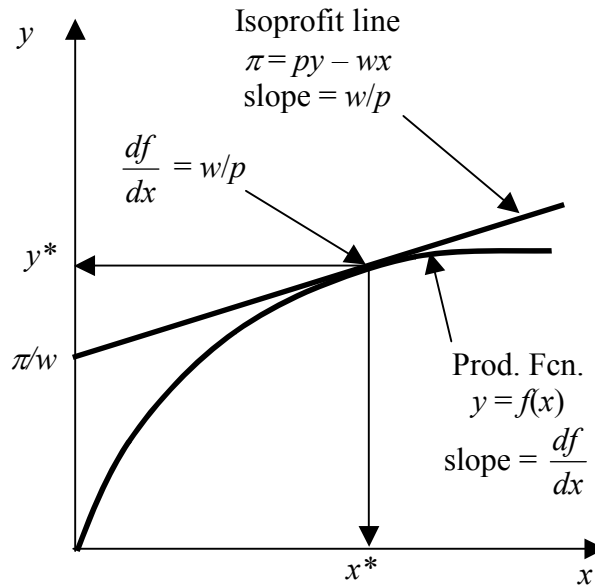
Figure 2.3.2.1. Profit maximization.

We can define a variant of the firm's problem, where a firm strives to minimize its costs while producing a specified level of output (y_0)

$$\text{Minimize } \sum_{n=1}^{N} w_n x_n$$

subject to $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.3.2.3)$

$$f(x_1, \dots, x_N) = y_0$$

The **Lagrangian** in this case is

$$L(\boldsymbol{x}, \lambda) = \sum_{n=1}^{N} w_n x_n - \lambda[f(x_1, \dots, x_N) - y_0] \qquad (2.3.2.4)$$

The **first-order optimality conditions** are

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$$\frac{\partial L}{\partial x_n} = w_n - \lambda \frac{\partial f}{\partial x_n} = 0 \qquad n = 1,...,N \tag{2.3.2.5}$$

$$\frac{\partial L}{\partial \lambda} = f - y_0 = 0 \tag{2.3.2.6}$$

Writing the first condition for two products i and j, we have

$$\frac{w_i}{w_j} = \frac{MP_i}{MP_j} = -TRS_{ij} \tag{2.3.2.7}$$

That is, the technical rate of substitution equals the price ratio. The tangency condition for optimality in this case is illustrated in Figure 2.3.2.2.
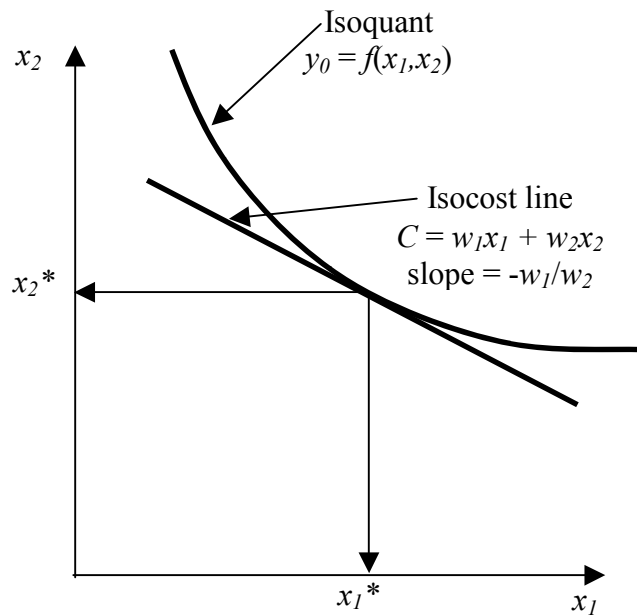
Figure 2.3.2.2. Cost minimization.

The total cost of producing at output level, y, is

$$TC(y) = \min\{\vec{w} \cdot \vec{x} : y = f(\vec{x})\} \tag{2.3.2.8}$$

The firm's total cost is comprised of *fixed* (*FC*) and *variable* (*VC*) *costs*

$$TC(y) = FC + VC(y) \tag{2.3.2.9}$$

The firm's *average cost* is the cost per unit to produce y units of output, or

$$AC = \frac{TC(y)}{y} \tag{2.3.2.10}$$

The firm's *marginal cost* is the cost of producing an additional unit of output

$$MC = \frac{dTC}{dy} = \frac{dVC}{dy} \tag{2.3.2.11}$$

Example: How much water should a water industry firm sell (produce) and at what price? The firm's problem can be defined as

Maximize $py - TC(y)$ (2.3.2.12)

The **first-order optimality conditions** are

$$MR(y) = \frac{dp}{dy}y + p = MC(y) \tag{2.3.2.13}$$

Marginal Revenue = Marginal Cost

An increase in output has two effects (1) adding p units to the benefits, and (2) causing the value placed on each unit of output to change by $p' = dp/dy$ (Dorfman, 1962). If the firm is **competitive**, then it has no market power, and $p' = dp/dy = 0$, and the price p is constant and fixed by the market. In this case, the first-order conditions are

$$p = MC \tag{2.3.2.14}$$

as illustrated in Figure 2.3.2.3.
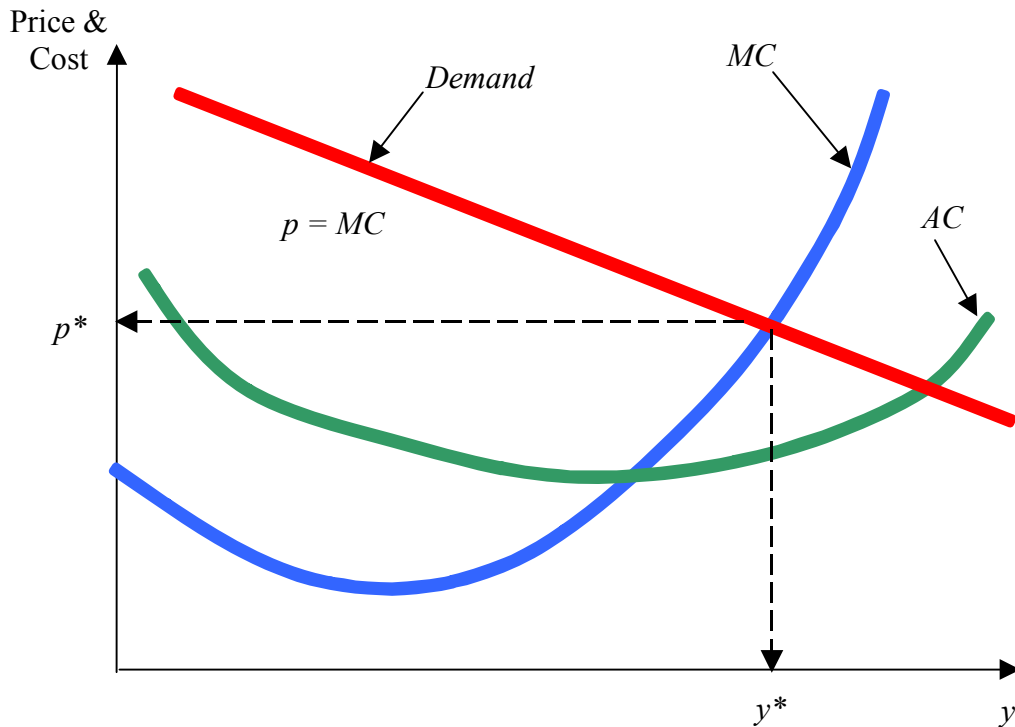
Figure 2.3.2.3. Average cost (AC), and Marginal cost (MC) curves. The optimal level of production occurs where the demand intersects the MC curve.

However, if the firm is **monopolistic**, it is unlikely to take the output price as given, since the monopolistic firm recognizes its influence over market price. The firm is free to choose the price and level of output so as to maximize its profit. Since for a monopoly the price is not constant, but is a function of output, we have

$$MR = \frac{dp}{dy} y + p = MC \qquad\qquad (2.3.2.15)$$

If the monopolistic firm chooses to **maximize profit**, then its chosen price and level of output will be p_M and y_M in order to set $MR = MC$. However, the firm knows that consumers are willing to pay a price $p > MC$ (see Figure 2.3.2.4). Since the MR curve lies below the demand curve, the monopolistic firm will produce (y_M) which is less than the amount (y_C) which a competitive firm would produce. That is, the price will be higher and the output lower for the monopolistic firm. Government regulatory commissions often have substantial power over the prices charged by public utilities. Without regulation, the firm will charge the price p_M and produce y_M. By setting a maximum price of p_C, the commission can make the monopolist increase output, thus making price and output correspond more closely to what they would be if the industry were organized competitively. Commissions often set prices at the level at which it equals average cost.
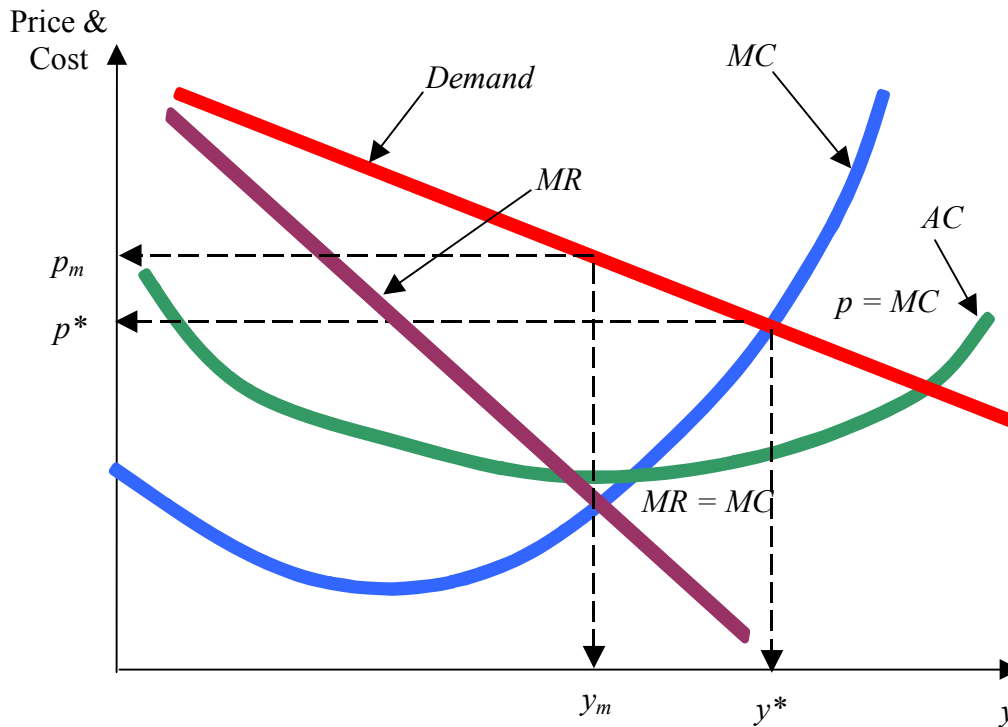
Figure 2.3.2.4. Optimal production level and price for a competitive firm and a monopolistic firm producing a good with a linear demand curve $p(q) = a - bq$.

2.3.3 Crop Production Functions with Water

2.3.3.1 Introduction

The fundamental building block for the estimation of the demand for and value of water in the agricultural sector is a production function that relates crop production to the use of water and other inputs. An ideal crop-water production model should be flexible enough to address issues at the crop, farm, or basin levels. The production function should allow the assessment of policy-related problems, and results should be transferable between locations. In addition, the model should be simple to operate, requiring a small data set; easily adjustable to various farming conditions; and sufficiently comprehensive to allow the estimation of externality effects. In addition, the interaction between water quantity and quality and the water input/production output should be clearly defined (Dinar and Letey 1996).

Existing modeling approaches to crop-water relationships (for example, surveys by Hanks 1983 and Vaux and Pruitt 1983) address economic, engineering, and biological aspects of the production process. These surveys conclude that crop-water relationships are very complicated and that not all management issues have been fully addressed in one comprehensive model. In the following, the advantages and disadvantages of alternative production functions are summarized.

2.3.3.2 Types of Production Function Models

Four broad approaches to production functions can be identified:

1. Evapotranspiration and transpiration models:
2. Simulation models;
3. Estimated models; and
4. Hybrid models that combine aspects of the first three types.

The following overview on production functions related to water use draws heavily on Dinar and Letey (1996), chapters 2 and 3, for the first three types of models.

Evapotranspiration and Transpiration Models

Evapotranspiration models are physical models that predict crop yield under varying conditions of salinity levels, soil moisture conditions, and irrigation strategies. They assume a linear yield-evapotranspiration relationship and are usually site-specific and very data intensive (see also Hanks and Hill 1980).

A basic yield-seasonal evapotranspiration relationship is represented by:

$$Y/Y_{max} = 1 - kc*(1 - E/E_{max})$$
(2.3.2.1)

where

Y	=	actual yield (ton/ha)
Y_{max}	=	maximum dry matter yield (ton/ha)
kc	=	crop coefficient
E	=	actual evapotranspiration (mm)
E_{max}	=	maximum evapotranspiration (mm)

The parameter E can be estimated by

$$E = w + r + \Delta q - o - d$$
(2.3.2.2)

where

w	=	applied water (mm)
r	=	rainfall (mm)
Δq	=	change in soil water storage (mm)
o	=	runoff
d	=	drainage

Transpiration models use a similar approach but measurement of transpiration is more difficult because it is difficult to separate it from evaporation. Although evapotranspiration and transpiration

models capture important aspects of crop-water relationships, they have limited ability to capture the impacts of non-water inputs, and are of limited use for policy analysis.

Simulation Models

Within the category of simulation models, Dinar and Letey (1996) distinguish between *holistic* simulation models, that simulate in detail the production process of one crop and *specific* models, that focus on one production input or the subsystems associated with a particular production input.

Detailed, data-intensive holistic models have been developed for most of the basic crops and a series of other agricultural production features (e.g., peanuts potatoes, maize, soybeans, and spring wheat). See also the CAMASE register, which currently includes more than 200 agro-ecosystem models or similar registers (CAMASE 1997). COTMOD, a model for cotton, for example, can be used to simulate the effects of various irrigation schedules, fertilizer application rates, and other management practices on cotton yield (Marani 1988). The relatively complicated data generation through field experiments and calibration procedures prevents the easy transferability of this model.

Dinar and Letey (1996) specify a model, in which annual applied water, irrigation water salinity, published coefficients relating crop sensitivity to salinity, the relationship between yield and evapotranspiration, and the maximum evapotranspiration for the area are the input parameters. Outputs include crop yield, amount of drainage water, and salinity of the drainage water. It is assumed that all nonwater-related inputs are applied at the optimum level. Water is the only limiting factor in the production process.

Estimated Production Function Models

Estimated production functions are more flexible than other model types. However, specification and estimation procedures must comply with plant-water relationships: (1) plant yield increases as water quantity increases beyond some minimum value; (2) yield possibly decreases in a zone of excessive water applications; (3) yields decrease as the initial level of soil salinity in the root zone or the salt concentration in the applied irrigation water increase beyond some minimum value; and (4) the final level of root zone soil salinity decreases with increasing irrigation quantities (except for possible increases, where relatively insufficient water quantities have been applied) (Dinar and Letey 1996). In order to meet these requirements, polynomial functions have been applied in many production functions. Dinar and Letey (1996) present the following quadratic polynomial form in the case of three production inputs:

$$Y/Y_{\max} = a_0 + a_1\,w + a_2\,s + a_3 u + a_4\,w\cdot s + a_5\,w\cdot u + a_6\,s\cdot u + a_7\,w^2 + a_8 s^2 + a_9\,u^2 \quad (2.3.2.3)$$

where

$$\begin{aligned}
Y &= \text{yield} \\
Y_{max} &= \text{maximum potential yield} \\
w &= \text{water application to potential evapotranspiration,}
\end{aligned}$$

s	= salinity of the irrigation water,
u	= irrigation uniformity,
a_i	= estimated coefficients (*i*=1,..9)

The quadratic form implies that an increase in the level of one of the decision variables results in a constant change in the level of the dependent variable up to a point. Any further increase results in an opposite response (positive-diminishing marginal-productivity zone on the production surface), followed by a zone of negative marginal productivity.

Hybrid Production Function Models

Hybrid models, which draw on the strengths of each production function approach, may offer considerable advantages to the three types of approaches taken individually. As noted above, each of the three basic methodologies for production functions have some weaknesses. Particularly limiting may be the data requirements for any given approach. It is likely that, for some relationships embodied in the model, available experimental and non-experimental data, especially on the interrelationships of water use, resource degradation, and production, may be inadequate. Several reasons can account for this. Non-experimental data (cross-section and time series data) collected by government agencies or targeted surveys rarely can adequately measure or control for water and important environmental variables (like water table depth and soil and water quality). Generation of this type of data can also be difficult, expensive, and often impractical, if not impossible, to achieve.

In many instances, however, data are not entirely absent. If data are relatively sparse, the available observations may not be adequate for statistical analysis but can be useful in calibrating generalized versions of simulation models. When important bio-physical and environmental variables in the study are inadequate or unavailable, simulation models can be used to generate pseudo-data. Pseudo-data are not true historical data, but rather are derived from process models replicating the real-world processes in computer experiments. Observations are generated by repeatedly solving the model for different initial values, and by parametrically varying input or output quantities and values. Simulation models are practical substitutes for complex biophysical experiments (or even non-experimental data), where it is often difficult to isolate the impacts of important policy, management, or environmental variables on output variables. In simulation models, the analyst can control institutional, technological and environmental factors, which is not possible with real-world experiments.

2.3.3.3 Example: Production of wheat in the Maipo basin, Chile

This example is adapted from *Rosegrant et al.* (2000). In agricultural production, water is allocated to crops according to their water requirements and economic profitability. Water demand can be determined in an optimization model based on empirical agronomic production functions for agriculture. The relationship between crop yield and seasonal applied nonsaline water provides values of crop yield under various water application, irrigation technology, and irrigation water salinity. The production function can be used directly used in an optimization model to calculate

crop yields with varying water application, salt concentration, and irrigation technology. The crop yield (production) function is specified as follows:

$$y = y_{max}[a_0 + a_1(x'/E_{max}) + a_2 \ln(x'/E_{max})] \qquad (2.3.3.1)$$

where

$$a_0 = b_0 + b_1 u + b_2 c$$
$$a_1 = b_3 + b_4 u + b_5 c \qquad (2.3.3.2)$$
$$a_2 = b_6 + b_7 u + b_8 c$$

and

y	crop yield (metric tons [mt]/ha),
y_{max}	maximum attainable yield (mt/ha)
a_0, a_1, a_2	coefficients,
$b_0 - b_8$	coefficients,
x'	infiltrated water (mm)
E_{max}	maximum evapotranspiration (mm)
c	salt concentration in water application (dS/m). Use the factor 1.14 to convert dS/m to g/L, and
u	Christiensen Uniformity Coefficient (CUC).

Uniformity (CUC) is used as a surrogate for both irrigation technology and irrigation management activities. The CUC value varies from approximately 50 for flood irrigation, to 70 for furrow irrigation, 80 for sprinklers, and 90 for drip irrigation, and also varies with management activities. By including explicit representation of technology, the choice of water application technology can be determined endogenously. The coefficients for the function in Eq. 2.3.3.1 as estimated by *Rosegrant et al.* (2000) are shown in Table 2.3.3.1. Using these coefficients in Table 2.3.3.2.

Table 2.3.3.1. Coefficients for the Production Function.

Coefficient	Wheat		
B1	0.284973	B5	-0.81096
B2	1.153264	B6	0.030845
B3	0.183139	B7	0.141539
B4	-0.05615	B8	1.181461
		B9	-0.03203

Table 2.3.3.2. Coefficients for the Production Function.

Coefficient	Wheat
a0	1.037233
a1	-0.35431
a2	0.937176

For the wheat produced in the Maipo basin in Chile, we have

$E_{max} = 535.5$ mm/m2/yr
$y_{max} = 6$ mt/ha

A typical crop yield function for wheat in the Maipo river basin is shown in Figure 2.3.3.1 and the data are shown in Table 2.3.3.3. In this figure and the table, the input, denoted x, is actually

$$x = x' * E_{max} * 10,000 m^2 / ha \qquad (2.3.3.3)$$
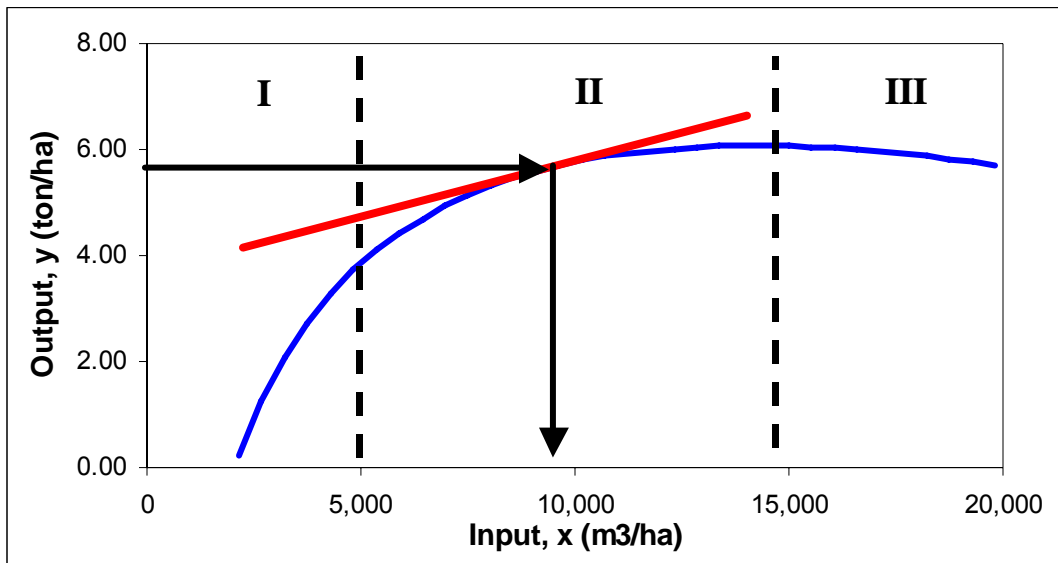
Figure 2.3.3.1. Production function for wheat as a function of applied water (CUC=0.7, salinity = 0.7).

In Fig. 2.3.3.1, it is evident that a certain amount of water must be applied to the crop before any production can result. Output increases at an increasing rate as the first few units of input are added; it continues to increase at a decreasing rate at higher input levels (Law of Diminishing Returns).

Average product is obtained by dividing the output by the input

$$AP = \frac{y}{x} \qquad (2.3.3.4)$$

and it measures the efficiency of the input used in the production. *Marginal product* is the change in the output resulting from a unit increment in input, that is

$$MP = \frac{dy}{dx} = Y_{max} \left(\frac{a_1}{E_{max}} + \frac{a_2}{x} \right) \qquad (2.3.3.5)$$

40

AP and *MP* are shown in Fig. 2.3.3.2.
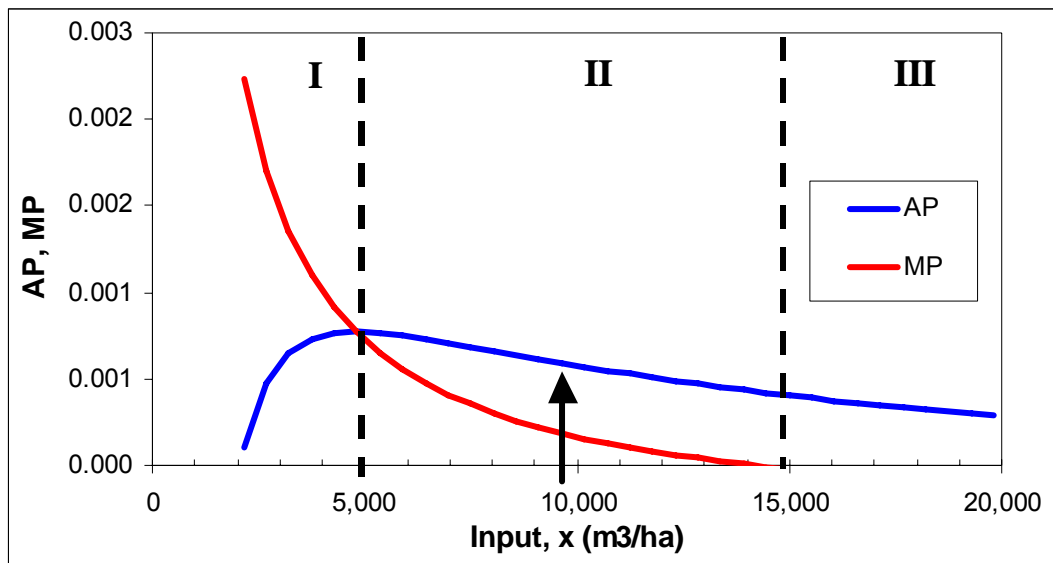
Figure 2.3.3.2. Average and marginal productivities versus input.

The production function can be broken into three regions, depending on the efficiency of resource use:

Region I $MP > AP$, not enough input is being used

If the product has value, input use should be increased until Region II is reached, since the physical efficiency of the input increases throughout Region I. It is not reasonable to cease using the input while efficiency is increasing.

Region II   *MP* is decreasing and $MP < AP$, just enough input is being used

Region II defines the area of economic relevance and optimal input use must be in this range. The exact level of production and resource use depend on the input and output prices.

Region III $MP < 0$, too much input is being used

Even if the input is free, it will not be used in this stage, since maximum output occurs at the boundary of Region II and further inputs simply decrease output.

Table 2.3.3.3 includes the costs of production for this example. *Fixed costs*, *FC*, are $100/ha and include ground preparation and other costs. *Variable cost*, *VC*, is computed by multiplying the input, *x*, by the unit price of the input, *w* ($0.05/m³ in this example)

$$VC = wx \qquad\qquad\qquad (2.3.3.6)$$

The shape of the *VC* curve depends on the shape of the production function (see Fig. 2.3.3.3). *Total cost, TC*, is simply the sum of *FC* and *VC*.

Average costs (*AC*) follow in the same way that average production does, they are the costs divided by the amount of the output

$$AC = \frac{VC}{y} = \frac{wx}{y} \qquad (2.3.3.7)$$

Average variable cost (*AC*) is inversely related to the average product, attaining a minimum when *AP* is at a maximum. When *AC* is decreasing, the efficiency of the input is increasing and efficiency is maximum when *AC* is minimum. *Marginal cost* (*MC*) is the change in the cost per unit of input (see Fig. 2.3.3.4)

$$MC = \frac{dVC}{dy} = \frac{w}{\dfrac{dy}{dx}} \qquad (2.3.3.8)$$

This is the slope of the cost curve cost and its value is the cost of producing an additional unit of output.
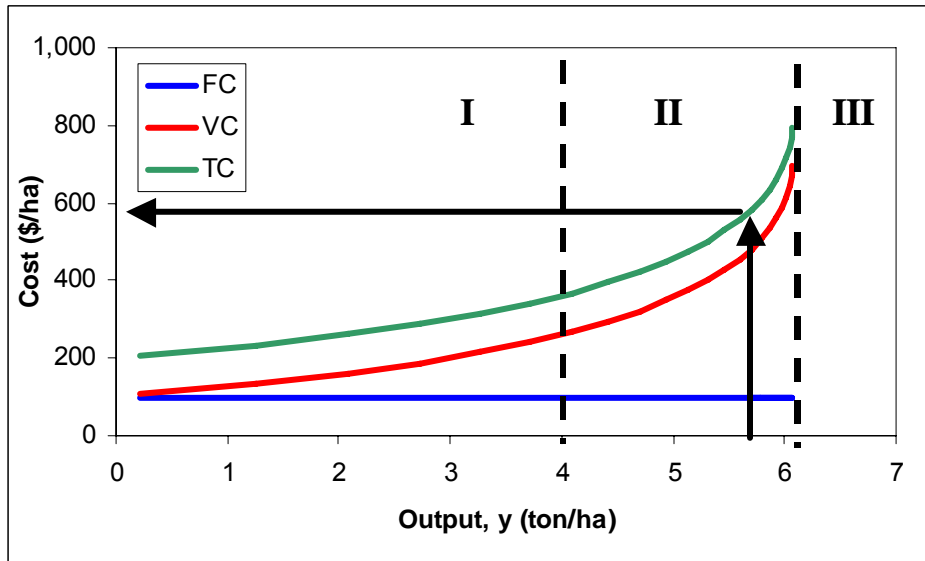
Figure 2.3.3.3. Fixed, variable and total costs versus output.
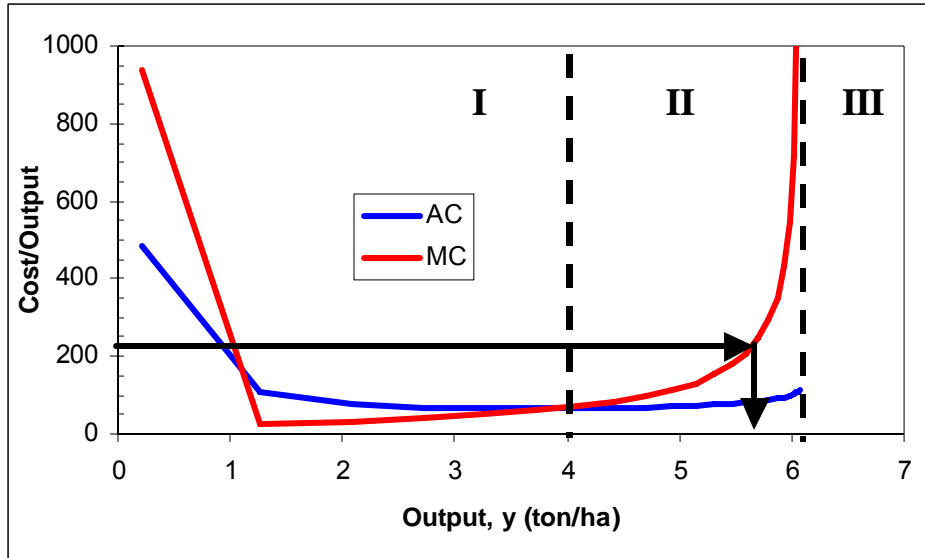
Figure 2.3.3.4. Average and marginal costs and the point where $p = MC$.

To determine the most profitable level of input, or the most profitable level of output, profit

$$\pi = py - wx \qquad (2.3.3.9)$$

is maximized, where w is the price of the produced good (wheat in this example and $p = \$230/mt$). The problem is to find the level of x which maximizes this function. The derivative with respect to input must be set equal to zero

$$\frac{d\pi}{dx} = p\frac{dy}{dx} - w = 0 \qquad (2.3.3.10)$$

We can rearrange this expression to yield

$$\frac{dy}{dx} = \frac{w}{p} \qquad (2.3.3.11)$$

or

$$p = \frac{w}{\dfrac{dy}{dx}} \qquad (2.3.3.12)$$

resulting in

$$p = MC \qquad (2.3.3.13)$$

This condition is shown in Fig. 2.3.3.4.
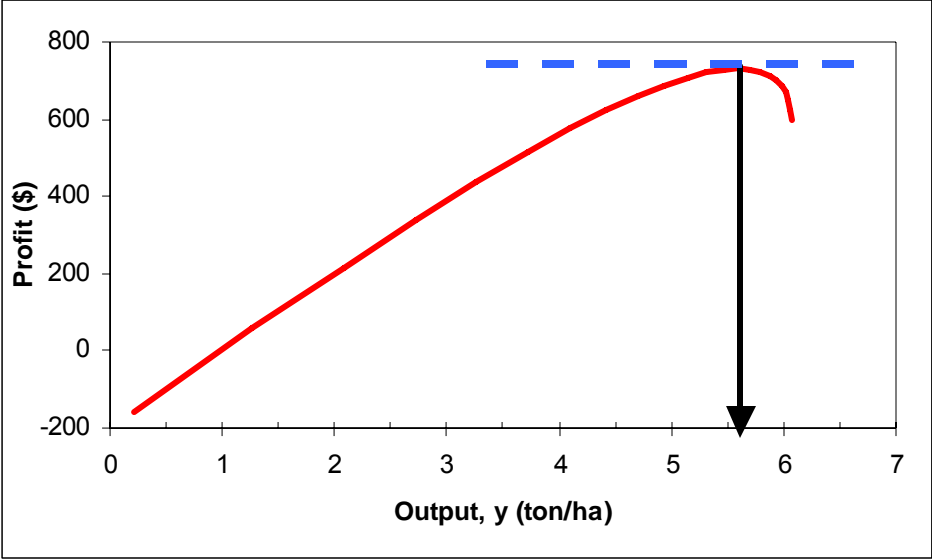
Figure 2.3.3.5. Profit versus output.

Table 2.3.3.3. Various production and cost data for wheat.

Input	Product	Ave. Prod.	Marg. Prod.	Fix. Cost	Var. Cost	Total Cost	Ave. FC	Ave. VC	Ave. TC	Marg. Cost	Total Val. Prod.	Profit
X	Y	AP=Y/X	MP	TFC	TVC	TC	AFC=FC/Y	AVC=AC/Y	ATC=TC/Y	MC	TVP= p_y*Y	
(m3/ha)	(mt/ha)	(mt/m3)	(mt/m3)	($)	($)	($)	($/mt/ha)	($/mt/ha)	($/mt/ha)	($/mt/ha)	($/ha)	($/ha)
2142	0.22	1.030E-04	4.122E-04	100	107.1	207.1	453.1	485.3	938.3	938.3	50.8	-156.3
3213	2.08	6.460E-04	1.518E-03	100	160.7	260.7	48.2	77.4	125.6	32.9	477.4	216.7
4284	3.27	7.628E-04	1.005E-03	100	214.2	314.2	30.6	65.5	96.1	49.7	751.6	437.4
5355	4.10	7.652E-04	7.094E-04	100	267.8	367.8	24.4	65.3	89.7	70.5	942.4	574.7
6426	4.70	7.310E-04	5.167E-04	100	321.3	421.3	21.3	68.4	89.7	96.8	1080.4	659.1
7497	5.14	6.855E-04	3.812E-04	100	374.9	474.9	19.5	72.9	92.4	131.2	1182.0	707.2
8568	5.46	6.378E-04	2.807E-04	100	428.4	528.4	18.3	78.4	96.7	178.1	1256.9	728.5
9639	5.70	5.916E-04	2.032E-04	100	482.0	582.0	17.5	84.5	102.1	246.0	1311.5	729.5
10710	5.87	5.480E-04	1.416E-04	100	535.5	635.5	17.0	91.2	108.3	353.0	1349.9	714.4
11781	5.98	5.076E-04	9.151E-05	100	589.1	689.1	16.7	98.5	115.2	546.4	1375.4	686.4
12852	6.04	4.703E-04	4.992E-05	100	642.6	742.6	16.5	106.3	122.9	1001.6	1390.2	647.6
13923	6.07	4.359E-04	1.486E-05	100	696.2	796.2	16.5	114.7	131.2	3365.0	1395.9	599.7
14994	6.06	4.042E-04	-1.510E-05	100	749.7	849.7	16.5	123.7	140.2	-3311.3	1394.0	544.3
16065	6.02	3.749E-04	-4.100E-05	100	803.3	903.3	16.6	133.4	150.0	-1219.7	1385.4	482.1
17136	5.96	3.479E-04	-6.360E-05	100	856.8	956.8	16.8	143.7	160.5	-786.1	1371.1	414.3
18207	5.88	3.228E-04	-8.351E-05	100	910.4	1010.4	17.0	154.9	171.9	-598.7	1351.7	341.3
19278	5.77	2.995E-04	-1.012E-04	100	963.9	1063.9	17.3	167.0	184.3	-494.2	1327.8	263.9

2.3.4 Opportunity Cost

Private firms operating in a market value productive resources as the cost to procure them in the market. We need a somewhat broader concept of cost here. The question we ask ourselves is: "What could have been produced with these productive inputs had they not been used in the current alternative?" This is the opportunity cost of using the inputs in the current alternative being considered. It is the maximum value of the other outputs we could have produced had we not used the resources to produce the item in question (Field, 1994). Opportunity costs include the out-of-pocket expense that the private firm operating in a competitive market considers, but they are broader than this.

Example: A manufacturing process may produce waste products that are discharged to a nearby stream. Downstream these production residuals produce environmental damage, which are the real opportunity costs of the production process, even though they do not show up as costs in a profit-and loss statement.

2.3.5 Average Cost Pricing

The demand curve (*D-D*) of the consumers of a water utility and the utility's Average Cost (*AC*) and Marginal Cost (*MC*) curves are given in Figure 2.3.5.1. Recall that the *MC* is less than the *AC* where the latter is declining and greater than *AC* where the latter is rising. If a single price is charged so as to "cover" costs, while clearing the market, that price can only be equal to OT, since at a price OT, the quantity OA would be demanded, the production of which involves an average cost of OT. At this solution, zero profits are earned; price equals unit cost. But this is not the solution that corresponds to the best use of society's resources. To see this, consider the units of output between OB and OA. For each of these units the marginal cost--the additional cost of producing the unit considered--is greater than the amount anyone is willing to pay for the extra unit supplied---the consumers' marginal value in use (*D-D*, the marginal willingness to pay). The quantity OB is demanded at price OU, and, if any larger quantity is to be taken by consumers, the price will have to be reduced below OU. But the marginal cost is higher than OU throughout the range being considered, which means that there are alternative uses of the resources entering into this *MC* which consumers value more highly than what those resources can produce in the use considered here. The solution for best use of resources is to produce just up to the point where the *MC* begins to exceed the price that consumers are willing to pay for the additional unit produced; that is, the correct output is OB at the marginal-cost price OU. (Hirshleifer et al., 1960)
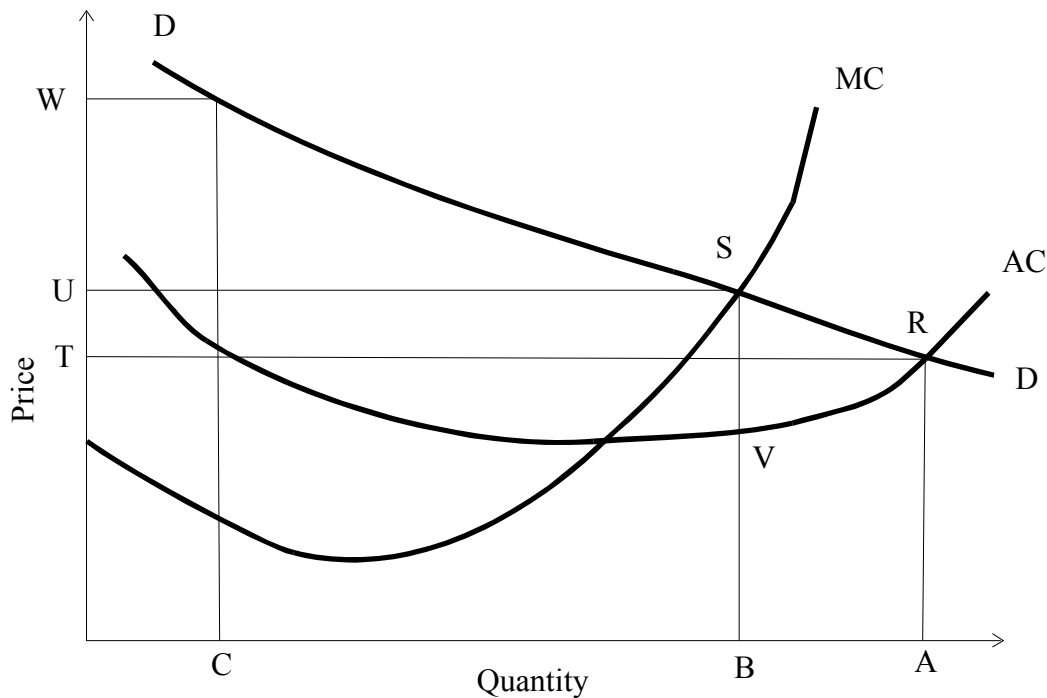
Figure 2.5.1. Average cost (AC), and Marginal cost (MC) curves. The optimal level
of production occurs where the demand intersects the MC curve.

2.3.6 Criteria for Decision Making

Efficiency can be defined as an allocation of resources where the net benefits from the use of those resources is maximized. Net benefits are the excess of benefits over costs. How do we measure benefits and costs?

Environmental goods and services have costs associated with them even when they are produced without human input. Opportunity costs are the net benefits foregone because resources providing services can no longer be used in the next most beneficial use. For example, consider a river. Possible uses include (1) white-water rafting, and (2) hydroelectric power production. Constructing a dam for the purpose of power production would flood rapids that are used for the purpose of rafting. The opportunity cost of saving the river for rafting is the net benefit of the power production that is foregone. The marginal opportunity cost curve is the supply curve for a good in a competitive market. The total cost equals the area under the marginal cost curve. The net-benefits are the area under the demand curve above the supply (marginal cost) curve. The net-benefits will be maximized when the output level and price are set at the point where the marginal benefit and marginal cost curves intersect.

This section is largely adapted from Hutchens and Mann[2]. Figures 2.3.6.1 – 2.3.6.4 illustrate the roles of demand and supply functions, in the theoretical derivation of consumers' and producers' surpluses, and show the proportioning of each as a result of market interaction. Figure 2.3.6.1 presents a demand curve and the total utility derived by consumers in the consumption of quantity x_0. The negative slope of the demand curve is derived from the definition of demand, which states that for any commodity that can be purchased in a market, the quantity demanded in a given period of time varies inversely with the price, other things equal. The demand curve consists of the locus of points of marginal utility associated with each incremental unit of a commodity consumed. Consequently, total utility is the integral represented by the area under the demand curve.

The area under the demand curve within the points O, p_1, A, and x_0 represents the maximum amount consumers would be willing to pay for the consumption of x_0 units of the commodity rather than go without it. This maximum willingness to pay reflects the total utility or benefit to the consumer. However, resources were expended to produce that output and the value of those expended resources must be deducted from the total benefit to determine the net benefit. Figure 2.3.6.2 illustrates the cost of resources (factors of production) required to produce x_0. The supply curve represents the locus of marginal cost associated with producing each increment of commodity x. The integral of that function, represented by the area under the supply curve delineated by points O, p_2, A, and x_0, is the total value of the resources required to produce x_0. This cost represents the minimum amount that the producer will accept for x_0 units and, therefore, the minimum amount that the consumer must pay.

Price, p

p_1

A

Total
Utility

Demand

0

x_0

Output, y
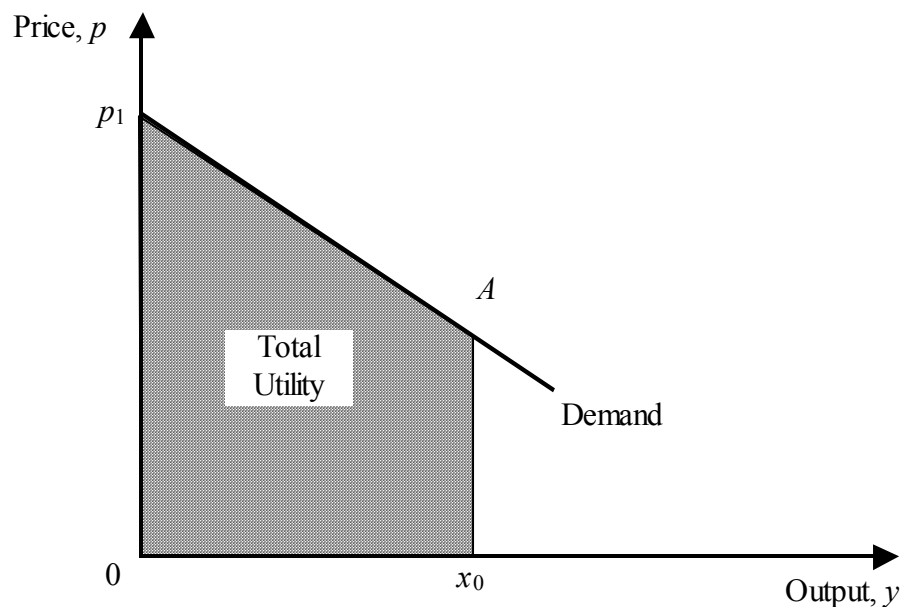
Figure 2.3.6.1. Net-benefits for a linear demand function and constant marginal costs.

2 Hutchens, A.O., and P.C. Mann , Review of Water Pricing Policies, Institutions and Practices in Central Asia, Environmental Policy and Technology Project, US Agency for International Development, Almaty, Kazakhstan, 1998.
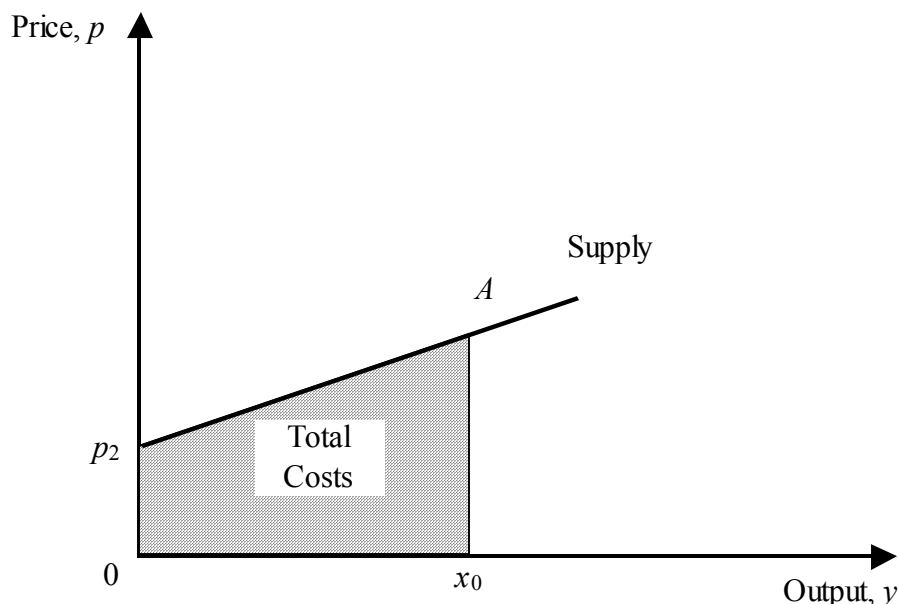
Figure 2.3.6.2. Marginal Opportunity Cost Curve for a constant marginal costs.

Figure 2.3.6.3 presents the results of superimposing Figure 2.3.6.2 onto Figure 2.3.6.1. The total utility or benefit illustrated in Figure 2.3.6.1 minus the total factor cost in Figure 2.3.6.2 yields the total surplus net of resource costs delineated by p_2, p_2, A. This, then, represents the difference between the <u>maximum the consumer would be willing to pay</u> rather than go without and the <u>minimum he must pay</u> in order to cover costs of production. It can also be viewed as the total net benefit to society.

The crucial issue of how this surplus or net benefit is shared or proportioned between producers and consumers is determined by the interaction of supply and demand in the market to determine the market price, which is illustrated in Figure 2.3.6.4. The area O, p_0, A, x_0, represents the amount that the consumer actually pays and, also, the amount that the producer actually receives. Therefore, the price line p_0, A, divides the total surplus into the amount the consumer would have been willing to pay, but did not have to, (consumers' surplus) and the amount in excess of what the producer would have been willing to accept, but was able to realize more (producers' surplus). <u>Total costs shared in proportion to producers' and consumers' surpluses will be shared in proportion to benefits received, which satisfies the economic equity criterion</u>**.** For most commodities, this would be automatically taken care of if there were an open competitive market; however, water and the necessary infrastructure to control it tend to exhibit some common property inflexibility and irreversibility characteristics that hinder a purely competitive market reallocation of water.

Output from the Egyptian Agricultural Sector Model (EASM) was used to derive estimates of consumers' and producers' surpluses under both financial and economic (free market) prices (Huchens and Mann, 1998). A run of EASM89 model derived the following estimates of producers' and consumers' surpluses:

	Base Case (1986)		Free Market (EASM89)	
Consumers' surplus	10067	55	6767	32
Producers' surplus	8236	45	14662	68
Total surplus	18303	100%	21449	100%

This shows that under 1986 financial price conditions, i.e., actual price controls and subsidies, consumers realized 55% and producers 45% of the "surplus value" in the agricultural sector. Under free market conditions, i.e., elimination of price, area and procurement controls, consumption subsidies, input subsidies and trade barriers, the proportions were estimated to be 32% consumers' surplus and 68% producers' surplus. The point to be realized from this is that, under free market conditions where farmers are not restricted by production quotas and administratively set prices, they will realize financial benefits that will enable him to pay for water services.
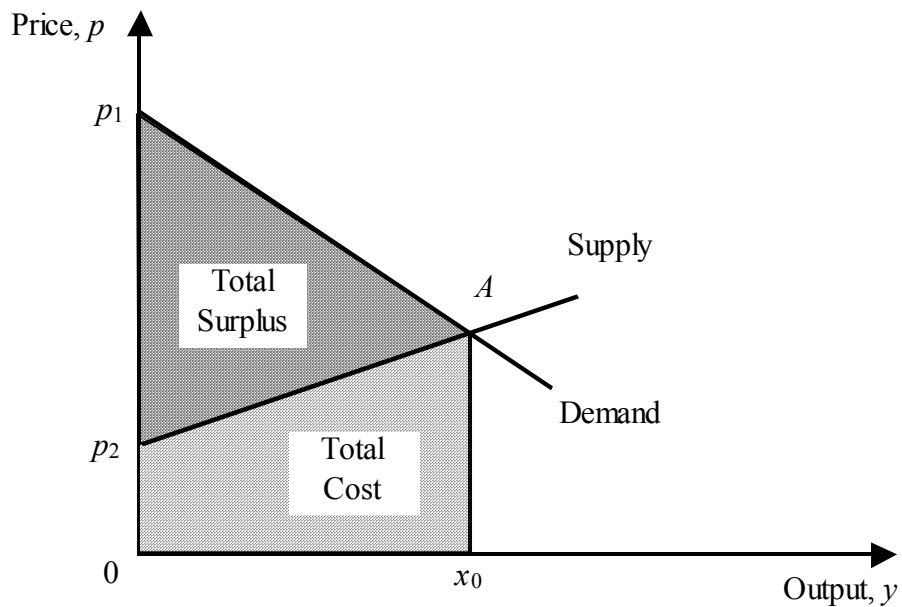
Figure 2.3.6.3. Maximum net-benefits for linear demand and constant marginal costs.
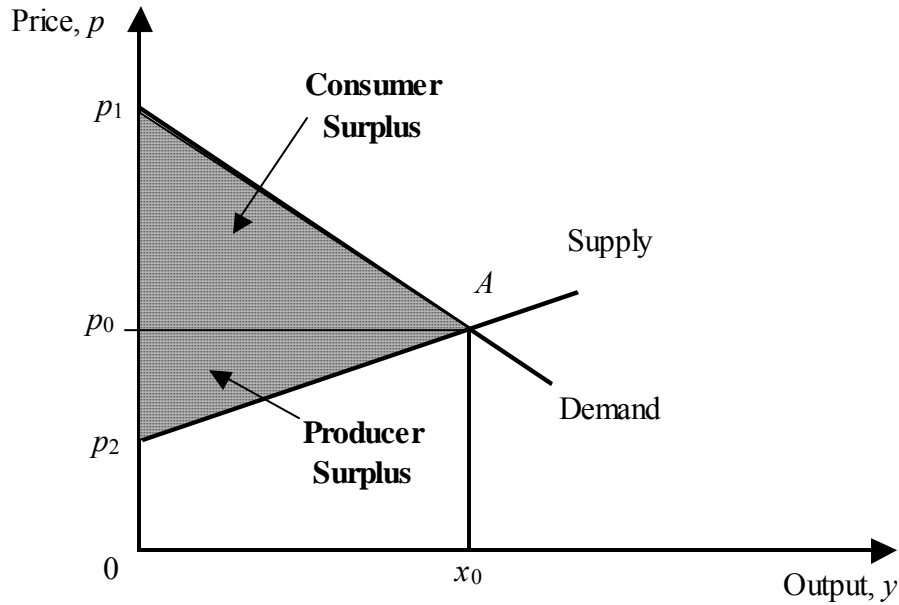
Figure 2.3.6.4. Maximum net-benefits for linear demand and constant marginal costs.

2.3.7 Externalities

The exclusivity of property rights is often violated, e.g., when a decision maker does not bear all of the consequences of a decision. Consider an example where a factory is producing a product and discharging waste to a nearby river. A hotel downstream of the factory uses the river for recreation. If there are different owners for the factory and the hotel, then an efficient use of the water in the river is not likely to occur. That is, the factory owner may not bear the cost of reducing business at the hotel as a result of the production decisions and resulting effluent discharge. The factory is likely to discharge too much effluent for a socially optimal solution.

An externality exists whenever the welfare of some agent (firm or consumer) depends on its own activities and the activities of some other (external) agent as well. In the above example, the additional cost to the hotel as a result of the factory discharge is an externality.
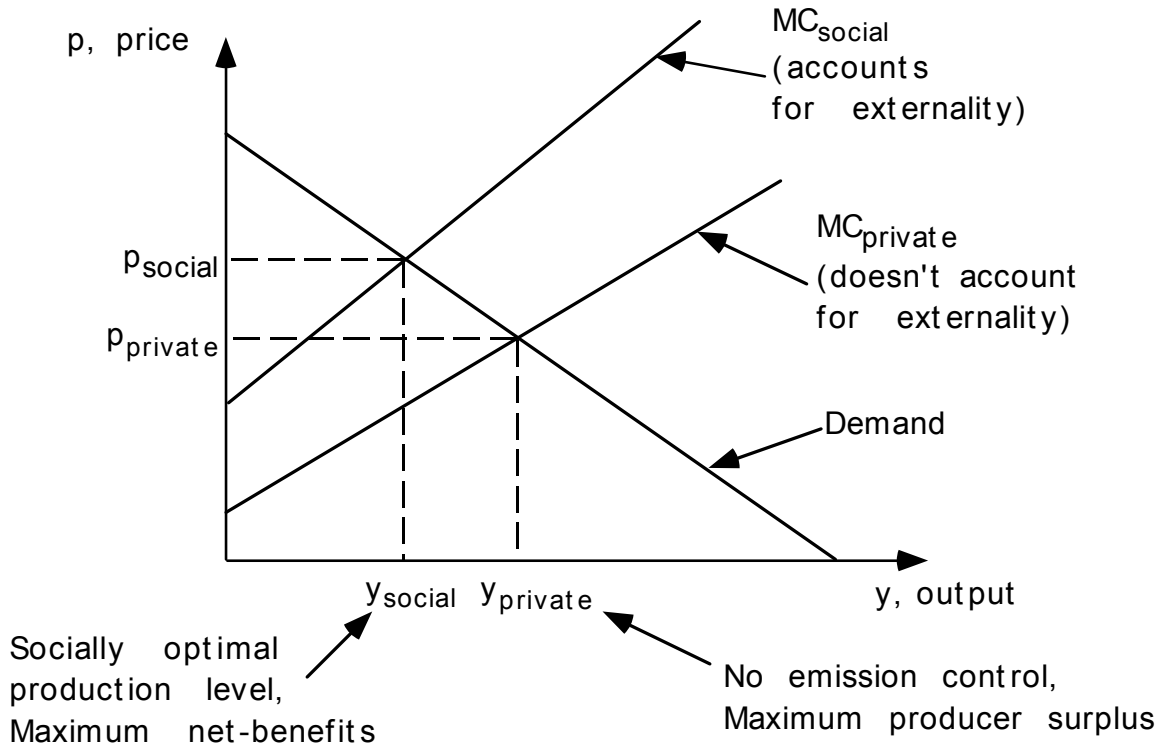
Figure 2.3.7.1. Effect of externality on production.

2.3.8 Production of Multiple Outputs

Previously, we have dealt with the production of a single output from multiple inputs. Suppose one input, x (water, say), can be used to produce two products, y_1 (irrigation, say) and y_2 (recreation, say) and that all other inputs to produce the outputs are fixed. So, the resource manager must decide how much input to allocate to the production of each output. If input x is unlimited, then the answer is found from equating the price of the input to the value of the marginal product of the input in production. When the input is limited, then the optimum amount of input can not be used in production of each output.

Production possibility curves (product transformation curves) represent the combinations of products that can be produced with a given set of inputs. Each point on the curve represents combinations of outputs produced using equal amounts of the input. A production possibility curve can be derived from two production functions (see Figure 2.3.8.6 and Table 2.3.8.1). The production functions use the same input x (water). Suppose that 10,000 m^3/ha of water are available. By using all 10,000 units of input on y_1, we can produce 5.8 mt/ha of wheat, or if all 10,000 units are used in y_2, 7.5 mt/ha of corn can be produced. We can consider many combinations between these two extremes. These combinations represent some of the production possibilities for 10,000 units of input; they are presented in Table 2.3.8.1 and Figure 2.3.8.2.
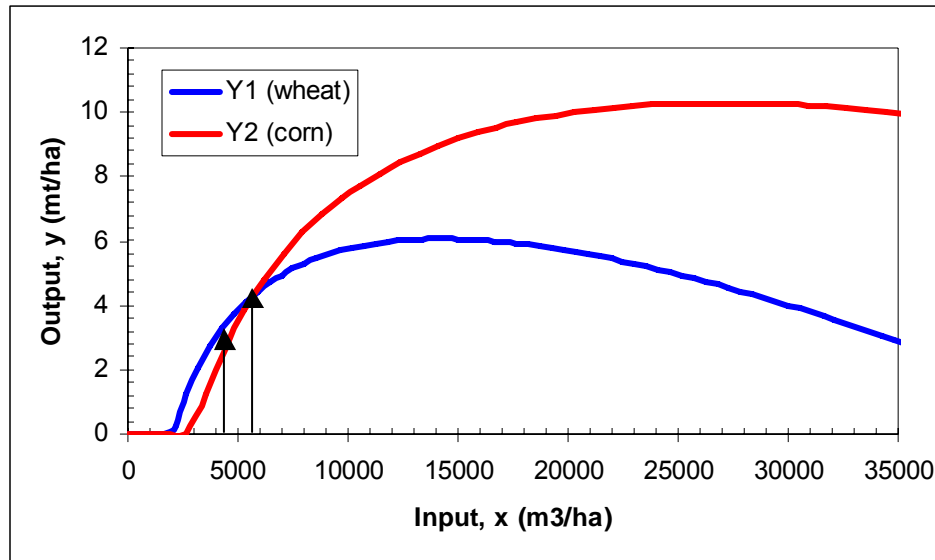
Figure 2.3.8.1. Production functions for wheat and corn.

Table 2.3.8.1. Production Functions for Wheat and Corn.

Production functions				**Production Possibilities** **for x = 10000 m3/ha**	
for wheat and corn					
Water	Wheat	Water	Corn	Wheat	Corn
x	y_1	x	y_2	y_1	y_2
(m3/ha)	(mt/ha)	(m3/ha)	(mt/ha)	(mt/ha)	(mt/ha)
0	0.00	0	0.00	5.8	0.0
536	0.00	883	0.00	5.6	0.0
1071	0.00	1765	0.00	5.3	0.0
1607	0.00	2648	0.00	4.9	0.5
2142	0.22	3530	1.21	4.5	2.0
3213	2.08	5295	3.87	3.8	3.5
4284	3.27	7060	5.60	3.0	4.6
5355	4.10	8825	6.83	1.8	5.5
6426	4.70	10590	7.74	0.2	6.3
7497	5.14	12355	8.43	0.0	6.9
8568	5.46	14120	8.96	0.0	7.5
9639	5.70	15885	9.37		
10710	5.87	17650	9.68		
11781	5.98	19415	9.91		
12852	6.04	21180	10.08		
13923	6.07	22945	10.19		
14994	6.06	24710	10.26		
16065	6.02	26475	10.28		
17136	5.96	28240	10.27		
18207	5.88	30005	10.23		
19278	5.77	31770	10.17		

Figure 2.3.8.2. Production possibilities curve for wheat and corn with input of 10,000 m³/ha of water.

Now, let's consider the problem of the farmer in this case. The decision to be made is how to allocate the scarce resources, $x = (x_1,...,x_J)$, to the production of various combinations of outputs, $y = (y_1,...,y_I)$. The objective is to maximize the farm income while satisfying the constraints of the production function. In general, we can write this problem as (*Willis and Finney*, 2000)

$$\text{maximize } U(y,x)$$
$$x$$
$$\text{subject to}$$
$$f(y,x) = 0$$

(2.3.8.1)

Form the Lagrangian function

$$L(y,x,\lambda) = U(y,x) - \lambda f(y,x)$$

(2.3.8.2)

The optimality conditions are

$$\frac{\partial L}{\partial x_j} = \frac{\partial U}{\partial x_j} - \lambda \frac{\partial f}{\partial x_j} = 0, \qquad j = 1,...,J$$

$$\frac{\partial L}{\partial y_i} = \frac{\partial U}{\partial y_i} - \lambda \frac{\partial f}{\partial y_i} = 0, \qquad i = 1,...,I \tag{2.3.8.3}$$

$$\frac{\partial L}{\partial \lambda} = f(\boldsymbol{y}, \boldsymbol{x}) = 0$$

Now, if the second of these equations is written for two products i and k, we have

$$\frac{\partial U}{\partial y_i} - \lambda \frac{\partial f}{\partial y_i} = 0$$

$$\frac{\partial U}{\partial y_k} - \lambda \frac{\partial f}{\partial y_k} = 0 \tag{2.3.8.4}$$

or

$$\frac{\partial U / \partial y_i}{\partial U / \partial y_k} = \frac{\partial f / \partial y_i}{\partial f / \partial y_k} \tag{2.3.8.5}$$

but, from the total derivative of the production function, we have

$$\frac{\partial f}{\partial y_i} dy_i + \frac{\partial f}{\partial y_k} dy_k = 0 \tag{2.3.8.6}$$

or

$$\frac{\partial f / \partial y_i}{\partial f / \partial y_k} = -\frac{dy_k}{dy_i} \tag{2.3.8.7}$$

so

$$\frac{\partial U / \partial y_i}{\partial U / \partial y_k} = -\frac{dy_k}{dy_i} = MRS_{i,k} \tag{2.3.8.8}$$

where $MRS_{i,k}$ is the marginal rate of substitution of product i for product k which is the slope of the production possibilities curve, and $\partial U / \partial y_i$ is the marginal benefit from producing an additional unit of product i, that it, its price p_i. Total revenue is the value of the output produced:

$$TR = p_1 y_1 + p_2 y_2 \tag{2.3.8.9}$$

For various given values of total revenue (*TR*) this relationship is a straight line (*isorevenue line*). The distance of the isorevenue line from the origin is determined by the value of *TR*. As *TR*

increases the line moves away from the origin. The slop of the isorevenue line is determined by the prices of the outputs.

Total costs are constant for all combinations of outputs on the production possibility curve. Profits will be maximized if the output combinations with the maximum *TR* is selected. This will be achieved at the point where the slope of the isorevenue line and the production possibility curve coincide.

Consider the case where

$$\text{Maximize } p_1 y_1 + p_2 y_2 + FC$$
$$\text{subject to}$$
$$f(x, y_1, y_2) = 0$$

(2.3.8.10)

The optimality conditions are

$$\frac{\partial L}{\partial x} = p_1 \frac{\partial y_1}{\partial x} + p_2 \frac{\partial y_2}{\partial x} - \lambda \frac{\partial f}{\partial x} = 0$$

$$\frac{\partial L}{\partial y_1} = p_1 - \lambda \frac{\partial f}{\partial y_1} = 0$$

$$\frac{\partial L}{\partial y_2} = p_2 - \lambda \frac{\partial f}{\partial y_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = f(x, y_1, y_2) = 0$$

(2.3.8.11)

Now, from the second and third equations, we have

$$\frac{p_1}{p_2} = \frac{\partial f / \partial y_1}{\partial f / \partial y_2}$$

(2.3.8.12)

but, from the total derivative of the production function, we have

$$\frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2 = 0$$

(2.3.8.13)

or

$$\frac{\partial f / \partial y_1}{\partial f / \partial y_2} = -\frac{dy_2}{dy_1}$$

(2.3.8.14)

so

$$\frac{p_1}{p_2} = -\frac{dy_2}{dy_1} = MRS_{1,2} \qquad (2.3.8.15)$$

Thus, we see from these results that the point of optimal production is where a line with slope equal to the ratio of the prices of the outputs is equal to the slope of the production possibilities curve. This is illustrated in Figure 3.8.2 for our example of wheat and corn production. The prices are $p_1 = $230/mt$ (wheat) and $p_2 = $160/mt$ (corn). The resulting outputs are 3.3 mt of wheat (which uses 4284 m^3/ha of water) and 4.2 mt/ha of corn (which uses 5606 m^3/ha of water). The resulting total revenue is $1450/ha.

2.4 Water Rights and Markets

2.4.1 Introduction

As the costs of water supply development increase it is increasingly important that supplies be allocated more efficiently than in the past. Systems of water allocation with nontransferable water rights can lead to rigid, inflexible, and inefficient allocations of water (Gibbons, 1986; Howe et al., 1986). Several advantages of private ownership and market exchange over bureaucratic control and allocation of water often exist. Markets have been established which have been successful in transferring water from low-valued to higher-valued uses over time (Rosegrant and Binswanger, 1994), where value is defined as the maximum amount a user would be willing to pay for the use of water (Gibbons, 1986). However, the establishment of water markets are often inhibited by the presence of externalities (third-party effects) such as increased pollution or changes in return flows. These effects must be accounted for in deciding a water right transfer and losing parties must be compensated for these effects (Howe, 1996).

Water is a necessary and scarce resource for the sustenance of human society and culture. However, the allocation of water to beneficial uses is a difficult problem, quite different from other resource allocation problems commonly considered in economics (Tregarthen, 1983). The quantity of water available in a river basin will fluctuate year-to-year. The interactions between surface water and groundwater are complex and not well known in many basins. Economies of scale exist in developed storage and distribution systems, encouraging the development of large systems that may be inflexible and non-robust over the long run. Externalities often exist and may require public sector intervention. The consumptive use of water in many cases may lead to the degradation of in-stream flow values and the loss of those benefits.

Several policy options exist that may lead to increases in water use efficiency while reducing environmental degradation and releasing water for increasing demands in other economic sectors (Rosegrant and Binswanger, 1994):

1. **Technological solutions:** (a) construction of new water resource systems, and (b) rehabilitation and modernization of existing systems, e.g., canal and drainage lining, and field drainage in irrigation systems.

2. **Reform of public management of water resource systems:** (a) modification of water distribution methods, (b) implementation of water pricing policies, and (c) reform of water management bureaucracies.

3. **Communal water resource system management** which involve water users more directly in both the process of system management and improvement.

4. **Establishment of tradable property rights** in water and development of markets in these rights. Market allocation of resources may be efficient given well-defined and nonattenuated (completely specified, exclusive, transferable, and enforceable) initial property rights allocation and low transaction costs.

Options 1 - 3 have been widely used by international lending institutions and national governments. Option 4 is somewhat new and is explored below.

2.4.2 Water Rights

Water law differs from country to country and within the US it varies from state to state. Traditionally, water law has been based on common law, but more recently it has shifted a bit toward legislative law. Most common law is based on protecting the quantity, not the quality of water. Common law is comprised of traditional legal aspects laid down by court decisions and it is based on precedent set by older cases. Common law can be overturned as the needs of society change over time. In the US, common law derives its legitimacy from the constitutions (US and States). Legislative law, on the other hand, is comprised of statutory law, where the legislature passes laws that regulate water, and administrative law, where the legislature may enable administrative bodies to write rules and regulations that have the power of law.

A well defined system of water rights is a necessary condition for the development of water markets. Water rights can be vested with individual citizens or with the government (local, state or national). Water rights are essentially a bundle of entitlements defining a water right owner's:

1. rights;
2. privileges; and
3. limitations

for the use of the water. Water rights are generally treated as real property with the right holder having a usufructuary right to make use of the water but not a right to physical possession of the water (Hirshleifer, et al., 1960). These rights must be well defined and exclusive to the person or entity owning them. Such a system of water rights must completely specify the (Howe, et al., 1986):

1. quantity of water that may be diverted;
2. quantity of water that may be consumed;
3. timing of the water delivery;
4. quality of the delivered water;
5. place of diversion; and

6. place of application.

Changes in any of these characteristics will likely affect other water users in the basin or system.

An efficient water rights structure should include:

- **Universality:** Rights must be privately owned, and entitlements must be completely specified.

- **Exclusivity:** Benefits and costs accrued as a result of owning and using the rights accrue to the owner (and only the owner). This property of water rights is often violated, e.g., when a user does not bear all of the costs of a water allocation, creating an externality that is not paid for by the user. An externality may exist whenever the welfare of some user depends on its own activities and the activities of some other user(s) as well.

- **Transferability:** The transfer of rights from one user to another must be entirely voluntary. The inability to transfer the rights would prevent the owner of the right from recognizing the true opportunity cost of the water, i.e., the value that another person may place on it.

- **Enforceability:** Right owners must be secure from involuntary seizure of their rights or encroachment on their rights.

Usually, systems for water rights (implicit or explicit) fall into one of several categories (Howe, 1996):

- **Non-tradable permits for water from undeveloped (natural) supplies**
  Non-tradable permits or rights are typically specified by laws or regulations, they are for specific or defined periods and they are not tradable. Problems associated with this type of arrangement include the fact that this method of water allocation does not consider the economic efficiency or equity of the use, and allocations may be inflexible and unresponsive to changes in social values.
- **Contracts for water from developed supplies**
  Developed supplies usually provide storage and distribution facilities and water is allocated to customers by contracts (as opposed to non-tradable permits where water is distributed by water right). Contracted water supply is usually for a specific use. Problems associated with this type of arrangement include the fact that the economic efficiency or equity of the water use is not considered.

In many cases, water demand is estimated from the projected requirement or need for water, that is, the farmer's ability to put water to use (Gardner, 1983). This method of demand estimation results in a maximization of physical yield rather than profit or social benefits. This can lead to the development of new water supplies rather than using economic incentives and market mechanisms to allocate water to its best uses. Systems of this type are common in the states of California (Gardner, 1983) and Texas (TWDB, 1997).

In other cases, a fixed cost per hectare of crop is charged for water. This violates economic principles since the price is not related to the quantity of water applied or used, and there is little

incentive to conserve water. This type of system is often justified by the difficulty and expense of determining how much water is delivered to a farm.

2.4.2.1 Riparian water rights systems

Under riparian water rights systems, the owner of land bordering a stream or lake has the right to take water for use on the land. The right to use the water exists solely because of the relation of the land to the water and resides in the ownership of the land. The first riparian user acquires no priority over those who may use the stream at a later date; the rights of upstream and downstream users are viewed as being coequal (Hirshleifer et al., 1960).

Under riparian systems, the owners of lands bordering water bodies may have "reasonable use" of the waters, provided that the water is returned undiminished in quality or quantity (Howe et al., 1986). That is, the withdrawal must be reasonable with respect to the requirements of the other riparians. The determination of what constitutes reasonable use is left to the courts. A riparian right subject to the reasonable use doctrine has no guaranty to a definite quantity of water. Under the riparian system, the transfer of water rights between competing uses by a market system is severely hampered (Hirshleifer et al., 1960).

Riparian rights are most appropriate for humid regions. Where water is truly scarce and/or where water quality problems are important, the riparian doctrine simply doesn't work (Howe, 1996).

2.4.2.2 Appropriative water rights systems

The doctrine of appropriation gives no preference to the use of water by riparian landowners. Appropriative (or prior) systems tend to exist in areas of water scarcity where users are located away from water bodies. Scarcity means that each succeeding appropriation results in fewer or less valuable resources available for other users. Scarce resources come to be appropriated in their natural state according to the principles of priority of right and beneficial use (Cuzan, 1983).

The earliest water right on a given watercourse has preference over later users, "first in time means first in right." Once the appropriation is granted, it becomes senior to subsequent appropriations. In times of shortage senior or older rights have precedence over junior or newer rights. That is, senior rights have first call on available water. Appropriative rights are a right to use, not a right to own, and the beneficial use of the water is required. Beneficial use has been described as use of water in a useful industry or to supply a well-recognized want (Tregarthen, 1983). In many cases, the owner of an appropriation may lose the right as a result of failure to put it to beneficial use.

The two rules of appropriative water rights, priority and beneficial use, result in the separation of rights to water from the rights to land. Persons can mobilize capital to build water supply works and transport water to wherever it is most productively used (Cuzan, 1983). Appropriative rights may be a system in which rights are clearly defined and transferable subject to the stipulation of "no injury" (Tregarthen, 1983). It is the severability of appropriative rights that causes them to be transferable.

Often under appropriative rights systems water is owned by the public and appropriators are granted the right to use the water but ownership of the resource remains with the state (Cuzan, 1983). Often this state expropriation of water rights leads to a system of controls which makes it difficult for water to be transferred privately through sales. These systems can generate pressure for monumental water schemes by governmental agencies which subsidize low-value water uses.

Economic efficiency requires that the marginal value of water used be equal in each use, net of transport costs, and assuming that marginal values include both private and social benefits and costs (Gardner, 1983; Howe et al., 1986). Assuming that water is homogeneous, i.e., no quality variations, water prices should vary among users only by the cost of moving it from one user to another (Hirshleifer, et al., 1960).

Two main types appropriative rights systems are common: priority rights and proportional rights systems

Priority rights

Priority rights operate on the doctrine of "first in time, first in right." If the flow in a river is sufficient to provide only x% of the water appropriated, then a call for water from the senior water rights holders can shut off diversions to the lowest (100 - x)% priority rights holders. Senior water rights holders have less risk than junior rights holders, but senior rights holders may place a lower value on the last unit of water than junior rights holders. In this case, a trade should occur, the senior rights holder selling water rights to junior rights holders, thus reducing the risk to the junior appropriators (Tregarthen, 1983). Priority rights allow different degrees of water supply reliability to be purchased, but the heterogeneous nature of the rights makes it difficult to organize markets.

Proportional rights

A proportional rights system shares available water among users according to a set of percentages determined by the number of rights owned, e.g., if a user owns 10 rights out of 100, then the owner is entitled to 10% of available water. Proportional rights systems require the purchase of more shares to reach any given level of assurance of water supply. The homogeneity of proportional rights makes it much easier to create markets than under the priority system.

2.4.3. Water Markets

2.4.3.1 Right to divert or consume

The right to transfer water may not be the amount of water appropriated to the use, but the "duty of water" at the point of use (Tregarthen, 1983). This concept limits the transfer of water rights, based on the consumptive use by the seller and the prospective consumptive use of the buyer. In many situations it is desirable to protect downstream users from a loss of water due to an upstream water rights trade. When rights are transferred, the use of the water may change, and with it the amount of

water that is consumed. The result may be a change in the amount of water available to downstream users. Often it is desirable to limit transfers so that the amount of water consumed is not changed. Consider the following example adapted from Gisser and Johnson (1983) where there are three users along a river and flow into the system is 1000 units per time period (see Fig. 2.4.3.1.1). User 1 diverts $S_1 = 1000$ units and has a return flow coefficient (R_1) of 0.5, that is, User 1 consumes $C_1 = 500$ units of water. Downstream, User 2 diverts $S_2 = 500$ units and has a return flow coefficient (R_2) of 0.5, and User 3 diverts $S_3 = 250$ units and has a return flow coefficient (R_3) of 0.5. The total diversion is 1,750 units of water, a greater amount than the initial streamflow.

$\hat{S} = 1000$ $\overline{S}$

| $S_1=1000$ | R_1*S_1 $=500$ | $S_2=500$ | $R_2*S_2=250$ | $S_3=250$ | $R_3*S_3=125$ |

User 1   User 2   User 3

$C_1=(1-R_1)*S_1=500$ $C_2=(1-R_2)*S_2=250$ $C_3=(1-R_3)*S_3=125$
$B_1(S_1)=\$1000$ $B_2(S_2)=\$500$ $B_3(S_3)=\$250$

Total Diversion = 1,750
Total Benefit=$1,750
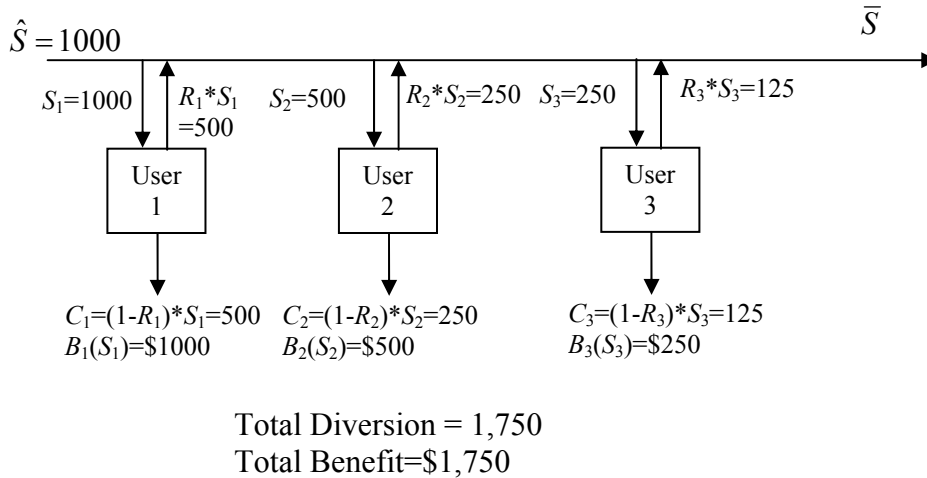
Figure 2.4.3.1.1. River system for water allocation.

Now consider that User 1 decides to sell his/her entire diverted amount to another user outside the basin for $1.1 per unit. The net result is that User 1 is no better off than before, but Users 2 and 3 have been left without any water to divert and there is an overall net loss for the basin of $650

$\hat{S} = 1000$ $\overline{S}$

| $S_1=1000$ | R_1*S_1 $=0$ | $S_2=0$ | $R_2*S_2=0$ | $S_3=0$ | $R_3*S_3=0$ |

User 1   User 2   User 3

$C_1=(1-R_1)*S_1=1000$ $C_2=(1-R_2)*S_2=0$ $C_3=(1-R_3)*S_3=0$
$B_1(S_1)=\$1100$ $B_2(S_2)=\$0$ $B_3(S_3)=\$0$

Total Diversion = 1,000
Total Benefit=$1100
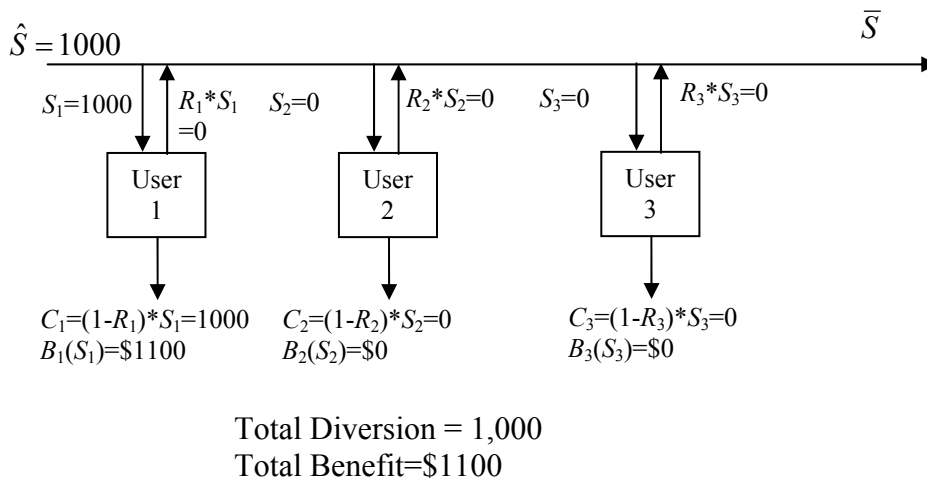
Figure 2.4..3.1.2. River system for water allocation without consideration of consumptive use.

An appropriator (user) may own a right to divert a given quantity of water but the user can only transfer this right according to the amount of water consumed. Determining the consumptive use of water and the amount of water that returns to the river or canal can be difficult and costly. Consider again the above example, but now User 1 decides to only sell the amount of his/her previous consumptive use (500 units) for a price of $1.1 per unit. In this case the downstream users continue to receive their water and may divert as before.

$\hat{S} = 1000$ $\overline{S}$

$S_1=500$ $R_1*S_1=0$ $S_2=500$ $R_2*S_2=250$ $S_3=250$ $R_3*S_3=125$

User 1	User 2	User 3

$C_1=(1-R_1)*S_1=500$ $C_2=(1-R_2)*S_2=250$ $C_3=(1-R_3)*S_3=125$
$B_1(S_1)=\$550$ $B_2(S_2)=\$500$ $B_3(S_3)=\$250$

Total Diversion = 1,250
Total Benefit=$1,300
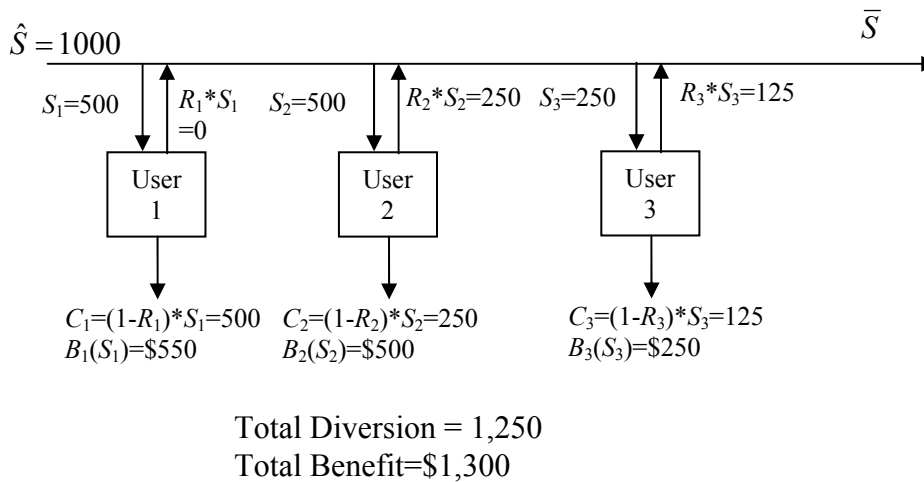
Figure 2.4.3.1.3. River system for water allocation with consideration of consumptive use.

Several authors have suggested that it is better to define water rights according to the consumptive use system rather than the diversion rights system. Using the consumptive rights system the ownership of the right is clear, transfers do not require litigation, the incentive to conserve water exists, and this conservation leads to water that can be sold to downstream users (Tregarthen, 1983). However, others have suggested that the diversionary rights system is preferable (Rosegrant et al. 1995).

The definition of the tradable portion of a water right depends on the method of handling return flows. In California, the tradable portion is limited to consumptive use (consumptive right) with protection of third-party rights to return flows. This method increases transaction costs because of the difficulty in measuring consumptive use and return flows. Consumptive use is defined to be the actual evapotranspiration of crops plus any water lost to deep percolation. Thus, the water available to trade includes water that would have been consumptively used and water that would be irretrievably lost to beneficial use. In Chile and Mexico, rights are proportional to streamflow (diversionary right) and rights to return flow are retained by the water authority. Return flows are made available to users at no charge, but no rights are assigned to these flows. Changes in return flows due to water rights trades are not actionable. This method has been demonstrated to reduce transaction costs. So the tradable water is the full diversion right which is proportional to stream flow.

What is the most appropriate method in developing countries? The transaction costs of enforcing consumptive rights increase but they protect third-parties against adverse impacts from water trades. If the lost benefits from not trading exceed the costs of adverse impact from lost return flows, then diversionary rights system is preferred. In general, the diversionary rights system will be preferred in developing countries so as to prevent high transaction costs, thus preventing the development of markets.

2.4.3.2 Tradable water rights markets

Tradable water rights are rights to use water that can be transferred all or in part, separately from the transfer of land (Rosegrant et al., 1996). Tradable water rights may be permanent, long-term, or even short-term. Tradable water rights markets may be capable of allocating water more effectively than other more restrictive and centrally controlled systems. Markets can operate most efficiently when the commodity being allocated is homogeneous (Howe et al., 1986). Heterogeneity of uses leads to difficulty in organizing a market, transmitting information to users, and matching sellers and buyers. Rights to water resources already exist in most countries (a) by custom, or (b) by law and regulation. Establishing tradable rights is a matter of reforming existing systems.

Characteristics of markets

Several desirable characteristics for water allocation mechanisms (regional, river basin or irrigation district level) have been described by Howe (Howe et al., 1986; Howe, 1996):

- **Flexibility over time**

  Water can be shifted from use to use and place to place as climate, demographics, and economic conditions change over time. Short-term (responding to climatic factors) and long-term (responding to demographic and economic factors) flexibility is necessary. It is important to note that not all water must be subject to reallocation, only a tradable margin must exist within each water-using area that is subject to low cost reallocation and this volume can be a relatively small part of the regional supply (Howe et al., 1986). Flexibility allows equating the marginal values in the water's various uses (Howe et al., 1986; Gisser and Johnson, 1983).

- **Security of tenure for established users**

  Water users must be assured of continued use or they will not invest in and maintain the water resource system. This encourages long-term investments that generate positive net benefits. In a market system, no one can be forced to sell.

- **Real opportunity costs of water**

  Valuation of water at its opportunity cost, the maximum value of outputs that could have been produced had inputs not been used to produce the item in question (Field, 1994), provides

incentives for users to shift from inefficient water uses and methods to more highly valued, less water-intensive uses and methods. Opportunity cost pricing can be implemented through the establishment of tradable water rights and development of markets in these rights (Rosegrant et al., 1996).

Water is often a scarce input to production and it is frequently priced well below its value in use (Gardner, 1983). Historically, the price of water, at most, has reflected the costs of its capture and distribution. The control of low priced water can provide access to enormous profits in many cases. A perpetual contract for the supply of water at a fixed price may fail to reflect changing opportunity costs involved in continued use. Water is one input to agricultural production, other inputs include land, capital, energy, chemicals, and labor. If production is to be profitable, all inputs used must be valued at least at their opportunity costs. A price for water established in a market and the ability to sell water (transfer or trade water rights) recognizes the real opportunity costs of the water in the use being considered. This prevents the acceptance of water uses that are less valuable than alternative uses.

It is desirable to maximize the scope of a water rights market so that transactions take place over as wide a geographical area and among as wide a variety of users as possible, subject to transaction cost limitations.

- **Differentiated risk-bearing**

  While old methods are familiar even if they are outmoded, new methods may increase uncertainty, even while they promise advantages. Predictability of the outcome of the transfer process is necessary to ensure that long-term investments that generate positive net benefits are encouraged.

- **Fairness to participants**

  Water users should not impose uncompensated costs (externalities) on other parties. Externalities occur whenever withdrawal, consumption, or quality changes by one user affect other water users. Parties giving up water should be compensated and those injured by changes in allocations should be compensated. Market transactions should guarantee fairness since no person will sell if they will not be made better off.

- **Protection of public values**

  Some values may be of little concern to individual water users and they may not be adequately reflected in the market exchange and these must be protected by social oversight, e.g., water quality and instream flows (Howe, 1996). Protection of public values will ensure that allocations will achieve the highest aggregate benefit level.

Benefits of tradable water rights markets

The benefits from establishing tradable water rights markets include (Rosegrant and Binswanger,

1994; Rosegrant et al., 1996):

- **Empowerment of water users** by requiring their consent to any reallocation of water and compensation for any water transferred;

- **Security of tenure** of water rights to the water users, which encourages investment in system efficiency improvements;

- Induces users to consider the **full opportunity cost** of water, including its value in alternative uses, providing incentives to efficiently use water and gain additional income through the sale of saved water;

- Provide incentives for users to **take account of external costs** imposed by their water use, reducing resource and environmental degradation;

- **Formalizes existing rights** to water; and

- **Provides maximum flexibility** in responding to changes in crop prices and water values.

Constraints of tradable water rights markets

Constraints to establishing tradable water rights markets leading to high transaction costs include (Rosegrant and Binswanger, 1994; Rosegrant et al., 1996):

- The unique physical, technological and economic **characteristics of water resources** systems pose problems;

- The **variable nature of water flow** makes achieving necessary certainty; and

- **Return flows** from water use can generate environmental degradation. Multiple reuse of water creates the likelihood of significant externalities imposed on third parties.

Policy considerations of tradable water rights markets

Policy considerations in developing tradable water rights markets include (Rosegrant et al., 1996):

- Definition of a method of initial allocation of water rights. This can be based on, among other things, historic water use (Chile and Mexico), fully appropriated existing rights (California);
- Type of rights, prior or proportional appropriative rights: Prior rights (California), Proportional (Chile and Mexico);
- Consumptive use or diversionary treatment of return flows;
- Indirect economic effects;
- Environmental protection;

- Water user associations;
- Infrastructure;
- Public and private institutions; and
- Regulations: Excessive regulation leads to high transaction costs, inadequate regulation leads to third-party costs or environmental degradation

2.4.3.3 Modeling tradable water rights markets

There are two fundamental strategies for dealing with water scarcity in river basins, supply management and demand management; the former involves activities to locate, develop, and exploit new sources of water, and the latter addresses the incentives and mechanisms that promote water conservation and efficient use of water (Rosegrant et al., 2000). Markets in tradable water rights can reduce information costs; increase farmer acceptance and participation; empower water users; and provide security and incentives for investments and for internalizing the external costs of water uses. Market allocation can provide flexibility in response to water demands, permitting the selling and purchasing of water across sectors, across districts, and across time by opening opportunities for exchange where they are needed. The outcomes of the exchange process reflect the water scarcity condition in the area with water flowing to the uses where its marginal value is highest (Rosegrant and Binswanger 1994; Rosegrant 1997). Markets also provide the foundation for water leasing and option contracts, which can quickly mitigate acute, short-term urban water shortages while maintaining the agricultural production base (Michelsen and Young 1993).

Water trading in a basin is constrained by the hydrologic balance in the river basin network; water may be traded taking account of physical and technical constraints of the various users, reflecting their relative profitability in trading prices; water trades reflect water scarcity in the basin that is influenced by both basin inflows and the water use plans of the users (Rosegrant et al., 2000).

The price that a water user would be willing to pay to acquire additional water must be determined for each user. This can be achieved by determining a shadow price – water withdrawal relationship can determined for each user. For this, a model must be run with varying water rights for each user as inputs and shadow prices or marginal values as output derived from the water balance equations (each user has a water balance equation in the model). If necessary, these shadow prices can be averaged over all uses for each user to obtain one shadow price for each water supply level for user. The Figure below shows the result of this for the problem of Exercise 1 below.
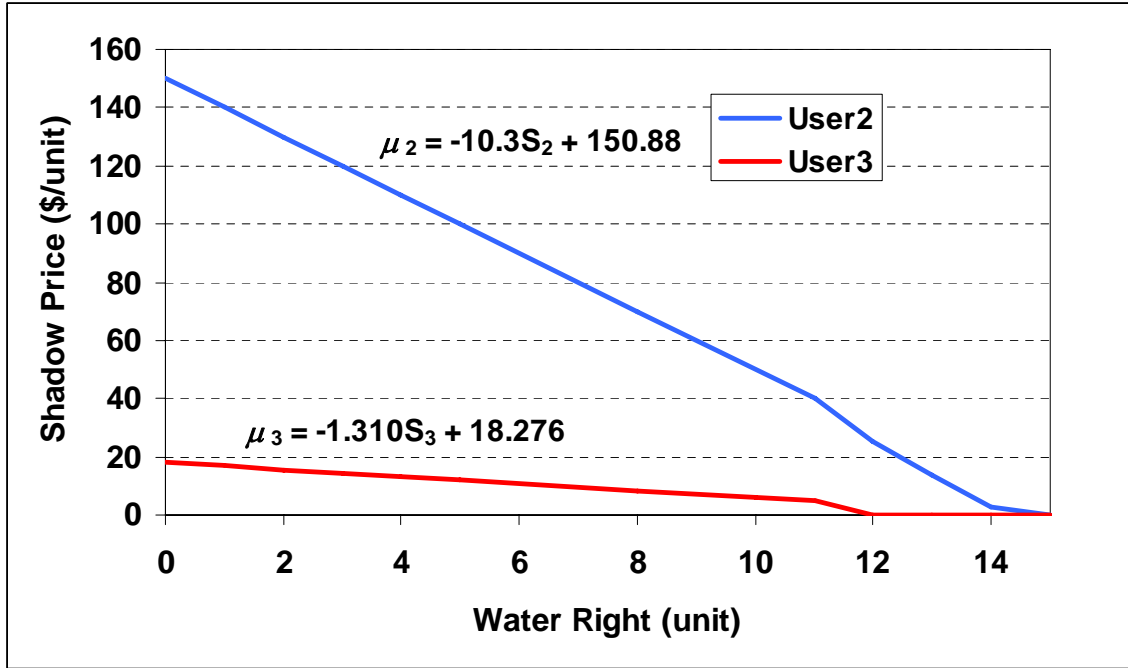
Figure 2.4.3.3.1. Shadow price for water users.

In the model of water trading, the objective is to maximize the combined benefits of all the users

$$\text{Maximize } B = \sum_i B_i = \sum_i (a_i S_i - b_i S_i^2 + \sum_j x_{i,j} \cdot wtp_j - \sum_j x_{j,i} \cdot wtp_i) \qquad (2.4.3.3.1)$$

where

B	Total benefit to all water users;
B_i	Benefit to User i, a quadratic benefit function is assumed here with coefficients a_i and b_i;
S_i	Water withdrawal by user i;
wtp_i	Water trading price for user i;
$x_{i,j}$	Water sold by user i to user j;

User i has access to water from water right or purchase

$$S_i \le wright_i - \sum_j x_{i,j} + \sum_j x_{j,i} \qquad (2.4.3.3.2)$$

where *wright* is water allocated to a user under prescribed water rights. Each user has shadow price for water which is a linear function of the amount of water demanded

$$wtp_i = am_i + bm_i S_i \qquad (2.4.3.3.3)$$

No user is allowed to sell more water than their water right

$$\sum_{j} x_{i,j} \leq wright_i \tag{2.4.3.3.4}$$

Trades are unidirectional, that is, if a user buys water from another user, then they can not sell water to the same user

$$x_{i,j} * x_{j,i} = 0 \tag{2.4.3.3.5}$$

2.5 Exercises

1. You are working with the manager of an irrigation facility who is interested in installing a more efficient pumping system. The proposed system costs $15,000 and you project that it will reduce the annual utility costs by $2,000. After five years, you expect to upgrade the system for $4,000. This upgrade is expected to further reduce utility costs by $1,000 annually. The annual effective interest rate is 7% and the life of the system, after upgrade is 50 years. What is the Present Value of the investment in the system?

2. You have a small excavation firm and wish to purchase a small backhoe. Based on your research, you need to have $54,000 to purchase one used. If your cost of capital is 0.50%/month and you want to recover your capital (on a Present Worth basis) in 20 months, how much profit must this backhoe generate each month.

3. (after North, 1985, Exercise 5.8) A flood control district can construct a number of alternative control works to alleviate the flood pattern in that area. These alternatives include dam A, dam B, and a levee system C. The levee system can be built alone or in combination with dam A or B. Both dams can not be built together but either one can function alone. The lofe of each dam is 80 years and the life of the levee system is 60 years. The cost of capital is 6 percent. Information on total investment, operation and maintenance costs, and average annual flood damage is given below. What form of flood control would be the most economical?

Table. Flood Control Project data

Project	Total Investment (million $)	Annual Operation and Maintainence (thous. $)	Average Annual Flood Damages (million $)
Dam A	6.2	93	1.10
Dam B	5.3	89	1.40
Levee C	6.7	110	0.80
Do nothing	0	0	2.15

4. (after Mays & Chung, 1992, Exercise 2.2.2) Four alternative projects can be used for developing a water supply for a community for the next 40 years. Use the incremental benefit-cost method to compare and select an alternative. Use a 6% interest rate.

Years	Project A	Project B	Project C	Project D
	Construction cost ($)			
0	40,000,000	30,000,000	20,000,000	10,000,000
10				10,000,000
20		10,000,000	20,000,000	10,000,000
30				10,000,000
	Operation and Maintenance Cost ($)			
0-10	100,000	110,000	120,000	120,000
10-20	120,000	110,000	130,000	120,000
20-30	140,000	120,000	140,000	130,000
30-40	160,000	140,000	150,000	130,000

5. (after James and Lee, 1971, Problem 2.6) The three alternatives described below are available for supplying a community water supply for the next 50 years when all economic lives as well as the period of analysis terminates.

Construction cost	Project A	Project B	Project C
Year 0	$20,000,000	$10,000,000	$15,000,000
Year 20	0	10,000,000	12,000,000
Year 35	0	10,000,000	0
O&M cost			
Year 1-20	70,000	40,000	60,000
Year 21-35	80,000	70,000	80,000
Year 36-50	90,000	90,000	90,000

Using $2,500,000 in benefits each year for each project, and a 4.5% discount rate where applicable, compare the projects using:

a. The present-worth method

The present-worth method selects the project with the largest present worth of the discounted sum of benefits minus the costs over its life

$$P_W = \sum_{t=1}^{T} \frac{B_t - C_t}{(1+i)^t}$$

where C_t is the cost and B_t is the benefit in year t, T is the period of analysis, and i is the discount rate.

b. The rate-of-return method

The rate-of-return is the discount rate at which the present worth as defined above equals zero as found by trial and error.

c. The benefit-cost ratio method

The benefit-cost ratio PW_b/PW_c is the present worth of the benefits PW_b divided by the present worth of the costs PW_c. Annual values can be used with out altering the ratio.

$$PW_b = \sum_{t=1}^{T} \frac{B_t}{(1+i)^t} \quad \text{and} \quad PW_c = \sum_{t=1}^{T} \frac{C_t}{(1+i)^t}$$

d. The annual-cost method

The annual-cost method converts all benefits and costs into equivalent uniform annual figures.

6. (after Thuesen et al., 1977, Problem 10.20) The federal government is planning a hydroelectric project for a river basin. In additionto the production of electric power, this project will provide flood control, irrigation, and recreation benefits. The estimated benefits and costs that are expected to be derived from the three alternatives under consideration are listed below:

Construction cost	Project A	Project B	Project C
Initial cost	$25,000,000	$35,000,000	$50,000,000
Annual benefits and costs			
Power sales	$1,000,000	$1,200,000	$1,800,000
Flood control savings	250,000	350,000	500,000
Irrigation benefits	350,000	450,000	600,000
Recreation benefits	100,000	200,000	350,000
O&M costs	200,000	250,000	350,000

The interest rate is 5% and the life of each of the projects is estimated to be 50 years.

a. Using the incremental benefit-cost method, determine which project should be selected.

b. Calculate the benefit cost ratio for each alternative. Is the best alternative selected if the alternative with the maximum benefit cost ratio is chosen?

c. If the interest rate is 8%, what alternative would be chosen?

7. (After D. P. Loucks, Course Notes, Engineering Economics, Cornell University.) Three mutually exclusive water resources projects, *A, B,* and *C,* are under consideration. Each project has a fixed initial cost ($FC_A, FC_B,$ and FC_C). Their (unequal) useful lives are $L_A, L_B,$ and $L_C,$ and during each year y of those lives they generate annual benefits of $B_{Ay}, B_{By},$ and $B_{Cy},$ and costs of $C_{Ay}, C_{By},$ and $C_{Cy}.$ Assume that an appropriate interest rate, $r,$ has already been determined for this analysis. Show how you would calculate each project's equivalent end-of-year annual benefits and costs, and based on these, their benefit-cost ratios.

8. Find the optimal levels of two goods purchased by a consumer with a utility function

$$u(x_1, x_2) = x_1^{1.5} x_2$$

and a budget constraint

$$3x_1 + 4x_2 = 100$$

9. Agricultural production is described by the quadratic equation

$$q_i = a_i + b_i x_i + c_i x_i^2$$

where q_i is the yield in hectares of crop i, a_i, b_i, and c_i are parameters of the production functionfor crop i, and x_i is the amount of water (m^3) applied to crop i. The unit cost of water is w; the unit market price of each crop is p_i. Develop a model, based on the theory of the firm, to determine the optimal water allocation to each crop. What is the demand function for water assuming the production function is a concave function of x_i (i.e., $c_i < 0$).

10. (After Mays and Tung, 1992, Problem 2.4.1; 2.4.4; 2.4.5)

(1) For the production process in the following Table, determine and plot the total, average, and marginal product curves for nitrogen fertilizer given that water is fixed at $x_1 = 7$ inches/acre.

(2) Determine and plot the cost curves for the production process. Assume that water is fixed at 7 inches per acre. Use input prices of $2.50 per pound of nitrogen fertilizer and $10 per acre-inch of water. Plot average fixed cost (AFC), average variable cost (AVC), average total cost (ATC), and marginal cost (MC) on one plot.

(3) Determine the profit for various levels of output for the production process. Assume that corn sells for $1.49 per bushel, and input prices are the same as in Part (2) and that irrigation water is fixed at 7 inches per acre. How much corn should be produced? How much nitrogen fertilizer is used in this production. What it the value of total product that maximizes profit?.

Table. Production Schedule of the Relationship Between Irrigation Water, Fertilizer & Yield (bushels per acre) of corn.

x_2**	x_1*													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	1.8	5.0	9.0	13.2	17.0	19.8	21.0	20.0	16.2	9.0	0.0	0.0	0.0
20	0.0	5.0	12.8	22.2	32.0	41.0	48.0	51.8	51.2	45.0	32.0	11.0	0.0	0.0
30	0.0	9.0	22.2	37.8	54.0	69.0	81.0	88.2	88.0	81.0	63.0	33.0	0.0	0.0
40	0.0	13.2	32.0	54.0	76.8	98.0	115.2	126.0	128.0	118.8	96.0	57.2	0.0	0.0
50	0.0	17.0	41.0	69.0	98.0	125.0	147.0	161.0	164.0	153.0	125.0	77.0	6.0	0.0
60	0.0	19.8	48.0	81.0	115.2	147.0	172.8	189.0	192.0	178.2	144.0	85.8	0.0	0.0
70	0.0	21.0	51.8	88.2	126.0	161.0	189.0	205.8	207.2	189.0	147.0	77.0	0.0	0.0
80	0.0	20.0	51.2	88.0	128.0	164.0	192.0	207.2	204.8	180.0	128.0	44.0	0.0	0.0
90	0.0	16.2	45.0	81.0	118.8	153.0	178.2	189.0	180.0	145.8	81.0	0.0	0.0	0.0
100	0.0	9.0	32.0	63.0	96.0	125.0	144.0	147.0	128.0	81.0	0.0	0.0	0.0	0.0
110	0.0	0.0	11.0	33.0	57.2	77.0	85.8	77.0	44.0	0.0	0.0	0.0	0.0	0.0
120	0.0	0.0	0.0	0.0	0.0	6.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
130	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

* x_1 = inches of water per acre
** x_2 = pounds of nitrogen fertilizer per acre

11. (After Loucks et al., Problem 4.2). Assume that a farmer's demand for water q is a linear function of the price p, i.e., $q(p) = a-bp$, where $a, b > 0$.

(1) Calculate the farmer's willingness-to-pay for a quantity of water q.

(2) If the cost of delivering a quantity of water q is $C(q) = cq$, $c > 0$, how much water should a public agency supply to maximize willingness-to-pay minus cost?

(3) If the local water district is owned and operated by a private firm whose objective is to maximize profit, how much water would they supply and how much would they earn?

(4) The farmer's consumer surplus is their willingness-to-pay minus what they must pay for the resource. Compare the farmer's consumer surplus in the two cases.

(5) Does the farmer lose more than the private firm gains by moving from the social optimum to the point that maximizes the firm's profit?

(6) Illustrate these relationships with a graph showing the demand curve and the unit cost c of water. Label the firm's profits and the farmer's consumer surplus?

12. Given the production functions for wheat and corn in the Maipo basin of Chile shown in the following Table, determine a Production Possibility curve if x = 15,000 m3/ha of water is available. If the prices are p_1 = \$230/mt (wheat) and p_2 = \$160/mt (corn), find the point of optimal production. What are the resulting outputs and amounts of water used for wheat and corn. What is the resulting total revenue?

Table. Production functions for wheat and corn.

Production functions

Water	Wheat	Water	Corn
x (m3/ha)	y_1 (mt/ha)	x (m3/ha)	y_2 (mt/ha)
0	0.00	0	0.00
536	0.00	883	0.00
1071	0.00	1765	0.00
1607	0.00	2648	0.00
2142	0.22	3530	1.21
3213	2.08	5295	3.87
4284	3.27	7060	5.60
5355	4.10	8825	6.83
6426	4.70	10590	7.74
7497	5.14	12355	8.43
8568	5.46	14120	8.96
9639	5.70	15885	9.37
10710	5.87	17650	9.68
11781	5.98	19415	9.91
12852	6.04	21180	10.08
13923	6.07	22945	10.19
14994	6.06	24710	10.26
16065	6.02	26475	10.28
17136	5.96	28240	10.27
18207	5.88	30005	10.23
19278	5.77	31770	10.17

13. (After Willis and Finney, 2000, Example Problem 4-8) Water quality pollution is an example of an externality, i.e., a "harmful effect on one or more individuals that emenates from the action of a different person or firm" (Samuelson, 1973). Downstream water users, for example, will have to treat water prior to use because of upstream firms don't consider the externalities in their decision making process.

A regional water management authority proposes to reduce point source wastewater discharges by imposing an effluent tax on each unit of waste discharged. These effluent charges are a method of internalizing the externalities created by the discharges. Investigate how the effluent charge affects the optimum production levels for a firm discharging waste. Assume:

1. the tax, τ, is expressed in $ per unit of output of the firm, q, and
2. the pollution generated by the firm, P, is a linear function of the production level, $P = \delta q$.

a. Find the first order optimality conditions for the firm production.

b. What is the effect of the tax on marginal revenue and marginal cost?

b.1 Plot a diagram of Price versus Quantity showing marginal revenue with and without the

tax (make whatever assumptions you need to to develop the graph).

b.2 What is the difference between the point of intersection of the marginal cost curve and the marginal revenue curve with and without the tax?

b.3 What is the difference in the production level with and without the tax?

c. What is the effect on the level of pollution produced with and without the tax?

14. A regional water management authority must pay an effluent tax on waste discharged. Consider

q	= firm's output (units)
$p(q) = \omega - \eta q$	= price of the firm's output (\$/unit)
$C(q) = \gamma q^2$	= firm's cost function (\$)
$D(q) = \alpha + \beta q$	= firm's pollution level (kg)
$t = p_c$	= pollution tax (\$/kg of waste produced)

$\gamma, \alpha, \beta, p_c, \omega,$ and η are just constants, but you and I do not know their values.

Part A: Suppose the firm is maximizing profits and price is given as $p = constant$ (that is, $p(q) = \omega = p, and \eta = 0$).

(1) If the tax is zero ($t = 0$), what level of output (q) will the firm select?

(2) If the tax is not zero ($t = p_c$), what level of output (q) will the firm select?

Part B: Suppose that the firm is socially conscious and wishes to maximize the benefits of the consumers and the inverse demand function (marginal willingness-to-pay) for its product is given by $p(q) = \omega - \eta q$. Be sure to continue considering the cost function in your model.

(1) If the tax is zero ($t=0$), what level of output (q) will the firm select?

(2) If the tax is not zero ($t = p_c$), what level of output (q) will the firm select?

15. The inverse demand curve for a depletable, nonrecyclable resource in year t is

$$p_t(q_t) = a - bq_t \quad t = 1, 2, ..., T$$

where q_t is the amount of resource demanded in year t, p_t is the price of the resource in year t, and $a, b > 0$ are constants. The marginal cost of extracting a unit of resource in any year is a constant ($= c$). The total amount of resource available ($= Q$) is less than the amount needed to satisfy demand over a T year planning horizon.

(a)   Determine the first order optimality conditions for resource extraction if the objective of the extraction is to balance the current and future uses of the resource by maximizing the present value of net benefits derived from the use of the resource over the T years. Assume that the discount rate is i.

(b) Calculate numerical values for the optimal extraction rates ($= q_t$, $t = 1,2, ..., T$) if
     $T = 2$, $a = 8$, $b = 0.4$, $c = 2$, $Q = 20$, $i = 0.10$.

16. Part A. Assume that stream flow ($\hat{S}$) is 26 million m³ per unit of time and there is an interstate agreement ($\overline{S}$) calling for 14.5 million m³ Also, initially there are two users on the river diverting S_2 and S_3 million m³. The benefits to each user are:

$$\text{User 2:}\quad B_2(S_2) = a_2 S_2 + b_2 S_2^2 = 150 S_2 - 5 S_2^2$$
$$\text{User 3:}\quad B_3(S_3) = a_3 S_3 + b_3 S_3^2 = 18 S_3 - 0.6 S_3^2$$

In addition, both users have the same return flow coefficient, $R_2 = R_3 = 0.5$. Write an optimization model to determine an efficient allocation that results in maximizing the value of water use in the basin and respects the interstate compact. What are the water allocations and the benefits to each user?

Part B. Now, assume that an additional user wants to divert S_1 million m³ of water from the river. User 1's benefit function is identical to User 2's benefit, i.e.,

$$\text{User 1:}\quad B_1(S_1) = a_1 S_1 + b_1 S_1^2 = 150 S_1 - 5 S_1^2$$

User 1 also has a return coefficient of $R_1 = 0.5$. Modify your optimization model to determine a new efficient allocation that results in maximizing the value of water use for all three users in the basin and respects the interstate compact. What are the water allocations and the benefits to each user?

Part C. Assume that the solution to Part A represents the initial water rights of Users 2 and 3, using the results obtained in Part B, what is the minimum payment that User 1 should pay to Users 2 and 3 in order to divert water from the river? What are the resulting net benefits to each of the three users?

Part D. Assume that the system described in Part B is modeled as a water market where users 1, 2, and 3 have water right allocations, 0.0, 4.0, and 7.5, respectively. Develop a model which will determine the optimal use of water by each user, assuming that the users are free to trade their water rights according to the model structure described in the text above.

17. Lewis and Clark Lake is a large reservoir in South Dakota created on the Missouri River by the Gavins Point Dam. It is located in an area where there are few natural bodies of water, and it has become very popular as a recreational area. Suppose that 10,000 families are potential users of the lake for recreational purposes and that each family's demand curve for recreational trips to the lake is as follows:

Price per trip ($) vs **No. trips per year**

Data points: (0, 20), (1, 16), (2, 12), (3, 10), (4, 8), (5, 6), (6, 4), (7, 3), (8, 2), (9, 1), (10, 0)
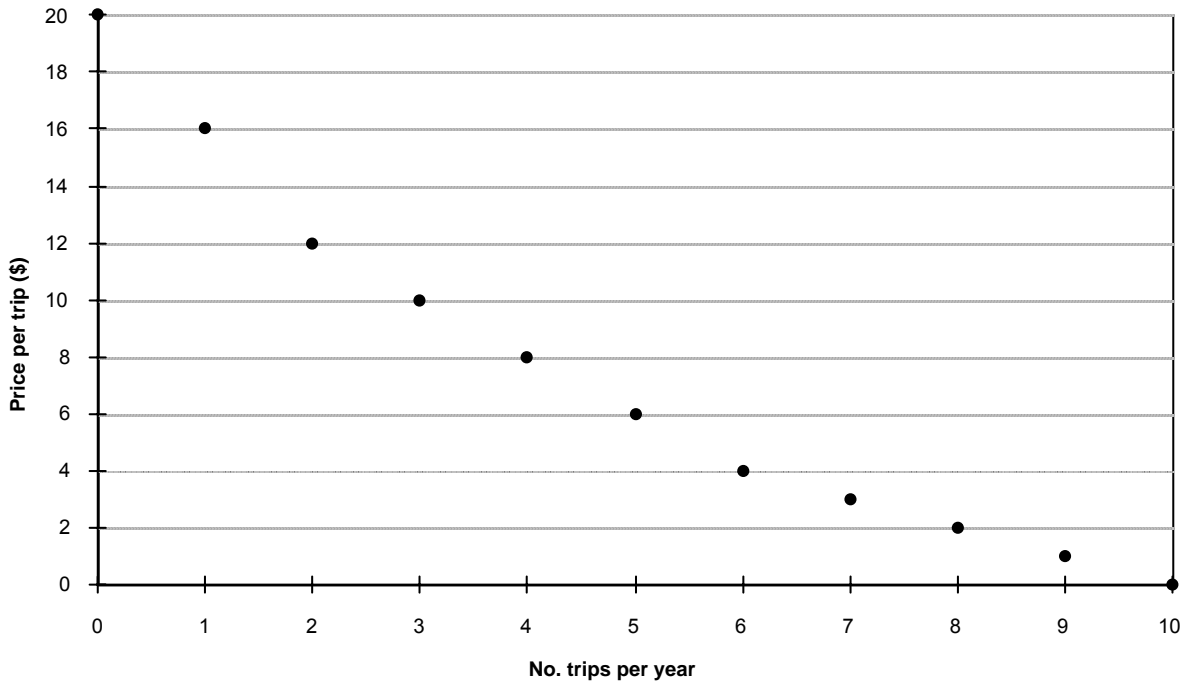
(1) If an ordinance were passed which limited each family to no more than 5 trips per year to the lake, what is the loss (in money terms) to each family?

(2) If an ordinance were passed which allowed a family to use the lake for recreational purposes only if it purchased a permit for $75 a year, would it be worthwhile for each family to buy a permit, if it could not use the lake without the permit (and it could use the lake as much as it liked with one)?

(3) How much is the consumer's surplus from each family's utilization of the lake if there is a charge of $8 for each trip to the lake?

18. (after North, 1985) **Part A.** Consider the following set of data regarding the production of lettuce

Acre-inch water	Production of lettuce	Price of lettuce/head
0	186	50
1	698	25
2	1185	20
3	1648	18
4	2085	17
5	2496	16
6	2883	15
7	3245	14
38	3582	13
9	3895	12
10	4184	10
11	4442	8
12	4679	6
13	4891	4

Determine and plot a schedule for:

(1)   Physical production (total, average, and marginal)

(2)   Cost functions (total, average, and marginal for both input and output)

(3)   Revenue functions (total, average, and marginal for both a competitive market price of 15 cents per head of lettuce and for the industry demand schedule given in column 3 below)

Part B. What are the firm equilibrium positions for both the competitive and monopolistic price structures, demonstrating total revenues, total costs, and net revenues.

Part C. What are the optimum levels of production and resource use under both pricing structures when water costs $40/acre-inch and fixed costs are $160.

19. (After Linsley et al., 1079) The average annual damage from floods in a river basin is estimated to be $400,000. Estimates have been made for several alternate proposals for flood mitigation works: channel improvements (25 yr life), two mutually exclusive dams (A and B, 100 yr lives), and various combinations of these. The table below shows the first cost, estimated annual damages, and the annual OM&R disbursements for each alternative, and the sum of the annual damages and annual costs.

Project	1st Cost $	Annual Damages, $	Annual OM&R Costs, $
Do nothing	0	400,000	0
I. Channel Improvement	500,000	250,000	100,000
II. Dam A	3,000,000	190,000	60,000
III. Dam B	4,000,000	125,000	80,000
IV. Dam A with Channel Improvements	3,500,000	100,000	160,000
V. Dam B with Channel Improvements	4,500,000	60,000	180,000

a. Compare the projects using an interest rate of 6 percent.
b. Compare the conclusions of the economic analysis with the interest rate of 3 percent used in Sec. 13-9 of Linsley and Franzini, Water Resources Engineering, with the 6 percent rates used in your solution to Part (1). What generalizations can you make regarding the influence of the interest rate on such studies?
c. Comment on the statement, "I would select the plan with the highest benefit cost ratio of all the plans."
d. Comment on the statement, "I would select the plan with the highest possible benefits for which benefits are greater than costs."

20. Assume that stream flow ($\hat{S}$) is 26 million m^3 per unit of time and there is an interstate agreement ($\overline{S}$) calling for 14.5 million m^3 Also, initially there are two users on the river diverting S_2 and S_3 million m^3. The benefits to each user are:

$$\text{User 2:} \quad B_2(S_2) = a_2 S_2 + b_2 S_2^2 = 150 S_2 - 5 S_2^2$$
$$\text{User 3:} \quad B_3(S_3) = a_3 S_3 + b_3 S_3^2 = 18 S_3 - 0.6 S_3^2$$

In addition, both users have the same return flow coefficient, $R_2 = R_3 = 0.5$. Write an optimization model to determine an efficient allocation that results in maximizing the value of water use in the basin and respects the interstate compact. What are the water allocations and the benefits to each user?

21. Now, assume that an additional user wants to divert S_1 million m^3 of water from the river. User 1's benefit function is identical to User 2's benefit, i.e.,

$$\text{User 1:} \quad B_1(S_1) = a_1 S_1 + b_1 S_1^2 = 150 S_1 - 5 S_1^2$$

User 1 also has a return coefficient of $R_1 = 0.5$. Modify your optimization model to determine a new efficient allocation that results in maximizing the value of water use for all three users in the basin and respects the interstate compact. What are the water allocations and the benefits to each user?

22. Assume that the solution to Part (1) represents the initial water rights of Users 2 and 3, using the results obtained in Part (2), what is the minimum payment that User 1 should pay to Users 2 and 3 in order to divert water from the river? What are the resulting net benefits to each of the three users?

23. Assume that the system described in Problem (2) is modeled as a water market where users 1, 2, and 3 have water right allocations, 0.0, 4.0, and 7.5, respectively. Develop a model which will determine the optimal use of water by each user, assuming that the users are free to trade their water rights according to the model structure described in Section 3.2.5 above.

24. A flood control district can construct a number of alternative control works to alleviate a flood problem in the area. These alternatives include dam A, dam B, and a levee system C. The levee system can be built alone or in combinations with dam A or dam B. Both dams can not be built together but either one can function alone. The life of each dam is 80 years and the life of the levee system is 60 years. The cost of capital (discount rate) is 6 percent. Information total investment costs, operating and maintenance costs, and average annual flood damages is given below. What form of flood control is preferred?

Project	Life (years)	Total investment ($1000)	Annual operation and maintenance ($1000)	Average annual flood damages ($1000)
No control at all	-	0	0	$2,150
Dam A	80	$6,200	$93	$1,100
Dam B	80	$5,300	$89	$1,400
Levees C	60	$6,700	$110	$750

25. (after Grant et al., 1976, p. 138) Just before a creek has its outlet into a salt water bay, it goes through an urban area. Because there have been occasional floods that have caused damage to property in this area, a flood control project has been proposed. Estimates have been made for two alternative designs, one involving channel improvements (CI) and the other involving a dam and reservoir (D & R). Economic analysis is to be based on an estimated 50-year project life assuming zero terminal salvage value and using a discount rate of 6%.

The expected value of the annual cost due to flood damages is $480,000 with a continuation of the present condition of no flood control (NFC). Alternative CI will reduce this figure to $105,000; alternative D&R will reduce it to $55,000.

Alternative CI has an estimated first cost of $2,900,000 and estimated annual maintenance costs of $35,000. Alternative D&R has an estimated first cost of $5,300,000, and estimated annual operation and maintenance costs of $40,000. Alternative D&R also has two types of adverse consequences related to the conservation of natural resources. These are to be treated in the economic analysis as disbenefits (negative benefits). The dam will cause damage to anadromous fisheries; this is priced at $28,000 per year. The reservoir will cause a loss of land for agricultural purposes including grazing and crop raising; this is priced at $10,000 per year.

Using Benefit – Cost analysis, which alternative is preferred?

3. WATER MANAGEMENT IN RIVER SYSTEMS

3.1 Allocation of Water to Users

(adapted from Loucks et al., 1981)

Consider a situation where a total quantity of water R is to be allocated to a number of different uses. Let the quantity of water allocated to each use be denoted by $x_i, i = 1,...,I$. The objective is to determine the allocations such that the total net benefits from all uses is maximized. Consider an example where there are three uses $I = 3$, as shown in Figure 3.1.
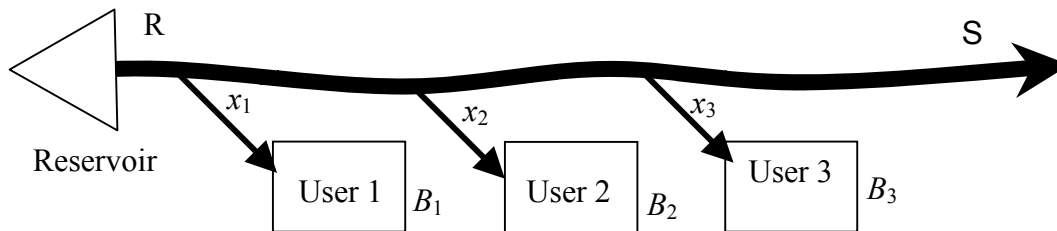
Figure 3.1. Reservoir release example.

The net-benefit resulting from an allocation of x_i to use i is (see Fig. 3.2)

$$B_i(x_i) = a_i x_i - b_i x_i^2 \qquad i = 1,2,3 \tag{3.1.1}$$

where a_i and b_i are given positive constants.
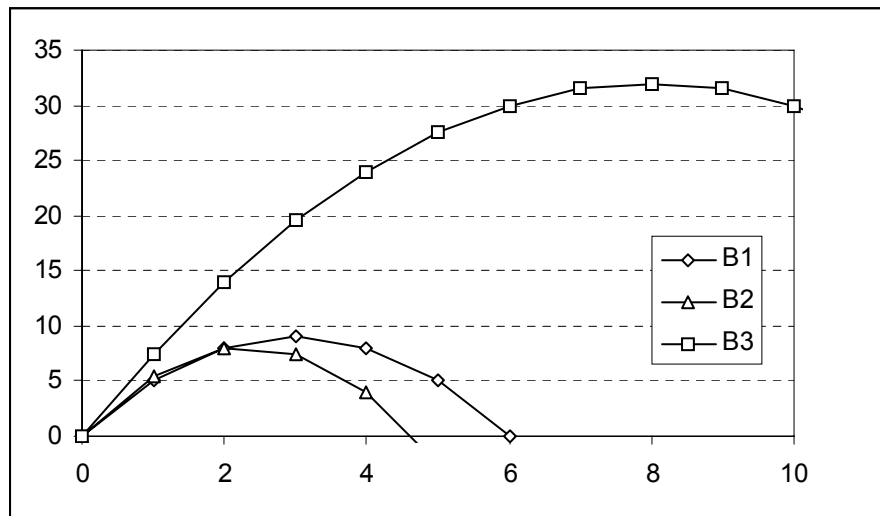
Figure 3.2. Net-benefit function for user i.

The three allocations x_i are unknown decision variables. The values that these variables can take on

are restricted between 0 (negative allocation is meaningless) and values whose sum, $x_1 + x_2 + x_3$, does not exceed the available supply of water R minus a required instream flow S. The optimization model is

$$maximize \atop \vec{x} \quad \sum_{i=1}^{3}(a_i x_i - b_i x_i^2) \tag{3.1.2}$$

$$subject\ to \tag{3.1.3}$$

$$\sum_{i=1}^{3} x_i + S - R = 0$$

The GAMS code to solve this problem is

```
SETS       i / 1, 2, 3/

SCALAR     r RELEASE    /10.0/;

PARAMETER
a(i) /1 6.0, 2 7.0, 3 8.0/
b(i) /1 -1.0, 2 -1.5, 3 -0.5/;

VARIABLES obj OBJECTIVE;

POSITIVE VARIABLES x(i) USE, s DOWNSTREAM FLOW;

S.lo=2.0;

EQUATIONS objective, cap;

objective..obj =E= SUM(i,a(i)*x(i)+b(i)*x(i)**2);

cap..sum(i,x(i))+s-r =E= 0.0;

MODEL user /ALL/ ;

SOLVE user USING NLP MAXIMIZE obj ;

FILE res /WaterUser.txt/
PUT res
PUT 'Release          ', PUT r,     PUT /
PUT 'Downstream flow ', PUT s.l,    PUT /
PUT 'Objective        ', PUT obj.l, PUT //
PUT 'i                    x(i) ' PUT /
loop( (i), PUT i.TL,  PUT x.l(i), PUT /)
PUT //, 'd(obj)/dr = ', PUT cap.m, PUT //
```

In this model, we know that the allocations to the water users (x_i) are non-negative, so we can define them as *positive variables* which allows the use of the *exponentiation* operator (**) in the benefit function. Also, note the use of comments in the file. This makes it easier for someone else to understand the model. Two other GAMS operators are used in this model. First is the '*.lo*' operator, which is used to set a lower bound on the variable s of 2.0 units of flow. The same effect could be achieved by defining a new equation as a lower bound constraint

```
EQUATIONS slo;
slo..s =G= 2.0;
```

However, computational savings are realized if the '.*lo*' operator is used. Upper bounds on variables can also be defined using the '.*up*' operator with a variable. Second, the *sum* operator is used in the objective function and in the resource constraint allowing us to construct the GAMS equations in the same form as the mathematical description of the model (Equations 3.1.2 and 3.1.3).

The result for a release of $R = 10$ is

```
Release              10.00
Downstream flow       2.00
Objective            41.41

i               x(i)
1               1.55
2               1.36
3               5.09


d(obj)/dr =     2.91
```

Figure 3.3 illustrates the result of running the model for various values of the reservoir release, *r*. Note that the water users continue to receive additional water up to the release of $r = 15.33$. After that point users are experiencing declining revenue from the use of the water and they stop demanding additional units and the marginal value of additional release of water $\lambda = d(obj)/dr = 0$. This marginal value (Lagrange multiplier) is printed in the output file using the '.*m*' operator.
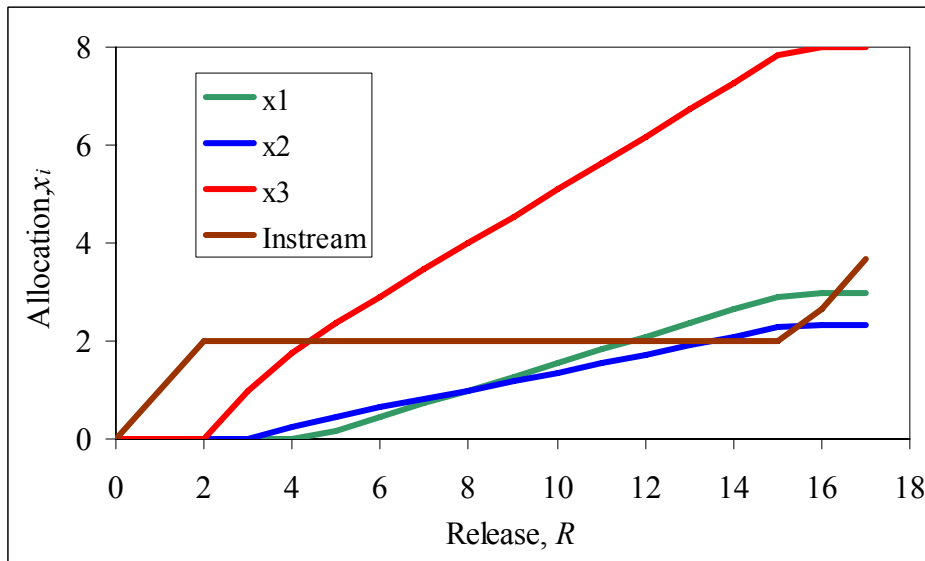
Figure 3.3. Release allocation to water users and downstream flow.

3.2 Management of a Single Reservoir

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Reservoir management is an important task for water managers throughout the world. Models are often constructed to simulate or optimize the performance of reservoir operations a well as to design reservoir or the associated facilities (spillways, etc.). In this section we consider two common tasks of reservoir modeling: (1) determining various coefficients of the functions that describe reservoir characteristics; and (2) the optimal mode of reservoir operation (storage volumes, elevations and releases) while satisfying downstream water demands.

3.2.1 Estimating a Reservoir Storage – Elevation Relationship Using the Method of Lease Squares

The objective for the first task is to find the coefficients (a, and b), for the function

$$s = a(h - h_0)^b \tag{3.2.1.1}$$

where s is the storage volume in the reservoir (L^3), h is the elevation of the water surface in the reservoir (L) and h_0 is the elevation of the tailwater at the outlet of the reservoir (L). In the estimation procedure, we use the method of least squares as indicated in the Section 2.2. As in that case, the difference (or residual) between the model value s (given in Equation 3.2.1.1 and which depends on the unknown parameter values a, and b) and the observed value $\hat{s}$ can be written as

$$e_i = s_i - \hat{s}_i = a[\hat{h}_i - h_0]^b - \hat{s}_i, \qquad i = 1, 2, ..., I \tag{3.2.1.2}$$

where
  i is the index of measurements (observed storage volumes and elevations);
  I is the total number of measurements;
  s_i is the i-th calculated storage volume in the reservoir;
  $\hat{s}_i$ is the i-th observed storage volume in the reservoir; and
  $\hat{h}_i$ is the i-th observed water elevation in the reservoir.

Numerical values of the coefficients are determined by minimizing the squared residuals

$$Minimize \ \sum_{i=1}^{I} (e_i)^2 \tag{3.2.1.3}$$

As an example, the volume and elevation data for Toktogul reservoir on the Naryn River in Kyrgyzstan are given the Table 3.2.1.1 and plotted in Figure 3.2.1.1.

Table 3.2.1.1 . Storage and Surface Area Versus Water Surface Elevation Data for Toktogul Reservoir.

Item	Volume (mln m^3)	Elevation (m)	Surface area (km^2)
i	$S(i)$	$h(i)$	$A(i)$
1	0	759	0
2	78	767	9.75
3	118	771	10
4	321	783	16.917
5	405	786	27.999
6	779	795	41.555
7	1204	802	60.713
8	1885	810	85.124
9	2297	814	102.997
10	3921	827	124.922
11	5585	838	151.271
12	5750	839	164.984
13	7717	850	178.817
14	8690	855	194.596
15	10336	863	205.747
16	11198	867	215.495
17	13003	875	225.622
18	14676	882	238.997
19	16961	891	253.886
20	19458	900	277.441
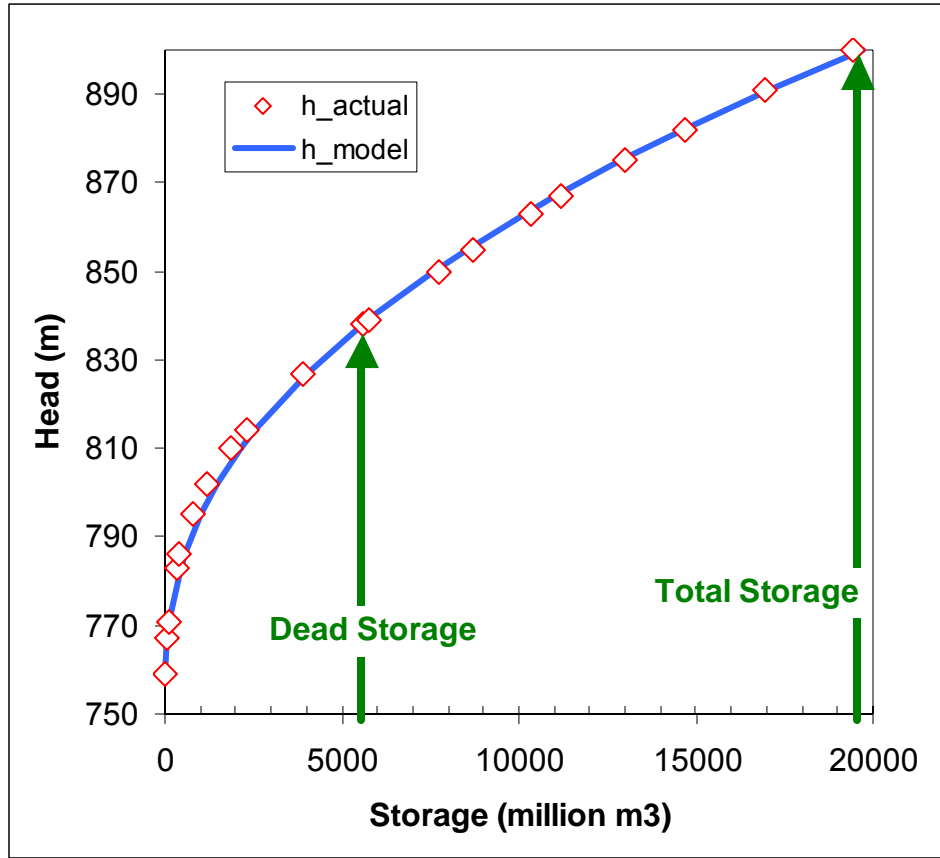
Figure 3.2.1.1. Storage versus water surface elevation relations for Toktogul reservoir.

The GAMS code for this model is shown below. Note that upper bounds on the coefficients have been added to prevent exponentiation overflow.

```
SETS
i    /1*20/

SCALAR h0  Minimum Elevation  /758/

PARAMETER
h(i) /
1          759
2          767
3          771
4          783
5          786
6          795
7          802
8          810
9          814
10         827
11         838
12         839
13         850
14         855
15         863
```

```
16        867
17        875
18        882
19        891
20        900
/;

PARAMETER
S_hat(i) /
1              0
2             78
3            118
4            321
5            405
6            779
7           1204
8           1885
9           2297
10          3921
11          5585
12          5750
13          7717
14          8690
15         10336
16         11198
17         13003
18         14676
19         16961
20         19458
/;

POSITIVE VARIABLES
a, b;

b.up=100;
a.up=100;

VARIABLES
e(i), obj;

EQUATIONS residual(i), objective;

residual(i).. e(i) =E= a*(h(i)-h0)**b - S_hat(i);

objective.. obj=E=sum(i,power(e(i),2));

MODEL hvs /ALL/;

SOLVE hvs USING  NLP MINIMIZING obj;

FILE res /River1a.txt/;
PUT res
PUT "Coefficients(a and b) in formula S = a*(h-h0)**b"/ ;
PUT "a = ", a.L, "   b = ", b.L//;
PUT "No.          Elevation   Volume(real)  Volume(calc)"/;
LOOP(i,PUT i.TL,  h(i), S_hat(i), ((a.L*(h(i)-h0)**b.L))/;);
```

The results are

```
Coefficients(a and b) in formula S = a*(h-h0)**b
```

```
a =          0.36   b =          2.20

No.              Elevation    Volume(real)   Volume(calc)
1                   759.00           0.00           0.36
2                   767.00          78.00          45.54
3                   771.00         118.00         102.29
4                   783.00         321.00         431.26
5                   786.00         405.00         553.40
6                   795.00         779.00        1021.84
7                   802.00        1204.00        1496.13
8                   810.00        1885.00        2160.78
9                   814.00        2297.00        2543.50
10                  827.00        3921.00        4026.47
11                  838.00        5585.00        5575.47
12                  839.00        5750.00        5729.98
13                  850.00        7717.00        7583.03
14                  855.00        8690.00        8519.56
15                  863.00       10336.00       10142.62
16                  867.00       11198.00       11012.33
17                  875.00       13003.00       12869.53
18                  882.00       14676.00       14624.88
19                  891.00       16961.00       17062.84
20                  900.00       19458.00       19707.16
```

A plot of the function versus the measured data are shown in Figure 3.2.1.1. The same procedure can be used to develop the function for surface area as a function of storage volume. The results are shown in Figure 3.2.1.2.
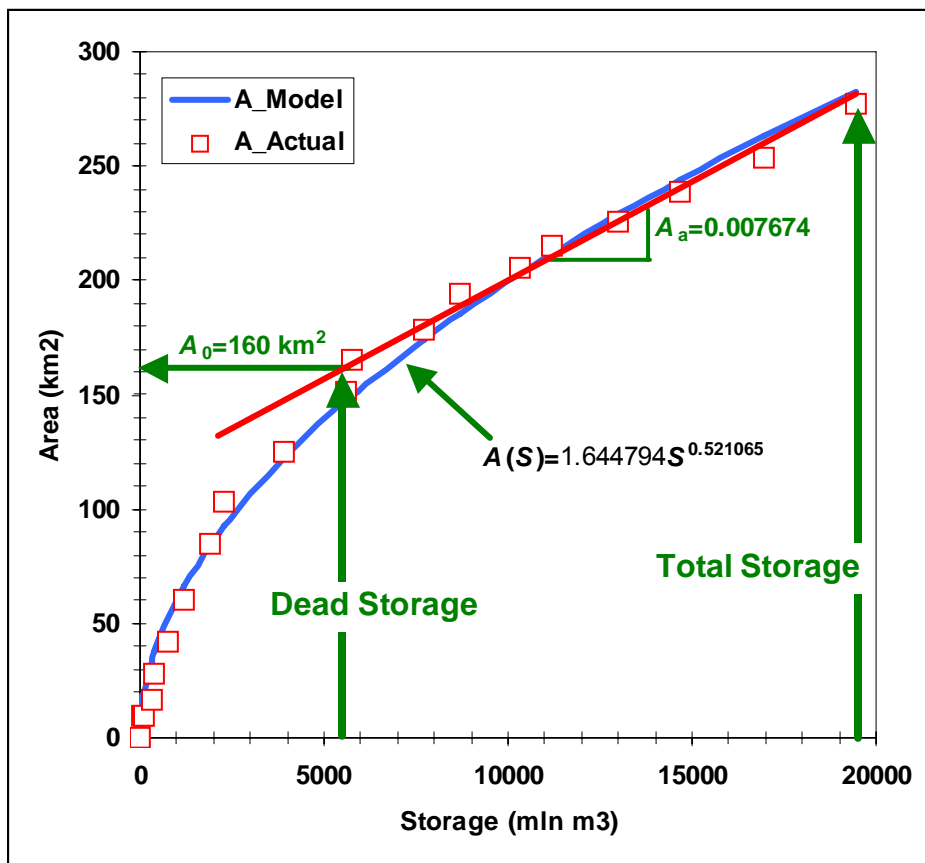
Figure 3.2.1.2. Storage versus reservoir surface area relations for Toktogul reservoir.

The model can be modified if the user would like to estimate

$$h = a(s)^b + h_0 \qquad\qquad (3.2.1.4)$$

In this case, the difference (or residual) between the model value h (given in Equation 3.2.1.4 and which depends on the unknown parameter values a, and b) and the observed value $\hat{h}$ can be written as

$$e_i = h_i - \hat{h}_i = a(\hat{s}_i)^b + h_0 - \hat{h}_i, \qquad i = 1,2,...,I \qquad\qquad (3.2.1.5)$$

The necessary modifications to the model are in the residual equation and the output:

```
residual(i).. e(i) =E= a*S_hat(i)**b + h0 -h(i);
.
.
.
PUT "No.             Elevation    Volume(real)  Elevation(calc)"/;
LOOP(i,PUT i.TL,  h(i), S_hat(i), ((a.L*S_hat(i)**b.L+h0))/;);
```

The results for the Toktogul reservoir example are:

```
Coefficients(a and b) in formula h = a*S**b+h0
a =        1.91   b =           0.43

No.              Elevation   Volume(real)  Elevation(calc)
1                  759.00         0.00          758.00
2                  767.00        78.00          770.70
3                  771.00       118.00          773.20
4                  783.00       321.00          781.49
5                  786.00       405.00          783.98
6                  795.00       779.00          792.52
7                  802.00      1204.00          799.72
8                  810.00      1885.00          808.69
9                  814.00      2297.00          813.24
10                 827.00      3921.00          827.69
11                 838.00      5585.00          839.27
12                 839.00      5750.00          840.30
13                 850.00      7717.00          851.53
14                 855.00      8690.00          856.48
15                 863.00     10336.00          864.19
16                 867.00     11198.00          867.95
17                 875.00     13003.00          875.33
18                 882.00     14676.00          881.67
19                 891.00     16961.00          889.70
20                 900.00     19458.00          897.80
```

3.2.2 Reservoir Operation

In the second model, the optimal operation of the reservoir is computed given a series of inflows and downstream water demand targets. Evaporation from the reservoir surface is calculated using the relation between storage volume and surface area is needed. This function is plotted in Figure 3.2.1.2. A linear approximation to the surface area – volume relation is used between the dead storage and total storage levels (Loucks et al., 1981). The model is

$$\text{Minimize } \sum_{t=1}^{T} \left[r_t - d_t \right]^2 \tag{3.2.2.1}$$

subject to

$$(1 + \alpha_t)s_t = (1 - \alpha_t)s_{t-1} + q_t - r_t - \beta_t \qquad t = 1,...,T \tag{3.2.2.2}$$

$$s_t \leq K \qquad t = 1,...,T \tag{3.2.2.3}$$

$$\alpha_t = \frac{A_a e_t}{2} \qquad t = 1,...,T \tag{3.2.2.4}$$

$$\beta_t = A_0 e_t \qquad t = 1,...,T \tag{3.2.2.5}$$

where:

- s_t Storage at the end of time period t, (L^3);
- s_{t-1} Storage at the beginning of time period t, (L^3);
- q_t Inflow volume during time period t, (L^3);
- r_t Release volume time period t, (L^3);
- e_t Evaporation rate, (L);
- d_t Demand, (L^3); and
- K   Capacity (total storage) of reservoir, (L^3)

The GAMS code for the model is shown below.

```
SCALAR K       /19500/;
SCALAR S_min   /5500/;
SCALAR beg_S   /15000/;

SETS
t    / t1*t12/;

$include River1B_Q_Dry.inc
$include River1B_D.inc
$include River1B_Evap.inc

POSITIVE VARIABLES
S(t), R(t);

S.UP(t)=K;
S.LO(t)=S_min;

VARIABLES
obj;

EQUATIONS   objective, balance(t);
```

```
objective.. obj =E= SUM(t,power((R(t)-D(t)),2));

balance(t).. (1+a(t))*S(t) =E= (1-a(t))*beg_S $(ord(t) EQ 1)
                                + (1-a(t))*S(t-1)$(ord(t) GT 1)
                                + Q(t) - R(t)- b(t);

MODEL Reservoir / ALL /;
SOLVE Reservoir USING NLP MINIMIZING obj;

FILE res /River1b.txt/;
PUT res
PUT "                   (mln.m3)    (mln.m3) (mln.m3)   (mln.m3)"/;
PUT "                   Storage      Input   Release    Demand"/;
PUT "t0            ", beg_S/;
LOOP(t,PUT t.TL, S.L(t), Q(t), R.L(t), D(t) /;);
```

Note that in this model we have used a couple of new GAMS operators. First, the '*$include*' operator is used to copy the contents of 3 data files containing:

(1) inflow data, *River1B_Q_Ave.inc:*

```
PARAMETER
Q(t) inflow (million m3)
* normal
/
t1   426
t2   399
t3   523
t4   875
t5  2026
t6  3626
t7  2841
t8  1469
t9   821
t10 600
t11 458
t12 413
/;
```

(2) demand data, *River1B_D.inc*:

```
Parameter
D(t) demand (million m3)
/
t1   1699.5
t2   1388.2
t3   1477.6
t4   1109.4
t5    594.6
t6    636.6
t7   1126.1
t8   1092.0
t9    510.8
t10   868.5
t11  1049.8
t12  1475.5
/;
```

and (3) evaporation data, *River1B_Evap.inc*:

```
Parameter
a(t) evaporation coefficient
/
t1  0.000046044
t2  0.00007674
t3  0.000180339
t4  0.000391374
t5  0.000602409
t6  0.000648453
t7  0.000656127
t8  0.000548691
t9  0.0003837
t10 0.000145806
t11 0.000103599
t12 0.000053718
/;

Parameter
b(t) evaporation coefficient
/
t1   1.92
t2   3.2
t3   7.52
t4  16.32
t5  25.12
t6  27.04
t7  27.36
t8  22.88
t9  16.00
t10  6.08
t11  4.32
t12  2.24
/;
```

into the model at the specified locations. This allows easy input and maintenance of data with a spreadsheet and the ability to quickly change complex input conditions from one case to another. This is illustrated below where the results of running the model with the file *River1B_Q_Dry.inc*

```
PARAMETER
Q(t) inflow (million m3)
* dry
/
t1    375
t2    361
t3    448
t4    518
t5   1696
t6   2246
t7   2155
t8   1552
t9    756
t10   531
t11   438
t12   343
/;
```

as the inflow condition instead of the previously used average inflow condition.

Second, the '$' or 'conditional' operator is used in the 'balance' equation in conjunction with the 'ord' operator to pick out the initial time period and make sure that the initial storage volume 'beg_s' is used as the previous month's storage. The effect of this is that when the condition after the '$' is true then the value before the '$' sign is used, otherwise it is omitted. The condition is that the ordinal value of the index of the set t must be 1 for the condition to be true in the first case or greater than 1 in the second case.

The results from running the model using the average inflow conditions are:

	(mln.m3) Storage	(mln.m3) Input	(mln.m3) Release	(mln.m3) Demand
t0	15000.00			
t1	13723.26	426.00	1699.50	1699.50
t2	12728.83	399.00	1388.20	1388.20
t3	11762.29	523.00	1477.60	1477.60
t4	11502.47	875.00	1109.40	1109.40
t5	12894.05	2026.00	594.60	594.60
t6	15837.78	3626.00	636.60	636.60
t7	17503.44	2841.00	1126.10	1126.10
t8	17838.17	1469.00	1092.00	1092.00
t9	18118.57	821.00	510.80	510.80
t10	17838.75	600.00	868.50	868.50
t11	17239.00	458.00	1049.80	1049.80
t12	16172.46	413.00	1475.50	1475.50

and the results from running the model using the dry inflow conditions are:

	(mln.m3) Storage	(mln.m3) Input	(mln.m3) Release	(mln.m3) Demand
t0	15000.00			
t1	13672.26	375.00	1699.50	1699.50
t2	12639.84	361.00	1388.20	1388.20
t3	11598.35	448.00	1477.60	1477.60
t4	10981.79	518.00	1109.40	1109.40
t5	12044.20	1696.00	594.60	594.60
t6	13609.93	2246.00	636.60	636.60
t7	14592.96	2155.00	1126.10	1126.10
t8	15013.84	1552.00	1092.00	1092.00
t9	15231.43	756.00	510.80	510.80
t10	14883.46	531.00	868.50	868.50
t11	14264.32	438.00	1049.80	1049.80
t12	13128.11	343.00	1475.50	1475.50

A comparison of the results from using the average and dry inflow conditions is illustrated in Figure 3.2.2.1.
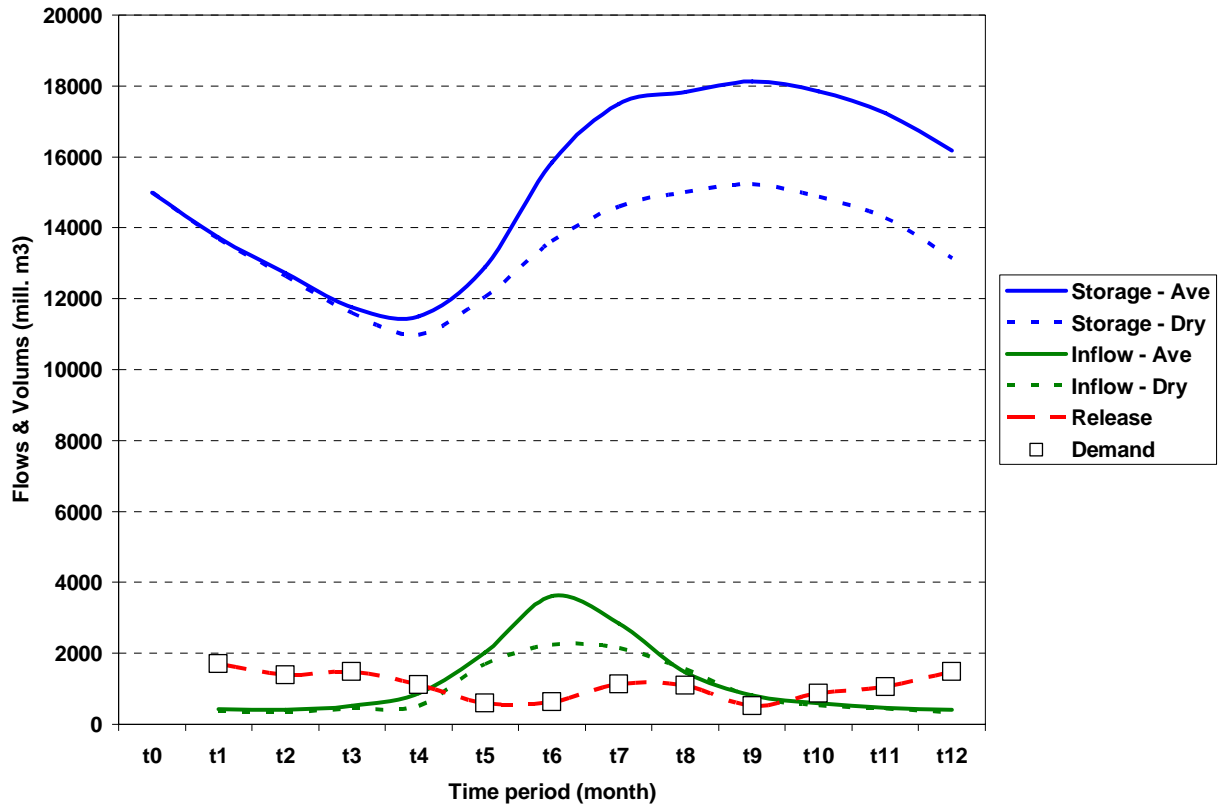
Figure 3.2.2.1. Reservoir model results using both average and dry inflow conditions.

3.3 Management of a River System

Figure 3.3.1 represents a simple river system, which includes water sources, reservoirs, consumers, and an estuary. It is desired to allocate the available water in such a way that demand is satisfied. The problem is based on graph theory, whereby conservative transport of water is carried out through a directed graph. At the nodes of the graph water resources change according to certain rules. For each type of node, the inflow (*Win*) and outflow (*Wout*) are calculated. *Win* is calculated as the sum of all releases (*Wout*) from all upstream nodes of the node under consideration. Withdrawal of water to consumers is subtracted at nodes under consideration. In this way a mass balance is maintained at each node in the system.
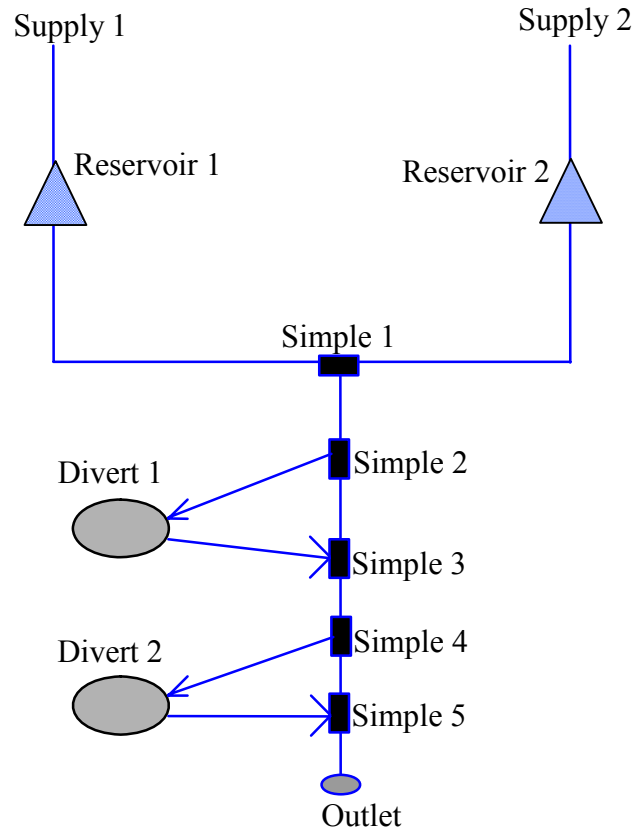
Figure 3.3.1. Structure of a River Network for the Water Management Task

Objective

$$Maximize \ \sum_{n}\sum_{t}\frac{Divert(n,t)}{Demand(n,t)} \qquad (3.3.1a)$$

where

Divert(n,t)	Diversion of water for consumer n in time period t;
Demand(n,t)	Demand for water by consumer n in time period t; and

This objective (percentage of met demand) is a linear objective and combined with the linear constraints presented below, results in a linear program. An alternative to this objective is one in which deviations of diversions from demands are minimized, resulting in a nonlinear (quadratic) program and requiring the use of a NLP solver in GAMS.

$$Minimize \ \sum_{n}\sum_{t}\left[Divert(n,t)-Demand(n,t)\right]^2 \qquad (3.3.1b)$$

Constraints

Simple nodes (junctions and other control nodes) $n \in nn$ and for each time period t, we have

$$\sum_{n \in out} R(n,t) = \sum_{n \in in} Q(n,t) + Supply(n,t) \qquad (3.3.2)$$

where

$R(n,t)$ Release from node n in period t (L^3). Releases are summed over all downstream nodes (*out*) that receive water released from the node;

$Q(n,t)$ Inflow to node n in period t (L^3). Inflows are summed over all upstream nodes (*in*) that release water to the node.

$Supply(n,t)$ Source of water at node $n \in ns$ and for each time period t.

Irrigation nodes $n \in nrr$, we calculate return flow as

$$R(n,t) = R_j * \sum_{n \in in} Q(n,t) \qquad (3.3.3)$$

where R_j is the return flow ratio for node n.

Reservoir nodes $n \in nl$, water balances in time period t are calculated as

$$S(n,t) = S(n,t-1) \sum_{n \in in} Q(n,t) - \sum_{n \in out} R(n,t) \qquad (3.3.4)$$

where $S(n,t)$ is the volume of water in reservoir n at the end of time period t (L^3);

The GAMS code is shown below.

```
SET n nodes
/ Supply_1, Supply_2,
  Simple_1*Simple_5,
  Divert_1, Divert_2,
  Res_1, Res_2,
  Outlet /;

ALIAS(n,n1);

SET
nn(n)      Simple nodes      /Simple_1*Simple_5/
ns(n)      Supply nodes      /Supply_1, Supply_2/
nr(n)      Water user nodes /Divert_1, Divert_2, Outlet/
nrr(n)     Irrigation nodes /Divert_1, Divert_2/
nl(n)      Reservoir nodes  /Res_1, Res_2/;

SET n_from_n(n,n1)  node n gets water from node n1 (any node)
/ Res_1.Supply_1,
  Res_2.Supply_2,
  Simple_1.Res_1,
  Simple_1.Res_2,
```

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```
      Simple_2.Simple_1,
      Divert_1.Simple_2,
      Simple_3.Simple_2,
      Simple_3.Divert_1,
      Simple_4.Simple_3,
      Divert_2.Simple_4,
      Simple_5.Simple_4,
      Simple_5.Divert_2,
      Outlet.Simple_5 /;

SET n_to_nr(n,n1)  node n diverts water to node n1 (user node)
/ Simple_2.Divert_1,
      Simple_4.Divert_2,
      Simple_5.Outlet /;

SET t months /Jan,Feb,Mar,Apr,May,Jun,Jul,Aug,Sep,Oct,Nov,Dec /;

PARAMETER beg_S(n)     initial storage
/ Res_1  1000,
   Res_2  300 /;

PARAMETER Ret(n)      return flow coefficients
/ Divert_1  0.5,
   Divert_2  0.5,
   Outlet    0.0 /;

TABLE Supply(n,t) water supplies (m3 per sec)
           Jan    Feb    Mar    Apr   May   Jun   Jul   Aug    Sep    Oct    Nov    Dec
Supply_1  128    125    234    360   541   645   807   512    267    210    181    128
Supply_2   39     39     52    121   168   144   105    78     49     44     45     39

TABLE Demand(n,t) water demands (m3 per sec)
          Jan  Feb  Mar  Apr     May    Jun    Jul     Aug    Sep  Oct  Nov
Dec
Divert_1   0    0    0   64.5   109.8  184.4  243.7   200.9  99.5   0    0
0
Divert_2   0    0    0   13.5    15.0   22.1   26.0    24.9  13.0   0    0
0
Outlet    500  500  500  100     100    100    100     500   500  500  500
500

POSITIVE VARIABLES
Divert(n,t)   Diversions
Q(n,t)        Inflows
R(n,t)        Releases
S(n,t)        Storages;

VARIABLE obj;

* Upper bound on diversions
Divert.up(n,t) = Demand(n,t);

* Upper bound of reservoirs
S.up('Res_1' ,t     )    = 1000;
S.up('Res_2' ,t     )    = 300;

* Final storage of reservoirs
S.lo('Res_1' ,'Dec' )    = 1000;
S.lo('Res_2' ,'Dec' )    = 300;

EQUATIONS
```

```
R_no(n,t)     Simple node
R_ns(n,t)     Source node
R_nr(n,t)     Irrigation node
R_nl(n,t)     Reservoir node
R_nn(n,t)     Simple node
Objective;

* Simple node:  Release = Inflow
R_no(n,t)$(nn(n)).. R(n,t) =e= Q(n,t);

* Source node:  Release = Supply
R_ns(n,t)$(ns(n)).. R(n,t) =e= Supply(n,t);

* Irrigation node:  Release = Return Flow = Percent of Diversion
R_nr(n,t)$(nr(n)).. R(n,t) =e= ret(n)*Divert(n,t);

* Reservoir node:  Release = Mass Balance
R_nl(n,t)$(nl(n)).. S(n,t) =e= beg_S(n)$(ord(t) EQ 1)
                             +  S(n,t-1)$(ord(t) GT 1)
                             + Q(n,t)-R(n,t);

* Simple node:  Inflow = Sum of Releases from Upstream - Sum of Diversions
R_nn(n,t).. Q(n,t) =e= sum(n1$(n_from_n(n,n1)),R(n1,t))
                      - sum(n1$(n_to_nr(n,n1)),Divert(n1,t));

*Sum over Irrigation Diversions/Demands + Downstream Flow/Demand
* objective.. obj =e= sum(t,sum(n$nr(n), (Divert(n,t)/(Demand(n,t)+1e-
9))));

*Sum over Irrigation Diversions/Demands + Downstream Flow/Demand
objective.. obj=e=sum(t,sum(n$nr(n), power((Divert(n,t)-Demand(n,t)),2) ));

MODEL Lakes /all/;

*SOLVE Lakes USING LP MAXIMIZING obj;
SOLVE Lakes USING NLP MINIMIZING obj;

file res /River2.txt/
put res;
put "Node   Divert_1 Demand_1 Divert_2 Demand_2   Outlet   Demand_O"/;
loop((t),put t.TL:6, Divert.L('Divert_1',t):9.2, Demand('Divert_1',t):9.2,
                     Divert.L('Divert_2',t):9.2, Demand('Divert_2',t):9.2,
                 Divert.L('Outlet',t)  :9.2, Demand('Outlet',t)  :9.2 /;);

put /"Reservoirs   "/;
put "                    Reservoir 1             Reservoir 2 "/;
put "           Inflow Storage Release  Inflow Storage Release  "/;
put "             Q-1      S-1     R-1    Q-2     S-2      R-2  "/;
loop((t),put t.TL:8, Q.L('Res_1',t):8.1, S.L('Res_1',t):8.1,
R.L('Res_1',t):8.1,
                     Q.L('Res_2',t):8.1, S.L('Res_2',t):8.1,
R.L('Res_2',t):8.1 /;);
```

Note that in this model we have used a couple of new GAMS operators:

1. *Subsets* of sets are used to partition the set of nodes into different types: simple nodes, supply nodes, water user nodes, irrigation nodes, and reservoir nodes. This facilitates the construction of a general model where different nodes serve different functions in the

network and transform water flows or other properties in different ways.
2. Alias is used to define another set of nodes exactly the same as the first set (*n* and *n1*). That way the two sets can be used to refer to two different nodes simultaneously in a two-dimensional set.
3. *Two-dimensional sets* are used to establish the topology of the network. That is, the upstream to downstream connections between nodes are identified in a two-dimensional set *n_from_n(n,n1)*. These connections are indicated by using the '*dot*' operator between two members of the two sets, e.g., *Res_1.Supply_1* where *Res_1* is from set *n* and represents the node for reservoir 1 and *Supply_1* is from set *n1* and represents the node for water supply 1.

In the example several parameters are specified that have a distinct effect on the solution:

1. The capacities of the two reservoirs in the system, 1000 and 300 units, respectively.
2. The initial volume of water stored in the reservoirs, both reservoirs at full capacity.
3. The final volume of water required to be stored in the reservoirs, both reservoirs at full capacity.
4. The instream flow requirement at the outlet, varies between 500 units in winter to 100 units in summer.

All of these factors have a distinct and significant effect on the solution and the reader can modify them and rerun the model to see the effect.

Using the nonlinear objective function, the results are:

```
Node    Divert_1 Demand_1 Divert_2 Demand_2   Outlet   Demand_O
Jan        0.00     0.00     0.00     0.00    500.00    500.00
Feb        0.00     0.00     0.00     0.00    500.00    500.00
Mar        0.00     0.00     0.00     0.00    500.00    500.00
Apr       64.50    64.50    13.50    13.50    100.00    100.00
May      109.80   109.80    15.00    15.00    100.00    100.00
Jun      184.40   184.40    22.10    22.10    100.00    100.00
Jul      243.70   243.70    26.00    26.00    100.00    100.00
Aug      193.27   200.90    17.27    24.90    484.73    500.00
Sep        6.00    99.50     0.00    13.00    313.00    500.00
Oct        0.00     0.00     0.00     0.00    254.00    500.00
Nov        0.00     0.00     0.00     0.00    226.00    500.00
Dec        0.00     0.00     0.00     0.00    167.00    500.00


               Reservoir 1              Reservoir 2
          Inflow Storage Release   Inflow Storage Release
            Q-1     S-1     R-1      Q-2     S-2     R-2
Jan      128.00  382.00  746.00    39.00  168.00  171.00
Feb      125.00  107.00  400.00    39.00  107.00  100.00
Mar      234.00    0.00  341.00    52.00    0.00  159.00
Apr      360.00    0.00  360.00   121.00    0.00  121.00
May      541.00    0.00  541.00   168.00  156.00   12.00
Jun      645.00  331.70  313.30   144.00  300.00    0.00
Jul      807.00 1000.00  138.70   105.00  300.00  105.00
Aug      512.00 1000.00  512.00    78.00  300.00   78.00
Sep      267.00 1000.00  267.00    49.00  300.00   49.00
Oct      210.00 1000.00  210.00    44.00  300.00   44.00
Nov      181.00 1000.00  181.00    45.00  300.00   45.00
Dec      128.00 1000.00  128.00    39.00  300.00   39.00
```

The percent of the satisfied demand is illustrated in Figure 3.3.2 and the reservoir operation is shown

in Figure 3.3.3. From these figures one can see the effect of requiring that the reservoirs return to the initial condition at the end of the year, stored water is not available to meet demand in September and downstream flow demands are unmet from July to December. One interesting modification to the model would be to apply weight factors to the three different demands and see how varying the weights affects the solution.
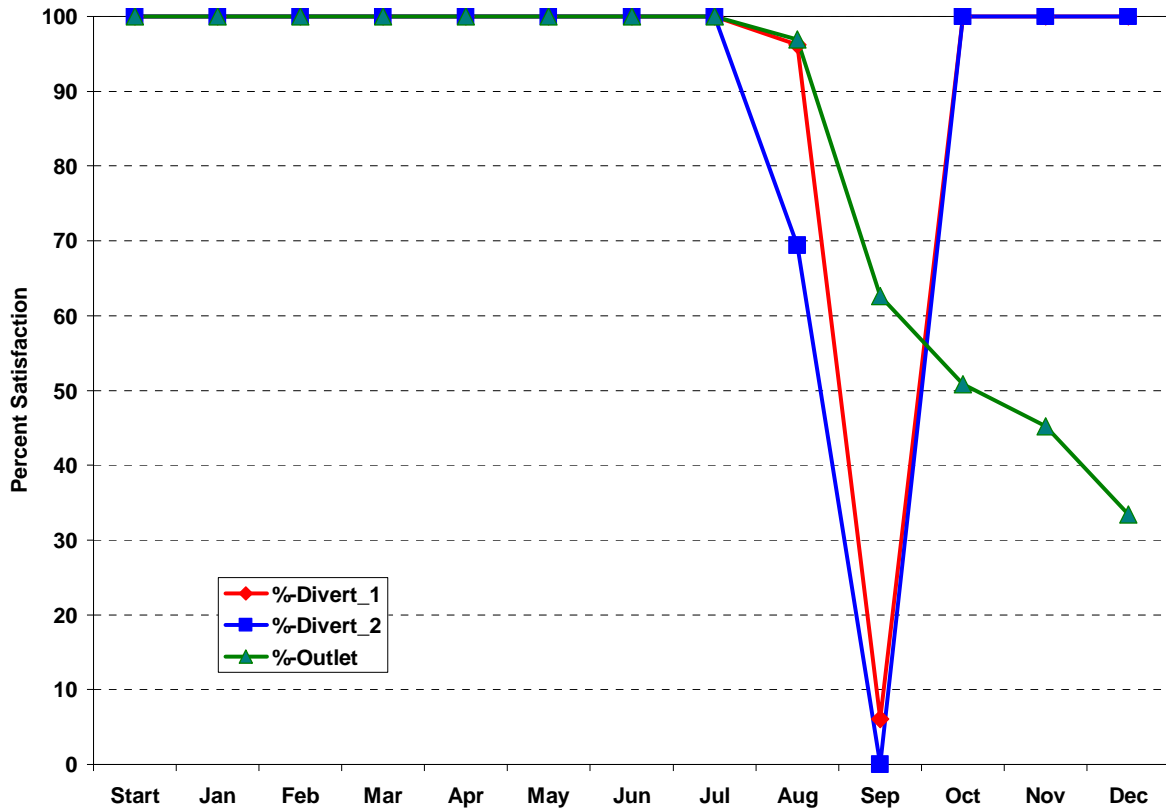
Figure 3.3.2. Demand satisfaction chart for river management example using nonlinear objective function.
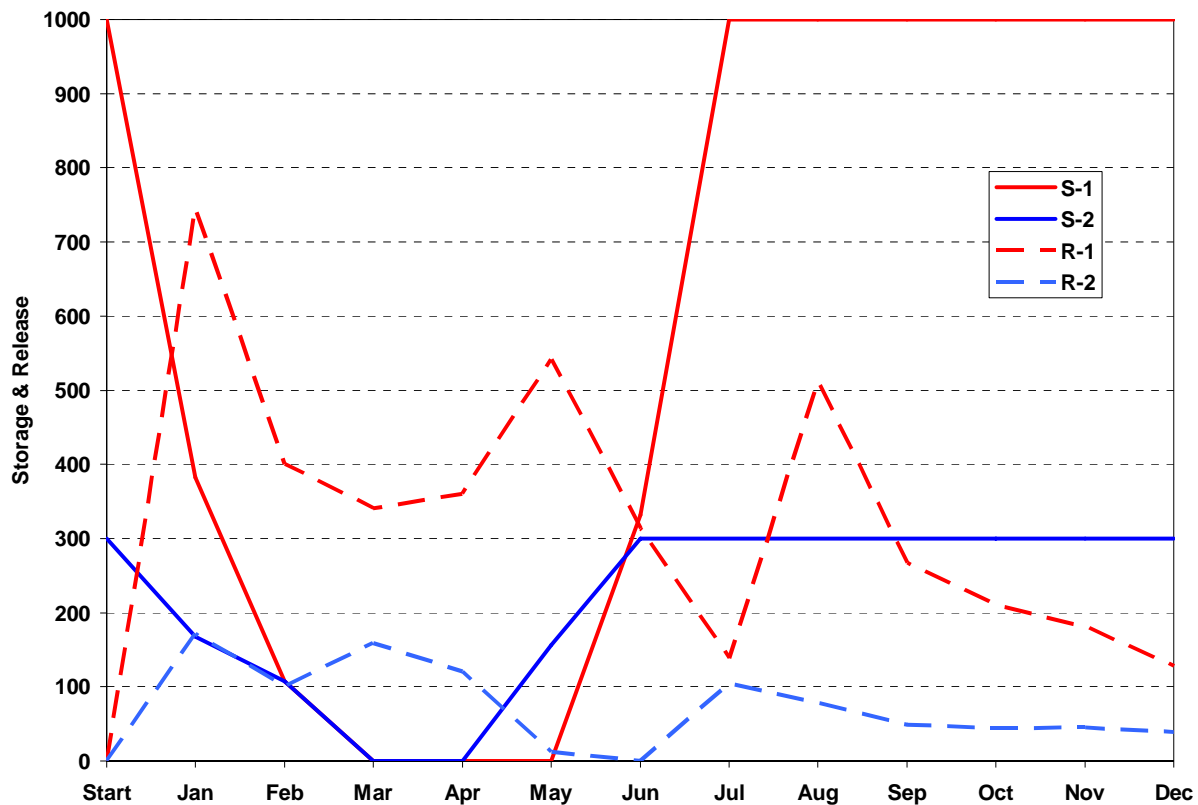
Figure 3.3.3. Reservoir storages and releases for river management example using nonlinear objective function.

This example served as the basis for the development of the models of optimal management of water resources in river systems (see N. Kipshakbaev and A. Tasybaev, Vol. 2, Section 1.1, "Optimization of the Syrdarya Water and Energy Uses under Current Conditions: Kazakhstan Part," McKinney, D.C. and A.K. Kenshimov (eds.), 2000).

3.4 Onstream and Offstream Reservoir Management

Consider the following example of solving a complex problem of using conditional operators with variables. We have two reservoirs. The first reservoir takes water from upstream (according to a hydrograph of inflows) and releases it to downstream demands (according to a schedule of demands). The second reservoir receives water from the first reservoir through a valve that opens when the volume in the first reservoir exceeds the volume in the second reservoir. The second reservoir sends water back to the first reservoir through a valve that opens when the volume in the first reservoir is insufficient to meet demands. In this case, water from the second reservoir will make up the deficit of the first reservoir through the return valve.

This is not an abstract example, but it is taken from real life. Tuyamuyun reservoir on the Amu Darya River in Central Asia is comprised of two parts: an onstream and an offstream part. Water comes to the offstream part of the reservoir by gravity flow when the water level of the onstream part exceeds the water level of the offstream part. On the other hand, water of the offstream reservoir comes to the onstream section when there is a shortage of irrigation water for agriculture. Water storage in the offstream reservoir is more preferable than in the onstream part, due to its structure. Hence, preferential storage of water is achieved in the offstream part of the reservoir.
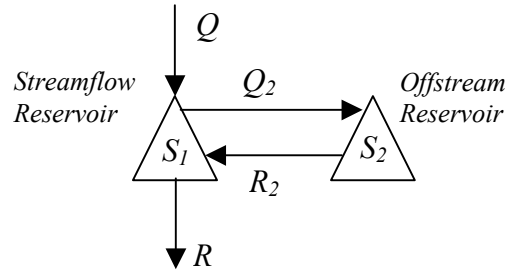
Figure 3.4.1. On- and off-stream reservoir system diagram.

The mathematical description of the problem can be written as:

1) $\quad S_1^t = S_1^{t-1} + Q^t - R^t - Q_2^t + R_2^t \qquad$ Balance equation for onstream reservoir (1)

2) $\quad S_2^t = S_2^{t-1} + Q_2^t - R_2^t \qquad\qquad$ Balance equation for offstream reservoir (2)

3) $\quad S_1^t = S_2^t \qquad$ if $Q^t > 0,$ and $R_2^t = 0$ Switch for flow from on- to off-stream

$$S_{1,min} < S_1^t < S_{1,max}$$

$$S_{2,min} < S_2^t < S_{2,max}$$

$$0 < Q^t < Q_{max}$$

$$0 < R_2^t < Q_{max}$$

$$Q^t = F^t, \qquad S_1 = S_1^0 \qquad \text{if } t = 0$$

$$Q_2^t = F_2^t, \qquad S_2 = S_2^0 \qquad \text{if } t = 0$$

4) $Minimize \sum R_2^t$

where

        Q - inflow to the streamflow reservoir,
        R - release from the streamflow reservoir
        Q_2 - inflow to the streamflow reservoir,
        R_2 - release from the streamflow reservoir
        S_1 - storage volume in streamflow reservoir
        S_2 - storage volume in streamflow reservoir

The third equation enters the calculation when the volumes in both reservoirs are equal. Equation 4 is the objective function which is to minimize the need for releases from the offstream reservoir.

How can we model this situation in GAMS? Since we can not take equation 3 out of the computation process using the conditional operator, we must eliminate its impact by isolating the elements of the equation through transforming it into an identity.

```
SETS
n reservoir  /tmgu,sult/;

SETS
t  time    / jan, feb, mar, apr, may,
             jun, jul, aug, sep, oct,
             nov, dec, eenndd /
tt(t) /jan/  ;

TABLE Q(n,t)   inflow to the onstream reservoir(mln.cub.m)
         jan feb mar apr may jun jul aug sep oct nov dec  eenndd
tmgu       128 125 234 360 541 645 807 512 267 210 981 928   250

TABLE R(n,t) required releases from the onstream reservoir (mln.cub.m);
         jan feb mar apr may jun jul aug sep oct nov dec  eenndd
tmgu       111 111 111 500 222 700 333 333 333 333 333 333   200

VARIABLES
            Q2(t),
            R2(t),
            S(n,t),
            obj;

EQUATION
balanc1(n,t),     water balance of the stream-flow reservoir;
balanc2(n,t),     water balance of the offstream reservoir;
rules1(n,t),      rules of filling the reservoirs;
ben;              objective function;

balanc1(n,t)$(not tt(t))..
          S('tmgu',t)-S('tmgu',t-1)
        =e=
          +Q('tmgu',t)-R('tmgu',t)
          -Q2(t)+R2(t);

balanc2(n,t)$(not tt(t))..
          S('sult',t)-S('sult',t-1)
        =e=
```

```
          Q2(t)-R2(t);

rules1(n,t)$(not tt(t))..
          (S('sult',t)-S('tmgu',t))      -
          (S('sult',t)-S('tmgu',t))      *
           (1.0-Q2(t) /(Q2(t)+0.00000001))
        =e=
          0.0;

ben.. obj =e= SUM(t$(not tt(t)),R2(t));

S.LO('tmgu ',t) = 1150; lower level of the stream-flow reservoir;
S.UP('tmgu ',t) = 4590; upper level of the stream-flow reservoir;
S.FX('tmgu ','jan') = 1200; initial level of the stream-flow reservoir;
S.LO('sult ',t    ) = 100 ; lower level of the offstream reservoir;
S.UP('sult ',t    ) = 4590; upper level of the offstream reservoir;
S.FX('sult ','jan') = 1200; initial level of the offstream reservoir;
R2.UP(t)= 1500;                   upper limits of release from offstream;
R2.LO(t)= 0.0;                    lower limits of release from offstream;
Q2.UP(t)= 1500;                   upper limits of diversion to offstream;
Q2.LO(t)= 0.0;                    lower limits of diversion to offstream;
Q2.L(t)= 0.00001;                 initial stage of finding a solution;

OPTION ITERLIM= 20000;       not more than 20000 iterations
OPTION LIMROW =12;           list all estimated time intervals;
OPTION OPTCR  =0.000001;     accuracy of optimal solution;

OPTION nlp  =minos5;

MODEL tmgu   /all/;

SOLVE tmgu   USING  NLP MINIMIZING obj;

PARAMETER A(t);
A(t)=(1-(Q2.l(t)/(abs(Q2.l(t))+0.00001)));

FILE res /River0.txt/
PUT res;
PUT " objective = ",obj.l:10:5;
PUT /"  ========balance ====================  "/;
PUT "                                         "/;
PUT " A       R2      Q2      S_sult    S_tmgu      Q      ";
PUT "      R          in-out   dS-tmgu   dS-sult "/;
LOOP((t)$(ord t ne card(t)),
 PUT   A(t):5:2,R2.L(t):7:1,Q2.L(t):7:1,S.L('sult',t):10:1,S.L('tmgu',t):10:1,
       Q('tmgu',t ):10:1,R('tmgu',t ):10:1,(Q('tmgu',t )-R('tmgu',t )):10:1,
       (S.L('tmgu',t)-S.L('tmgu',t-1)):10:1,(S.L('sult',t)-S.L('sult',t-1)):10:1
/;);
 PUT " ==================================== "/;
```

The result of the calculation will be represented in text file:

```
objective =   21.50000
  ========balance ====================

 A       R2      Q2      S_sult    S_tmgu       Q         R         in-out   dS-tmgu   dS-sult
 0.50    0.0    0.0     1200.0    1200.0     128.0     111.0      17.0    1200.0    1200.0
 0.00    0.0    7.0     1207.0    1207.0     125.0     111.0      14.0       7.0       7.0
 0.00    0.0   61.5     1268.5    1268.5     234.0     111.0     123.0      61.5      61.5
 1.00   21.5    0.0     1247.0    1150.0     360.0     500.0    -140.0    -118.5     -21.5
```

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```
0.00    0.0  111.0   1358.0   1358.0   541.0   222.0   319.0   208.0   111.0
1.00    0.0    0.0   1358.0   1303.0   645.0   700.0   -55.0   -55.0     0.0
0.00    0.0  209.5   1567.5   1567.5   807.0   333.0   474.0   264.5   209.5
0.00    0.0   89.5   1657.0   1657.0   512.0   333.0   179.0    89.5    89.5
1.00    0.0    0.0   1657.0   1591.0   267.0   333.0   -66.0   -66.0     0.0
1.00    0.0    0.0   1657.0   1468.0   210.0   333.0  -123.0  -123.0     0.0
0.00    0.0  229.5   1886.5   1886.5   981.0   333.0   648.0   418.5   229.5
0.00    0.0  297.5   2184.0   2184.0   928.0   333.0   595.0   297.5   297.5
====================================
```

From the results we see that the offstream reservoir was used during the third time interval, when the stream-flow reservoir was incapable of covering the required release.

3.5 Managing Water Quality in Streams

```
SETS  i reaches /1*7/
SETS  k treatment levels /1*4/

ALIAS(i,i1);
SET i_from_i(i,i1)  reach i gets water from node i1 (any node)
/
3.1,
3.2,
5.3,
5.4,
6.5,
7.6
/;

Parameter W(i) Waste input to plant (mg per L)
/
1 248,
2 408,
3 240,
4 1440,
5 0,
6 2180,
7 279
/;

Parameter E_init(i)  Initial treatment level of plant
/
1 0.67,
2 0.30,
3 0.30,
4 0.30,
5 0.0,
6 0.30,
7 0.30
/;

Parameter DO_Sat(i) Saturate DO level in reach (mg per L)
/
1 10.20,
2 9.95,
3 9.00,
4 9.70,
5 9.00,
6 8.35,
```

```
7 8.17
/;

Table C(i,k)   Treatment plant costs for various levels of treatment ($)
          1                 2                 3                 4
1      0            22100             77500            120600
2 630000           780000            987000           1170000
3 210000           277500            323000            378000
4 413000           523000            626000            698000
5 0                     0                 0                 0
6 500000           638000            790000            900000
7 840000          1072000           1232500           1350000

Parameter R(k) Possible treatment levels
/
1 0.60,
2 0.75,
3 0.85,
4 0.90
/;

Parameter Qw(i) Wastewater inflows (million m3)
/
1 19,
2 140,
3 30,
4 53,
5 0,
6 98,
7 155
/;

Parameter Dw(i) DO deficit in wastewater inflow (mg per L)
/
1 1.0,
2 1.0,
3 1.0,
4 1.0,
5 0.0,
6 1.0,
7 1.0
/;

Parameter Qi(i) Source inflow (million m3)
/
1 5129,
2 4883,
4 1120
/;

Parameter Dmax(i) Maximum allowed DO deficit on reach (mg per L)
/
1 3.2,
2 2.45,
3 2.00,
4 3.75,
5 2.50,
6 2.35,
7 4.17
/;
```

```
Parameter Di(i) DO Deficit in source flow (mg per L)
/
1 0.70,
2 1.95,
4 0.16
/;

Parameter Bi(i)  BOD level in source flow (mg per L)
/
1 1.66,
2 0.68,
4 1.0
/;

Parameter k1(i)  Deoxygenation coefficient (day-1)
/
1 0.93,
2 0.58,
3 0.68,
4 0.49,
5 0.90,
6 0.69,
7 0.16
/;

Parameter k2(i)  Coefficient (day-1)
/
1 0.06,
2 0.28,
3 0.23,
4 0.46,
5 0.09,
6 0.28,
7 0.78
/;

Parameter k3(i)  Reaeration Coefficient (day-1)
/
1 0.79,
2 0.45,
3 0.50,
4 0.83,
5 0.80,
6 0.86,
7 0.88
/;

Variables
obj         Objective value;

Positive variables
E(i)        level of treatment,
Bw(i)       BOD leaving treatment plant (mg per L),
B(i)        BOD in reach (mg per L),
D(i)        DO deficit in reach (mg per L),
B_minus(i) BOD at beginning of reach (mg per L),
D_minus(i) DO deficit at beginning of reach (mg per L),
B_plus(i)  BOD at end of reach (mg per L),
D_plus(i)  DO deficit at end of reach (mg per L),
Q(i)        Flow in reach (million m3);
```

```
Integer variables
Z(i,k)      integer variables indicating the level of treatment;

E.l(i) = E_init(i);
E.lo(i) = E_init(i);
E.fx('5') = 0.0;

Equations
Objective,
Zsum(i) Sum of integer variables,
Treatment(i) Level of treatment in terms of available levels,
BOD_Plant(i) Output BOD level of plant,
Reach_Flow(i) FLow in reach,
Reach_BOD(i) BOD in reach,
Reach_DO(i) DO deficit in reach,
BOD_Plus(i) BOD level at end of reach,
DO_Plus(i) DO deficit level at end of reach,
DO_Max(i) DO deficit constraint at begin of reach,
DO_Plus_Max(i) DO deficit constraint at end of reach
;

Objective..       obj       =E= sum(k,sum(i,C(i,k)*Z(i,k)));

Zsum(i)..         sum(k,Z(i,k)) =L= 1;

Treatment(i)..    E(i)      =E= sum(k,R(k)*Z(i,k));

BOD_Plant(i)..    Bw(i)     =E= W(i)*(1.0-E(i));

Reach_Flow(i)..  Q(i)       =E= Qi(i)      +sum(i1$(i_from_i(i,i1)),Q(i1))
+Qw(i);

Reach_BOD(i)..                              B(i)*Q(i)            =E=
Bi(i)*Qi(i)+sum(i1$(i_from_i(i,i1)),B_plus(i1)*Q(i1))+Bw(i)*Qw(i);

Reach_DO(i)..                                D(i)*Q(i)           =E=
Di(i)*Qi(i)+sum(i1$(i_from_i(i,i1)),D_plus(i1)*Q(i1))+Dw(i)*Qw(i);

BOD_Plus(i)..    B_plus(i) =E= B(i)*k1(i);

DO_Plus(i)..     D_plus(i) =E= B(i)*k2(i)+D(i)*k3(i);

DO_Max(i)..      D(i)      =L= Dmax(i);

DO_Plus_Max(i).. D_Plus(i) =L= Dmax(i);

Model waterquality /all/;

Solve waterquality using MINLP minimizing obj;

file WQ /WQ.txt/
put WQ
put 'Site   E(i)' put /
loop( i,  put i.tl, put E.l(i), put /)
put /
put 'Site   Q(i)' put /
loop( i,  put i.tl, put Q.l(i), put /)
put /
put 'Site   Bw(i)' put /
loop( i,  put i.tl, put Bw.l(i), put /)
put /
```

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```
put 'Site                 B(i)   B_Plus(i)' put /
loop( i,  put i.tl, put B.l(i), put B_Plus.l(i), put /)
put /
put 'Site                 D(i)   D_Plus(i)     Dmax(i)         DO_Plus(i)'
put /
loop( i,   put i.tl, put D.l(i), put D_Plus.l(i), put Dmax(i), put
(DO_Sat(i)-D_Plus.l(i)),put /)
```

SOLUTION

```
Site   E(i)
1                 0.85
2                 0.75
3                 0.60
4                 0.90
5                 0.00
6                 0.90
7                 0.85


Site   Q(i)
1             5148.00
2             5023.00
3            10201.00
4             1173.00
5            11374.00
6            11472.00
7            11627.00


Site   Bw(i)
1               37.20
2              102.00
3               96.00
4              144.00
5                0.00
6              218.00
7               41.85

Site               B(i)   B_Plus(i)
1                  1.79       1.67
2                  3.50       2.03
3                  2.12       1.44
4                  7.46       3.66
5                  1.67       1.50
6                  3.35       2.31
7                  2.84       0.45

Site               D(i)   D_Plus(i)   Dmax(i)     DO_Plus(i)
1                  0.70       0.66      3.20         9.54
2                  1.92       1.85      2.45         8.10
3                  1.25       1.11      2.00         7.89
4                  0.20       3.60      3.75         6.10
5                  1.37       1.24      2.50         7.76
6                  1.24       2.01      2.35         6.34
7                  1.99       3.97      4.17         4.20
```

3.6 Exercises

1. a. Revise the model of Section 3.1 to solve the water user allocation problem where the objective function is given in Loucks et al., Problem 2.9. Note that problem 2.9 does not include the downstream user and this component of the model will have to be take out. The coefficients in the objective function will have to be changed to those specified in the problem (12, -1; 8, -1; and 18, -3) and the release is equal to 10 and 15 units of water. To show that the marginal values are all equal to the dual variable (Lagrange multiplier) for the constraint, note that this marginal value is printed in the solution report file. You can plot the individual objectives, mark the points where the optimal solution occurs for each decision variable (x_i) and compute the slope of the objective function there (the slopes should be the same for each decision variable).

b. Add a fourth user of water (x_4) to the model of Section 3.1 and resolve problem 2.9. The coefficients for the objective function are: $a_4 = 15$ and $b_4 = -2$.

1a. (Adapted from Loucks (2004) Chapter 5, Section 4.1.3) Assume that the benefit each of three water-using firms (i = 1,2,3) receives is determined by the quantity of product it produces, yi, and the price per unit of the product that is charged, pi. These products require water and water is the limiting resource. The amount of product produced by each firm i is dependent on the amount of water, xi, allocated to it. The demand functions are

$$p_i = a_i - b_i y_i$$

The associated cost functions are

$$Cost_i = c_i y_i^{d_i}$$

The production functions

$$y_i = e_i x_i^{f_i}$$

where the coefficients are given in the following table.

Coefficient\Firm	1	2	3
a	12	20	28
b	1	1.5	2.5
c	3	5	6
d	1.3	1.2	1.15
e	0.4	0.5	0.6
f	0.9	0.8	0.7

Write a GAMS model to find the optimal water allocations, production levels, and prices that together maximize the total net benefit obtained from all three firms. The water allocations plus the amount that must remain in the river, R, cannot exceed the total amount of water Q available. Solve the model for available levels of water (Q – R) of 10, 20, 30, 38, and 38.3 units (see Table 5.2 in Loucks (2004) for the table of answers).

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2. (Adapted from Loucks et al., Problem 2-29(a)) A reservoir storage-yield function defines the maximum constant reservoir release that is available during a period of operation. Given a series of reservoir inflows, the yield from a reservoir is a function of the reservoir's active storage capacity. The reservoir storage-yield function can be determined by solving the following linear program:

$$\text{Maximize } R$$

subject to

$$S_{t+1} = S_t + Q_t - R_t - R \qquad t = 1,...,T; \; T+1 = 1$$

$$S_t \le K \qquad\qquad\qquad t = 1,...,T$$

where R is the yield of the reservoir (constant release in each time period), S_t is the volume of water stored in the reservoir in each period, Q_t is the inflow to the reservoir in each period, and R_t is the excess release (or spill) from the reservoir in each time period. Use linear programming to derive an annual storage-yield function for a reservoir at a site having the following record of annual flows of 5, 7, 8, 4, 3, 3, 2, 1, 3, 6, 8, 9, 3, 4, 9 units of flow. Find the values of storage required for yields of 2, 3, 3.5, 4, 4.5, and 5 units of flow.

3. Given 7 years of inflow values, the surface vs area data and relationship, and evaporation data for Toktogul reservoir in Kyrgyzstan (see Data Set 1), construct a storage-yield relationship for Toktogul. Show that the result for one point on the storage-yield relationship curve is the same for "Max R - Given K" and "Min K - Given R".

4. Consider the river network illustrated in Figure 3.5.1 below. The system contains one reservoir (Toktogul reservoir in Kyrgyzstan) and two irrigated zones (one in Uzbekistan and one in Kazakhstan). Four different crops can be grown in each irrigation zone: cotton, wheat, rice, and Lucerne. The profit per hectare for each crop is shown in Table 3.5.1 along with the maximum crop areas and the crop water requirements. The capacity of the reservoir is 19,500 million m3, and the dead storage capacity is 5,500 million m3. The initial storage volume is 14,000 million m3. Write a GAMS model to determine the optimal mix of crops to be grown by each of the countries if the ending storage in the reservoir must be 13,000 million m3. What impact does the ending storage volume have on the solution?

Table 3.5.1. Data for Rerervoir Operation Problem.

Crop	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
						Crop water demands (mm per month)						
Cotton	150	150	100	100	200	250	250	200	50	0	0	0
Wheat	0	0	0	100	300	300	0	0	200	0	0	0
Rice	0	0	0	450	700	800	800	700	50	0	0	0
Lucerne	0	0	0	50	200	350	350	250	50	0	0	0

Naryn River Inflow		Crop Profits		Maximum area per crop		
Month	Inflow	Crop	Profit	Crop	Uzbekistan	Kazakhstan

	(Million m3)		($/ha)		(ha)	(ha)
Jan	386	Cotton	350	Cotton	500,000	65,000
Feb	346	Wheat	70	Wheat	500,000	30,000
Mar	416	Rice	225	Rice	0	500,000
Apr	713	Lucerne	50	Lucerne	50,000	500,000
May	1532					
Jun	2373					
Jul	2157					
Aug	1431					
Sep	798					
Oct	582					
Nov	487					
Dec	420					

Maximum total area		Return Flow Coefficient	
Country	Area(ha)	Country	Coefficient
Uzbekistan	2,000,000	Uzbekistan	0.50
Kazakhstan	2,000,000	Kazakhstan	0.50
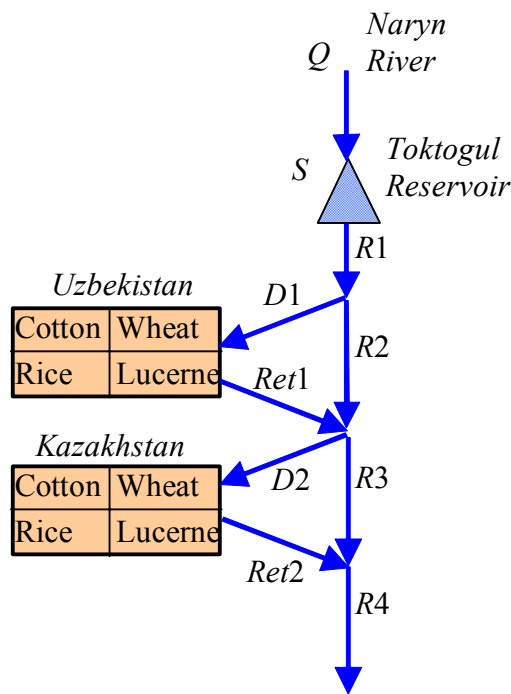
Figure 3.5.1. River Network for Reservoir Operation Task

3.5. Consider the river network illustrated in Figure 3.5.1 below. The system contains one long-term (over-year) storage reservoir (Toktogul reservoir in Kyrgyzstan), four constant volume (pass-through) reservoirs (Kurpsia, Tashkumur, Shamaldasai, and Uchkurgan, all in Kyrgyzstan), two irrigated zones (one in Uzbekistan and one in Kazakhstan). The crop information and initial and final storage information is the same as in Problem 3.4 above. Toktogul is the only reservoir in the

cascade for which the volume varies over time depending on the volume of water in storage. The equation expressing the head across the turbines as a function of the reservoir water elevation and storage volume for Toktogul reservoir is

$$H(t) = H_0 + aS(t)^b - H_{tail}$$

where H_0 is the elevation at dead storage, $S(t)$ is the volume of water in storage at time t, a and b are coefficients, and H_{tail} is the elevation of the water in the tail water of the reservoir. The numerical values are given in Table 3.5.1. For the other reservoirs in the cascade, the head across the turbines is a constant, given in the table. Write a GAMS model to determine the optimal mix of crops to be grown by each of the countries if the ending storage in the reservoir must be 13,000 million m3. In addition, compute the power generated from the releases of water.

Table 3.5.1. Data for Rerervoir Operation Problem with Power.

Reservoir	Efficiency	Installed Capacity	H_0	H_{Tail}	$H_{Constant}$
Toktogul	0.874	1200	758	720	-
Kurpsai	0.874	800	-	-	100
Tashkumur	0.874	450	-	-	53
Shamaldasia	0.874	240	-	-	18.5
Uchkurgan	0.874	180	-	-	28
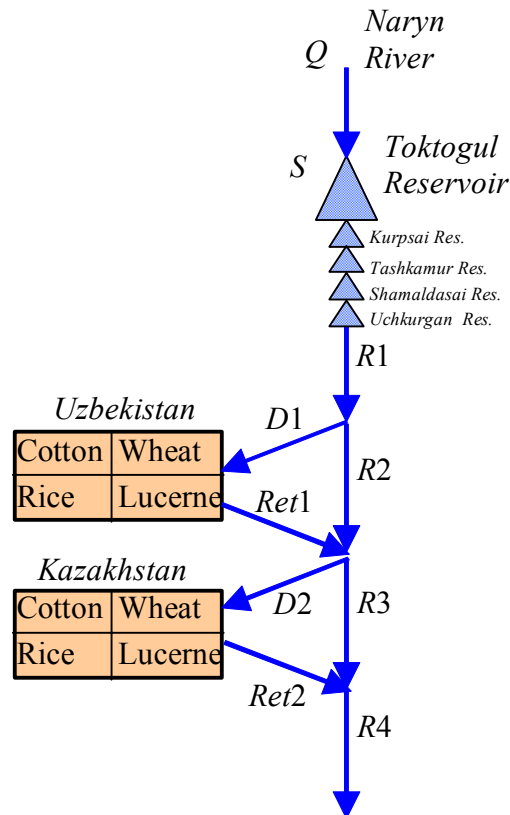
Figure 3.5.1. River network for reservoir operation fask, including power cascade

3.6. Consider the river network illustrated in Figure 3.5.1 above. Write a GAMS model to determine the optimal mix of crops to be grown by each of the countries if the ending storage in the reservoir must be 13,000 million m3. In addition, the Kyrgyz power demand must be satisfied by a combination of hydropower, thermal power, and regional grid power transfers. The total power produced is a combination of the hydropower produced, the thermal power generated by Kyrgyz power plants, and power transferred from the regional grid (see Figure 3.6.1)

$$P(t) = P_{thermal}(t) + P_{hydro}(t) + P_{transfer}(t)$$

where in any month t, $P(t)$ is the total power produced, $P_{thermal}(t)$ is the thermal power produced, $P_{hydro}(t)$ is the hydropower produced, and $P_{transfer}(t)$ is the power transferred from the regional grid.
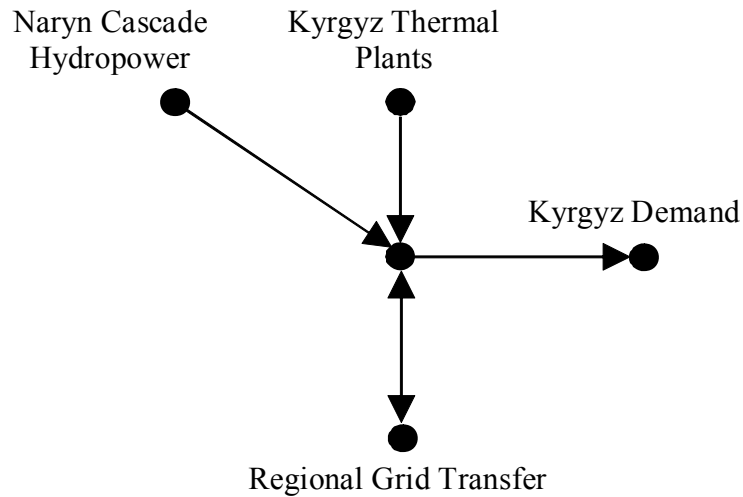
Figure 3.6.1. Power network for reservoir operation fask, including power cascade

The objective function can be formulated as

$$Z = \sum_{crops} gm(crop) * [A_2(crop) + A_2(crop)]$$
$$-\sum_{t} \{C_{thermal} * P_{thermal}(t) + C_{hydro} * P_{hydro}(t) + C_{transfer} * P_{transfer}(t) + [P(t) - E_{dem}(t)]^2\}$$

where $C_{hydro} = 0.001$ $/kWh is the cost to produce hydropower, $C_{thermal} = 0.012$ $/kWh is the cost to produce thermal power, $C_{transfer} = 0.013$ $/kWh is the cost to transfer power to or from the regional grid. The Kyrgyz power demand and the Kyrgyz thermal power generation capacity is limited according to the values in Table 3.6.1.

Table 3.6.1. Electricity Data for Rerervoir Operation Problem with Power.

		Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Electricity Demand	GWh	161 4 345	130 1 345	131 8 345	881 100	609 45	51 6 45	52 9 45	528 45	50 9 45	69 7 34	119 6 345	152 2 345
Thermal Generation (Max)	GWh										5		

3.7. (adapted from Loucks et al., Problem 5.14) Create and solve a GAMS model to find the solution which maximizes the total annual power production of the two reservoirs in series system shown in Figure 3.7.1.
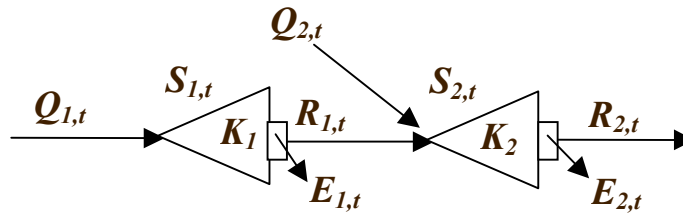
Figure 3.7.1. Two reservoirs in series.

The releases $R_{1,t}$ from the upstream reservoir (reservoir 1) plus the unregulated incremental flow $Q_{2,t}$ constitute the inflow to the downstream reservoir (reservoir 2). The flows $Q_{1,t}$ into the upstream reservoir in each of four seasons along with the incremental flows $Q_{2,t}$ and constraints on reservoir releases are given in the accompanying two tables:

Reservoir 1 (Upstream)

Season t	Inflow, $Q_{1,t}$	Minimum Release, $R_{1,t}^{min}$	Maximum Release Through Turbines $R_{1,t}^{max}$
1	60	20	90
2	40	30	90
3	80	20	90
4	120	20	90

Reservoir 2 (Downstream)

Season t	Incremental Flow, $Q_{2,t}$	Minimum Release, $R_{2,t}^{min}$	Maximum Release Through Turbines, $R_{2,t}^{max}$
1	50	30	140
2	30	40	140
3	60	30	140
4	90	30	140

Note that there is a limit on the quantity of water that can be released through the turbines for energy production in any season due to the limited capacity of the power plant and the desire to produce

hydropower during periods of peak demand.

Additional information that affects the operation of the two reservoirs are limitations on the fluctuations in the pool levels (head) and the storage-head relationships:

Item	Reservoir 1 (Upstream)	Reservoir 2 (Downstream)
Maximum Head	$h_{1,t}^{max} = 70$ m	$h_{2,t}^{max} = 90$ m
Minimum Head	$h_{1,t}^{min} = 30$ m	$h_{2,t}^{min} = 60$ m
Capacity	$K_1 = 150 \times 10^6$ m^3	$K_2 = 400 \times 10^6$ m^3
Storage - Head Relationship	$h_{1,t}(S_{1,t}) = h_{1,t}^{max}\left(\dfrac{S_{1,t}}{K_1}\right)^{0.64}$	$h_{2,t}(S_{2,t}) = h_{2,t}^{max}\left(\dfrac{S_{2,t}}{K_2}\right)^{0.62}$

Assume that the efficiencies of the hydropower units are both 70%, independent of either flow through the turbines or the head on the turbines. In calculating the energy produced in any season, use the average head during that season

$$\overline{h}_{i,t} = 0.5 * [h_{i,t}(S_{i,t}) + h_{i,t}(S_{i,t+1})], \quad i = 1,2$$

3.8. Assume that stream flow ($\hat{S}$) is 26 million m^3 per unit of time and there is an interstate agreement ($\overline{S}$) calling for 14.5 million m^3 Also, initially there are two users on the river diverting S_2 and S_3 million m^3. The benefits to each user are:

User 2: $B_2(S_2) = a_2 S_2 + b_2 S_2^2 = 150 S_2 - 5 S_2^2$

User 3: $B_3(S_3) = a_3 S_3 + b_3 S_3^2 = 18 S_3 - 0.6 S_3^2$

In addition, both users have the same return flow coefficient, $R_2 = R_3 = 0.5$. Write an optimization model to determine an efficient allocation that results in maximizing the value of water use in the basin and respects the interstate compact. What are the water allocations and the benefits to each user?

3.9. Now, assume that an additional user wants to divert S_1 million m^3 of water from the river. User 1's benefit function is identical to User 2's benefit, i.e.,

User 1: $B_1(S_1) = a_1 S_1 + b_1 S_1^2 = 150 S_1 - 5 S_1^2$

User 1 also has a return coefficient of $R_1 = 0.5$. Modify your optimization model to determine a new efficient allocation that results in maximizing the value of water use for all three users in the basin and respects the interstate compact. What are the water allocations and the benefits to each user?

3.10. Assume that the solution to Part (1) represents the initial water rights of Users 2 and 3, using the results obtained in Part (2), what is the minimum payment that User 1 should pay to Users 2 and 3 in order to divert water from the river? What are the resulting net benefits to each of the three users?

3.11. Assume that the system described in Problem (2) is modeled as a water market where users 1, 2, and 3 have water right allocations, 0.0, 4.0, and 7.5, respectively. Develop a model which will determine the optimal use of water by each user, assuming that the users are free to trade their water rights according to the model structure described in Section 3.2.5 above.

3.12. (from Loucks et al., 1981) A water user can withdraw water from a stream at a particular site where the cumulative distribution function of streamflow is $F(q) = \frac{q}{1+q}$ for $q \geq 0$. The withdrawal x at that location must satisfy a chance constraint $Pr\{x \leq Q\} \geq p$, or equivalently $Pr\{x \geq Q\} \leq 1\text{-}p$. Write the deterministic equivalents for each of the following constraints:

        a) $Pr\{x \leq Q\} \geq 0.90$

        b) $Pr\{x \geq Q\} \leq 0.80$

        c) $Pr\{x \leq Q\} \leq 0.95$

        d) $Pr\{x \geq Q\} \geq 0.75$

3.13. (from Loucks et al., 1981) Assume that there are N years of monthly streamflow records for a particular site at which a reservoir is to be built and from these records the probability distributions of streamflows can be estimated at the reservoir site, Qty , and downstream, Qty*.
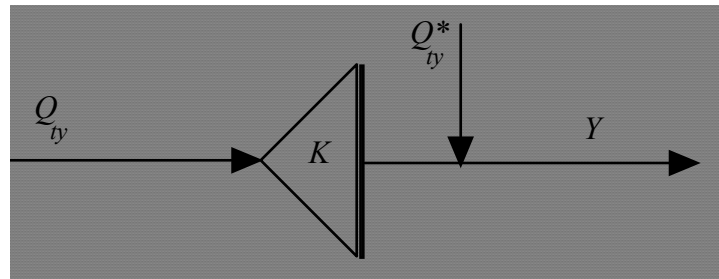
(a) Develop a chance-constrained model for estimating the minimum active storage capacity K required for a single reservoir to obtain a maximum constant yield Y each month with approximately 90% chance of being equaled or exceeded in every month of any year.

(b) Develop a yield model for estimating the minimum active storage capacity K required for a single reservoir to obtain the maximum constant yield Y each month which is 90% reliable in any year.

(c) Discuss the relative advantages and disadvantages of the two models developed in (a) and (b) above. Indicate which, in your opinion, would be the better screening model to use for this

problem and why. Typically, a screening model is an optimization model used to limit the number of possible designs considered for a particular project. The "screened" candidate designs are then evaluated using an extensive and detailed simulation model of the water resource system

3.14. Indicate whether the following are true or false

a) Hydroelectric power production is inversely proportional to the storage volume in a reservoir upstream of the turbines.

b) Reservoir storage volumes must be maintained below flood storage levels during all time periods not in the flood season.

c) The primary purpose of dead storage volume in a reservoir is to maintain downstream flows for fish and wildlife during extremely low flow periods.

d) Long-run benefits and losses from the operation of a water resources project are not affected by water flow or storage targets, whereas short-run benefits and losses are.

e) Minimizing the deviations of storage volumes from targets storage levels over a planning horizon is one possible purpose of a water resources project.

f) A typical constraint incorporated into a screening model for designing a reservoir is the maximization of hydroelectric power production.

g) When computing streamflow probabilities by a Markov chain no knowledge of previous streamflow values is required.

4. OPTIMAL MANAGEMENT OF GROUNDWATER

4.1 Finite-Difference Method

Analytical solutions are difficult for situations where you have irregular boundaries, various different boundary conditions in a region, heterogeneous and anisotropic porous media properties, or multiple phases. Often the aquifers and subsurface environments we want to model exhibit all of these characteristics. So what can we do? Numerical solutions to the governing equations of flow and mass transport in porous media provide a useful and convenient method of handling these complications. In this section, we will discuss the most common technique used to model groundwater flow and mass transport, the finite-difference method.

4.2 Numerical Differentiation

The forward Taylor series expansion of $f(x)$ in the neighborhood of a point x is

$$f(x+h) = f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots \tag{4.2.1}$$

where h is a small number. The backward Taylor series expansion of $f(x)$ about x can be written as

$$f(x-h) = f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{iv}(x)+\cdots \tag{4.2.2}$$

Solving the forward equation for the first derivative, $f'(x)$, we have

$$f'(x) = \frac{f(x+h)-f(x)}{h}-\frac{h}{2}f''(x)-\frac{h^2}{6}f'''(x)+\cdots \tag{4.2.3}$$

or

$$f'(x) = \frac{f(x+h)-f(x)}{h}+\mathcal{O}(h)$$

$$= \frac{f_{i+1}-f_i}{h}+\mathcal{O}(h) \tag{4.2.4}$$

where we will use the index notation $f(x)=f_i$ and $f(x+h)=f_{i+1}$ (see Figure 4.2.1).
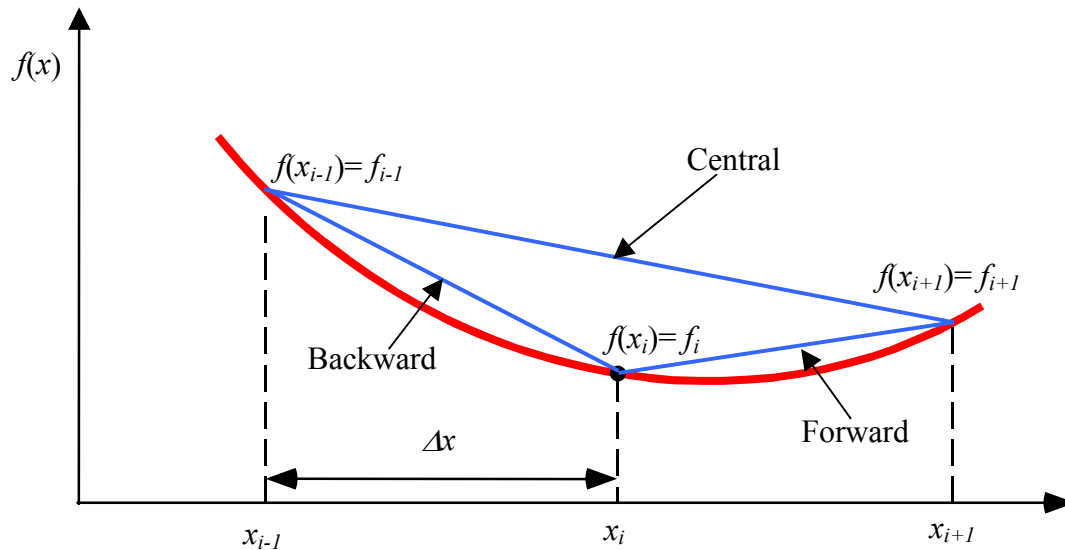
Figure 4.2.1. Forward, backward and central finite-difference approximations.

The error in this forward finite-difference approximation of the first derivative is

$$\mathcal{O}(h) = -\frac{h}{2}f''(\overline{x}) \tag{4.2.5}$$

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where $x \le \bar{x} \le x+h$. Similarly, we can solve the backward equation for the first derivative and we get

$$f'(x) = \frac{f(x)-f(x-h)}{h}+\mathcal{9}(h)$$

$$= \frac{f_i-f_{i-1}}{h}+\mathcal{9}(h) \tag{4.2.6}$$

which is a backward finite-difference approximation of the first derivative and it has the same (first-order) error as the forward approximation.

If we subtract the forward and backward Taylor series expressions, we have

$$f(x+h)-f(x-h) = 2hf'(x)+\frac{h^3}{3!}f'''(x)+\cdots \tag{4.2.7}$$

or, solving for $f'(x)$

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} +\mathcal{9}(h^2)$$

$$= \frac{f_{i+1}-f_{i-1}}{2h} +\mathcal{9}(h^2) \tag{4.2.8}$$

which is the central finite-difference approximation for the first derivative. An expression for the second derivative can be obtained by adding the forward and backward Taylor series expansions

$$f(x+h) = f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{iv}(x)+\cdots \tag{4.2.8}$$

and

$$f(x-h) = f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{iv}(x)+\cdots \tag{4.2.9}$$

to get

$$f(x+h)+f(x-h) = 2f(x)+h^2f''(x)+\frac{h^4}{12}f^{iv}(x)+\cdots \tag{4.2.10}$$

Solving for $f''(x)$, we have

$$f''(x) = \frac{f(x+h)-2\,f(x)+f(x-h)}{h^2}$$

$$= \frac{f_{i+1}-2f_i+f_{i-1}}{h^2} + \mathcal{O}(h^2) \qquad\qquad (4.2.11)$$

4.3 Grids and Discretrization

With the finite-difference method the governing partial differential equation (PDE) is replaced by a numerical approximation. In this process the continuous derivatives of the PDE are replaced by discrete approximations. The result of this discretization is a set of simultaneous algebraic equations that must be solved for the values of the unknown variables at discrete locations in the modeled domain. In order to accomplish the descretization process a mesh or grid must be defined that covers the domain. The grid consists of a series of intersecting, orthogonal, straight lines such as is illustrated in Figure 4.3.1.   The unknown state variable(s) are then solved for at either the intersections of the gridlines (mesh-centered) or at the centers of the gridblocks (block-centered).
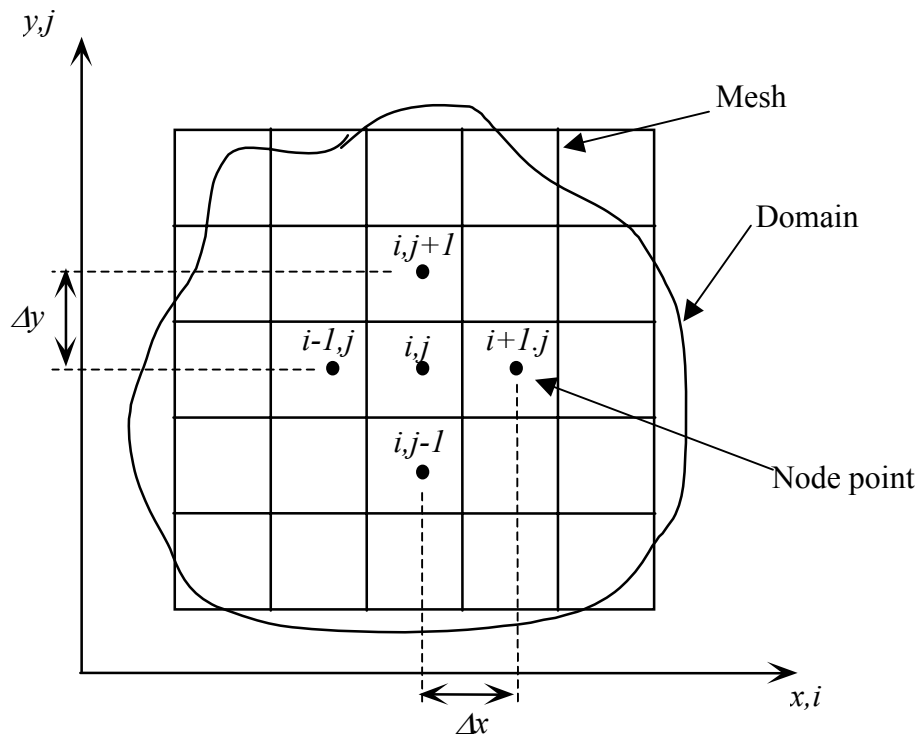
Figure 4.3.1. Finite-difference mesh.

4.4 One-Dimensional Flow

Steady-state flow

Consider steady, one-dimensional flow in a confined aquifer shown in Figure 4.4.1.
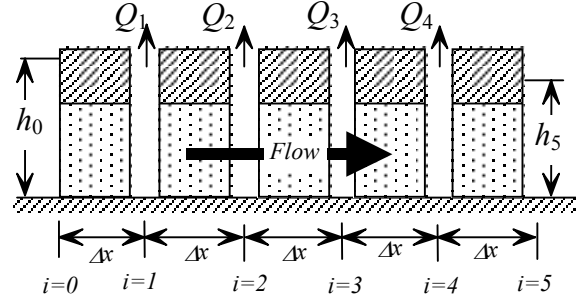
Figure 4.4.1. One-dimensional example aquifer.

The governing equation is

$$\frac{d}{dx}\left(T\frac{dh}{dx}\right) = Q \tag{4.4.1}$$

where $T\,[L^2/T]$ is the transmissivity of the aquifer and $Q(x)\,[L^3/T/L^2 = L/T]$ is the pumping rate per unit area of aquifer. When the transmissivity is homogeneous, and we apply a finite-difference approximation to the second derivative, we have

$$\frac{\left(T\frac{\partial h}{\partial x}\right)_{i+1/2,j} - \left(T\frac{\partial h}{\partial x}\right)_{i-1/2,j}}{\Delta x} - Q_{i,j} = 0 \tag{4.4.2}$$

$$\frac{\left(T\frac{h_{i+1,j} - h_{i,j}}{\Delta x}\right) - \left(T\frac{h_{i,j} - h_{i-1,j}}{\Delta x}\right)}{\Delta x} - Q_{i,j} = 0 \tag{4.4.3}$$

$$A_{i,j}(h_{i+1,j} - h_{i,j}) + B_{i,j}(h_{i,j} - h_{i-1,j}) - Q_{i,j} = 0 \tag{4.4.4}$$

where

$$A_{i,j} = \frac{T}{\Delta x^2}, \quad B_{i,j} = \frac{-T}{\Delta x^2} \tag{4.4.5}$$

and Q_i is the source/sink rate for a particular node or cell. This finite-difference equation must be written for each node i where the head is unknown (nodes $1 - 4$ in Figure 4.4.1).

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Example - Steady Flow in a 1-D Homogeneous System

(adapted from Mays and Tung, Problem 8.3.1)

Consider the situation where the transmissivity of the aquifer is $T = 1000$ m^2/day, the well are $\Delta x = 80$m apart, the heads on the left- and right-hand boundaries are constant values of 40m and 35m, respectively. The objective of the optimization model is to maximize the head distribution in the aquifer while supplying a minimum pumping rate of *Demand* $= 500$ m^3/d/m^2. The optimization model can be written as

$$\textit{Maximize} \quad \sum_{i=1}^{4} h_i$$

$$\textit{Subject to}$$

$$\sum_{i=1}^{4} Q_i \geq \textit{Demand}$$

$$A_i(h_{i+1} - h_i) + B_i(h_i - h_{i-1}) - Q_i = 0 \qquad i = 1,...,4$$

$$h_i = 40 \qquad\qquad\qquad\qquad\qquad\qquad\quad i = 0$$

$$h_i = 35 \qquad\qquad\qquad\qquad\qquad\qquad\quad i = 5 \qquad\qquad (4.4.6)$$

The GAMS code for solving this model is

```
SET
        I    /1*4/;
SCALAR
        T    transmissivity      /1000/
        DX   cell width          /80/
        QMIN minimum pumping      /10/
        H0   left boundary head  /40/
        H5   right boundary head /35/
        a    constant             ;

a=T/(DX*DX);

VARIABLES
        OBJ   objective function;

POSITIVE VARIABLE
        H(I)  head at well I
        Q(I)  production rate at well I;

EQUATIONS
        FD(I)       Finite difference equation,
        PRODRATE    Sum of pumping rates,
        TOTALHEAD   Sum of heads;

FD(I)..   a*H(I-1)$(ORD(I) gt 1)
        - 2*a*H(I)
        + a*H(I+1)$(ORD(I) lt 4)
        - Q(I)
    =E= - a*H0$(ORD(I) = 1)
        - a*H5$(ORD(I) = 4);
```

```
PRODRATE..  sum(I,Q(I)) =G= QMIN;

TOTALHEAD.. sum(I,H(I)) =E= OBJ

MODEL GW1 /ALL/ ;

SOLVE GW1 USING LP MAXIMIZING OBJ ;

file GW1_Out /GW1.txt/
put GW1_Out
put '                Head  Pumping',  put /
loop( I,
        put I.tl,
        put H.l(I):8.1,
        put Q.l(I):8.1,
        put /)
```

The solution of the model is:

```
Node              Head    Pumping
0                40.00
1                11.00    3.96
2                 7.33    0.00
3                 3.67    0.00
4                 0.00    6.04
5                35.00
```

4.5 Two-Dimensional Flow

Steady-State flow

Consider the steady, two-dimensional flow in a confined aquifer shown in Figure 4.5.1. The governing equation is

$$\frac{\partial\left(-Q^x\right)}{\partial x}+\frac{\partial\left(-Q^y\right)}{\partial y}=$$
$$\frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right)=Q \tag{4.5.1}$$

where $T\,[\mathrm{L^2/T}]$ is the transmissivity of the aquifer and $Q\,[\mathrm{L^3/T/L^2=L/T}]$ is the source/sink flow rate per unit area of aquifer.
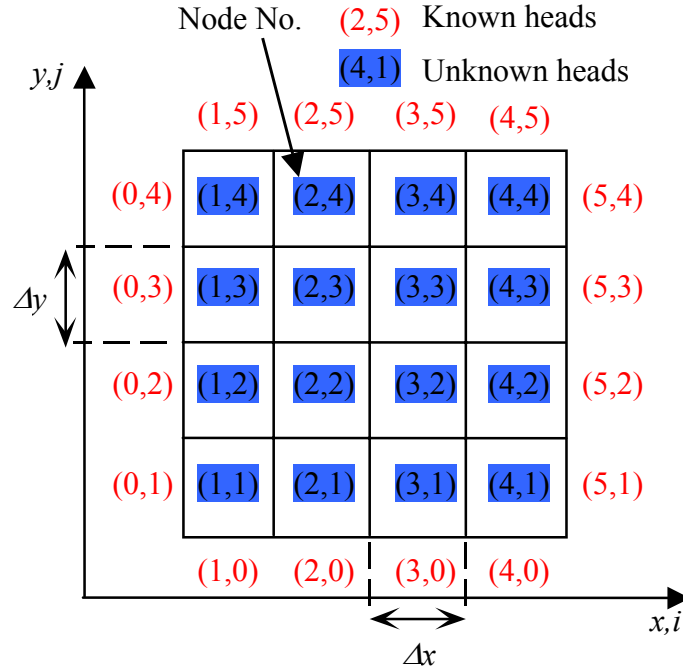
Figure 4.5.1. Two-dimensional example finite-difference grid.

Considering the cell (i,j) (see Figure 4.5.2) we can see that the flows in and out of the cell are represented by Darcy's Law written for aquifer flow. When the transmissivity is homogeneous, and we apply a finite-difference approximation to the second derivatives, we have

$$\frac{\left(T\frac{\partial h}{\partial x}\right)_{i+1/2,j} - \left(T\frac{\partial h}{\partial x}\right)_{i-1/2,j}}{\Delta x} + \frac{\left(T\frac{\partial h}{\partial y}\right)_{i,j+1/2} - \left(T\frac{\partial h}{\partial y}\right)_{i,j-1/2}}{\Delta y} - Q_{i,j} = 0 \tag{4.5.2}$$

$$\frac{\left(T\frac{h_{i+1,j} - h_{i,j}}{\Delta x}\right) - \left(T\frac{h_{i,j} - h_{i-1,j}}{\Delta x}\right)}{\Delta x} + \frac{\left(T\frac{h_{i,j+1} - h_{i,j}}{\Delta y}\right) - \left(T\frac{h_{i,j} - h_{i,j-1}}{\Delta y}\right)}{\Delta y} - Q_{i,j} = 0 \tag{4.5.3}$$

$$A_{i,j}(h_{i+1,j} - h_{i,j}) + B_{i,j}(h_{i,j} - h_{i-1,j}) + C_{i,j}(h_{i,j+1} - h_{i,j}) + D_{i,j}(h_{i,j} - h_{i,j-1}) - Q_{i,j} = 0 \tag{4.5.4}$$

where

$$A_{i,j} = \frac{T}{\Delta x^2}, \quad B_{i,j} = \frac{-T}{\Delta x^2}, \quad C_{i,j} = \frac{T}{\Delta y^2}, \quad D_{i,j} = \frac{-T}{\Delta y^2} \tag{4.5.5}$$

This equation must be solved for each node (i,j) on the interior of the solution domain.
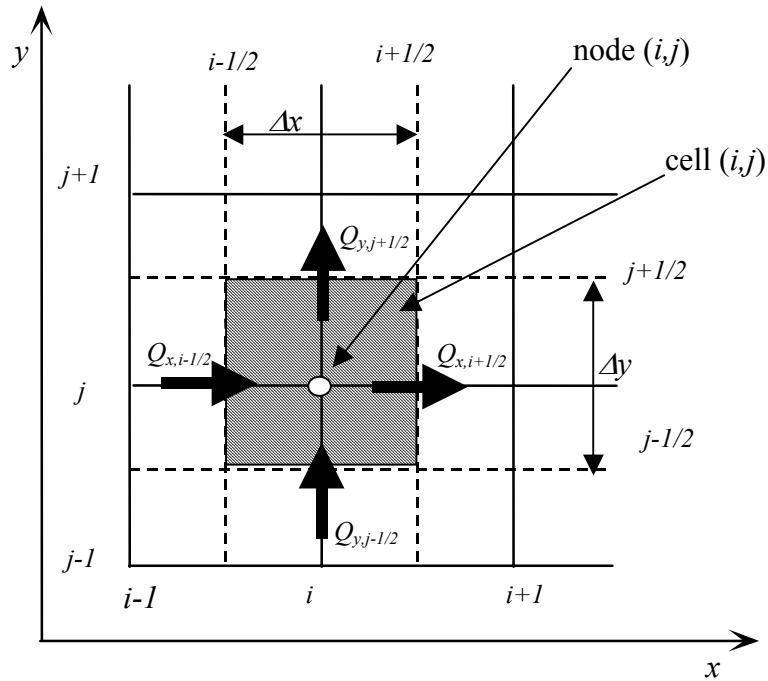
Figure 4.5.2. Finite-difference cell in two dimensional flow.

Boundary Conditions

Two types of boundary conditions are often encountered in groundwater flow problems:

(1) Constant or prescribed head (Dirichlet) conditions

$$h = f_1(x, y, t) \; on \, C_1 \tag{4.5.6}$$

In the previous example, if we write out the finite-difference equation for each node (i,j) where the head is unknown (nodes $(1,1)$ - $(4,4)$), we have, e.g., at node $(1,1)$

$$A_{1,1}(h_{2,1} - h_{1,1}) + B_{1,1}(h_{1,1} - h_{0,1}) + C_{1,1}(h_{1,2} - h_{1,1}) + D_{1,1}(h_{1,1} - h_{1,0}) - Q_{1,1} = 0 \tag{4.5.7}$$

where the heads $h_{0,1}$ and $h_{1,0}$ are known from the boundary conditions. For node $(2,3)$, we have

$$A_{2,3}(h_{3,3} - h_{2,3}) + B_{2,3}(h_{2,3} - h_{1,3}) + C_{2,3}(h_{2,4} - h_{2,3}) + D_{2,3}(h_{2,3} - h_{2,2}) - Q_{2,3} = 0 \tag{4.5.8}$$

(2) Prescribed flux (Neuman) conditions

$$q_n = f_2(x, y, t) \; on \, C_2 \tag{4.5.9}$$

or

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$$\frac{\partial h}{\partial n} = f_2(x, y, t) \ on \ C_2 \tag{4.5.10}$$

Consider the case where we have constant head boundary conditions on the west and east sides of an aquifer and no-flow boundary conditions on the north and south sides as shown in Figure 4.5.3.
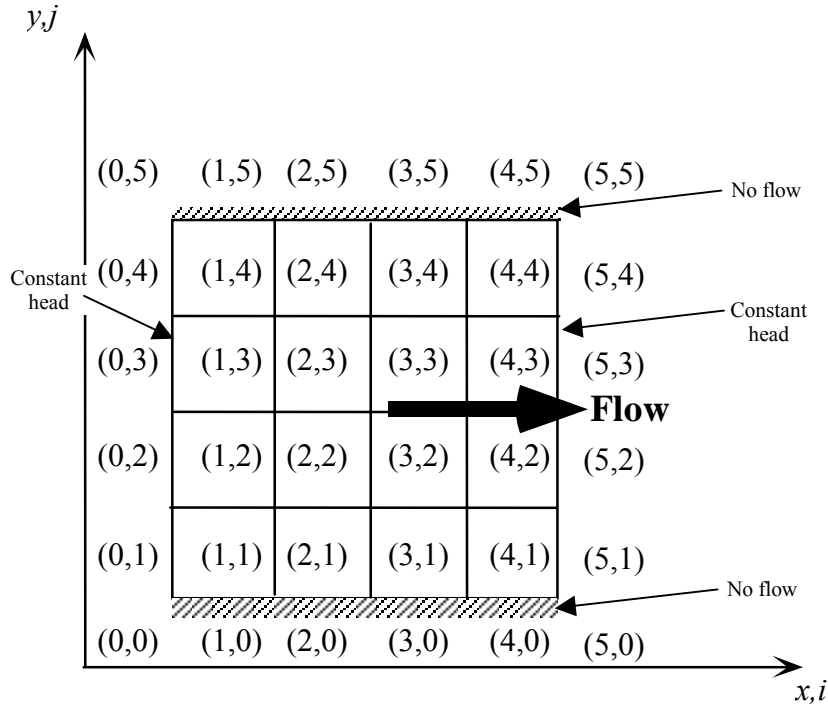
Figure 4.5.3. Two-dimensional example with no-flow boundary conditions.

The north and south side no-flow conditions can be written as

$$\left.\frac{\partial h}{dy}\right|_{i,j=1} = 0 \quad \text{and} \quad \left.\frac{\partial h}{dy}\right|_{i,j=4} = 0 \tag{4.5.11}$$

These conditions can easily be approximated by finite-differences, using a central difference approximation as

$$\frac{h_{i,0} - h_{i,1}}{\Delta y} = 0 \quad \text{and} \quad \frac{h_{i,5} - h_{i,4}}{\Delta y} = 0 \tag{4.5.12}$$

or

$$h_{i,0} = h_{i,1} \quad \text{and} \quad h_{i,5} = h_{i,4} \tag{4.5.13}$$

The resulting finite-difference equations for nodes along these no-flow boundaries are:

for j = 1 (south boundary):

$$A_{i,1}(h_{i+1,1} - h_{i,1}) + B_{i,1}(h_{i,1} - h_{i-1,1}) + C_{i,1}(h_{i,2} - h_{i,1}) + D_{i,1}(h_{i,1} - h_{i,0}) - Q_{i,1} = 0 \tag{4.5.14}$$

for j = 4 (north boundary):

$$A_{i,4}(h_{i+1,4} - h_{i,4}) + B_{i,4}(h_{i,4} - h_{i-1,4}) + C_{i,4}(h_{i,5} - h_{i,4}) + D_{i,4}(h_{i,4} - h_{i,3}) - Q_{i,4} = 0 \tag{4.5.15}$$

Substituting in the boundary information, we have

for j = 1 (south boundary):

$$A_{i,1}(h_{i+1,1} - h_{i,1}) + B_{i,1}(h_{i,1} - h_{i-1,1}) + C_{i,1}(h_{i,2} - h_{i,1}) + \ 0 \ - Q_{i,1} = 0 \tag{4.5.16}$$

for j = 4 (north boundary):

$$A_{i,4}(h_{i+1,4} - h_{i,4}) + B_{i,4}(h_{i,4} - h_{i-1,4}) + \ 0 \ + D_{i,4}(h_{i,4} - h_{i,3}) - Q_{i,4} = 0 \tag{4.5.17}$$

Example: Steady Flow in 2-D Homogeneous System

Consider the flow in the example aquifer shown in Figure 4.5.2, where the mesh size is 10m, western and eastern boundaries are held fixed at 100m and 50m respectively, the transmissivity is 200 m^2/day, and the maximum pumping is limited to 40 m^3/day/m^2. The optimization model for this problem can be written as

$$\textit{Minimize} \ \sum_{i=0}^{9} \sum_{j=1}^{8} Q_{i,j}$$

$\textit{Subject to}$

$$\sum_{i=0}^{9} \sum_{j=1}^{8} Q_{i,j} \geq \textit{Demand}$$

$$\begin{aligned}
&A_{i,j}(h_{i+1,j} - h_{i,j}) + B_{i,j}(h_{i,j} - h_{i-1,j}) + \\
&\qquad C_{i,j}(h_{i,j+1} - h_{i,j}) + D_{i,j}(h_{i,j} - h_{i,j-1}) - Q_{i,j} = 0 && i = 1,...,8; j = 1,...8 \\
&A_{i,1}(h_{i+1,1} - h_{i,1}) + B_{i,1}(h_{i,1} - h_{i-1,1}) + C_{i,1}(h_{i,2} - h_{i,1}) = 0 && i = 0; \ j = 1,...8 \\
&A_{i,4}(h_{i+1,4} - h_{i,4}) + B_{i,4}(h_{i,4} - h_{i-1,4}) + D_{i,4}(h_{i,4} - h_{i,3}) = 0 && i = 9; j = 1,...,8
\end{aligned}$$

$$\begin{aligned}
h_{i,j} &= 100 && i = 0,...,9; j = 0 \\
h_{i,j} &= 50 && i = 0....,9; j = 9
\end{aligned}$$

$$\tag{4.5.18}$$

The GAMS code to solve this linear program is given below. The reader should note that the grid orientation is rotated so that the origin is in the upper left-hand corner and the *x* axis runs

down the page.

```
SETS
        I row index of model cells        / I0*I10 /
        J column index of model cells     / J0*J10 /

SCALARS
        DX      grid size x (m)                          /   10 /
        DY      grid size y (m)                          /   10 /
        HBW     boundary head west (m)               / 100 /
        HBE     boundary head east (m)               / 100 /
        HBN     boundary head north (m)              / 100 /
        HBS     boundary head south (m)              / 100 /
        DEM     water demand (1000 m3 per day)       / 400 /
        QMAX    max pumping rate (m3 per d per m2) / 10 /
        QMIN    min pumping rate (m3 per d per m2) /    0 /
        T       Transmissivity m2 per day           / 200 /
         ;

* finite-difference coefficients
PARAMETER
        A(I,J)    finite-difference coefficient
        B(I,J)    finite-difference coefficient
        C(I,J)    finite-difference coefficient
        D(I,J)    finite-difference coefficient;

        A(I,J) =  T/(DX*DX) ;
        B(I,J) = -T/(DX*DX) ;
        C(I,J) =  T/(DY*DY) ;
        D(I,J) = -T/(DY*DY) ;

* finite-difference cell types
TABLE CELL(I,J) cell type

* -2 = north no-flow bndy
* -1 = south no-flow bndy
*  0 = interior cells
*  1 = west fixed potential
*  2 = east fixed potential
*  3 = north fixed potential
*  4 = south fixed potential

     J0  J1  J2  J3  J4  J5  J6  J7  J8  J9 J10
I0    1   3   3   3   3   3   3   3   3   3   2
I1    1   0   0   0   0   0   0   0   0   0   2
I2    1   0   0   0   0   0   0   0   0   0   2
I3    1   0   0   0   0   0   0   0   0   0   2
I4    1   0   0   0   0   0   0   0   0   0   2
I5    1   0   0   0   0   0   0   0   0   0   2
I6    1   0   0   0   0   0   0   0   0   0   2
I7    1   0   0   0   0   0   0   0   0   0   2
I8    1   0   0   0   0   0   0   0   0   0   2
I9    1   0   0   0   0   0   0   0   0   0   2
I10   1   4   4   4   4   4   4   4   4   4   2;

* Use for fixed head all around
*I0    1   3   3   3   3   3   3   3   3   3   2
*I10   1   4   4   4   4   4   4   4   4   4   2;
* Use for fixed head east and west
*I0    1  -2  -2  -2  -2  -2  -2  -2  -2  -2   2
```

```
*I10  1  -1  -1  -1  -1  -1  -1  -1  -1  -1   2;

VARIABLES
        OBJ       objective value
        Q(I,J)    pumping in cell ij [m3 per day]
        ;

POSITIVE VARIABLES
        H(I,J)    head in cell ij
        ;

*        boundary conditions
* Use for fixed head east and west
        H.fx(I,J)$(CELL(I,J) = 1) = HBW ;
        H.fx(I,J)$(CELL(I,J) = 2) = HBE ;
* Use for fixed head all around
        H.fx(I,J)$(CELL(I,J) = 3) = HBN ;
        H.fx(I,J)$(CELL(I,J) = 4) = HBS ;

*        maximum/minimum pumping contrtraints
        Q.lo(I,J)$(CELL(I,J) le 0) = QMIN;
        Q.up(I,J)$(CELL(I,J) le 0) = QMAX;

EQUATIONS
        CONT(I,J)      finite-difference continuity eqn.
        DEMAND         demand constraint eqn.
        TCOST          expected total cost ;

*        continuity equation (including no-flow boundaries)
CONT(I,J) ..
     ( A(I,J)*(H(I+1,J)-H(I,J))
      +B(I,J)*(H(I,J)  -H(I-1,J))
      +C(I,J)*(H(I,J+1)-H(I,J))
      +D(I,J)*(H(I,J)  -H(I,J-1)))$(CELL(I,J) eq 0)

    +( A(I,J)*(H(I+1,J)-H(I,J))
*      +B(I,J)*(H(I,J)  -H(I,J))
      +C(I,J)*(H(I,J+1)-H(I,J))
      +D(I,J)*(H(I,J)  -H(I,J-1)))$(CELL(I,J) eq -2)

    +(
*      A(I,J)*(H(I,J)  -H(I,J))
      +B(I,J)*(H(I,J)  -H(I-1,J))
      +C(I,J)*(H(I,J+1)-H(I,J))
      +D(I,J)*(H(I,J)  -H(I,J-1)))$(CELL(I,J) eq -1)
      -Q(I,J)
    =E= 0.0;

*        demand constraint
DEMAND ..  SUM((I,J), Q(I,J)) =E= DEM ;

*        objective function
TCOST ..  OBJ =E= (SUM((I,J), H(I,J))) ;

MODEL PUMPING least-cost pumping model /ALL/ ;

SOLVE PUMPING USING LP MAXIMIZING OBJ ;

File GW2D /GW2D_output.txt/
Put GW2D;
PUT ' Objective value = '; Put OBJ.l;
```

```
        Put /;
        Put 'Pumping'/; Put '           ';
        Loop(J, Put J.tl:6;) Put /;
        Loop(I, Put I.tl:6; Loop(J, Put Q.L(I, J):6:1;); Put /;);
        Put /;
        Put 'Head'/; Put '          ';
        Loop(J,Put J.tl:6;) Put /;
        Loop(I, Put I.tl:6; Loop(J, Put H.L(I, J):6:1;); Put /;);
        Put /;
```

The results of solving this model are shown in the following table and figure.

```
Objective value =     11566.11
Pumping
         J0     J1     J2     J3     J4     J5     J6     J7     J8     J9     J10
I0      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I1      0.0   10.0   10.0   10.0   10.0   10.0   10.0   10.0   10.0   10.0    0.0
I2      0.0   10.0   10.0    0.0    0.0    0.0    0.0    0.0   10.0   10.0    0.0
I3      0.0   10.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0   10.0    0.0
I4      0.0   10.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0   10.0    0.0
I5      0.0   10.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0   10.0    0.0
I6      0.0   10.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0   10.0    0.0
I7      0.0   10.0   10.0    0.0    0.0    0.0    0.0    0.0   10.0   10.0    0.0
I8      0.0   10.0   10.0   10.0    0.0    0.0    0.0   10.0   10.0   10.0    0.0
I9      0.0   10.0   10.0   10.0   10.0   10.0   10.0   10.0   10.0   10.0    0.0
I10     0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0

Head
         J0     J1     J2     J3     J4     J5     J6     J7     J8     J9     J10
I0     100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0
I1     100.0   96.0   94.6   94.5   94.5   94.5   94.5   94.5   94.6   96.0  100.0
I2     100.0   94.6   92.9   93.7   94.0   94.0   94.0   93.7   92.9   94.6  100.0
I3     100.0   94.4   93.7   93.6   93.7   93.7   93.7   93.6   93.7   94.4  100.0
I4     100.0   94.3   93.7   93.5   93.4   93.4   93.4   93.5   93.7   94.3  100.0
I5     100.0   94.1   93.4   93.1   93.0   93.0   93.0   93.1   93.4   94.1  100.0
I6     100.0   93.8   92.7   92.5   92.6   92.7   92.6   92.5   92.7   93.8  100.0
I7     100.0   93.4   90.9   91.7   92.3   92.5   92.3   91.7   90.9   93.4  100.0
I8     100.0   93.8   91.1   90.9   92.4   92.8   92.4   90.9   91.1   93.8  100.0
I9     100.0   95.6   93.8   93.3   93.6   93.7   93.6   93.3   93.8   95.6  100.0
I10    100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0  100.0
```

Heterogeneous Systems

Consider the a steady, two-dimensional flow in a heterogeneous, anisotropic, confined aquifer. The governing equation is (see Figure 4.5.3)

$$\frac{\partial\left(-Q^x\right)}{\partial x} + \frac{\partial\left(-Q^y\right)}{\partial y} =$$

$$\frac{\partial}{\partial x}\left(T^x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T^y \frac{\partial h}{\partial y}\right) = Q \tag{4.5.19}$$

where T^x and T^y [L^2/T] are transmissivities in the x and y directions, respectively. When we apply a finite-difference approximation to the derivatives, we have

$$\frac{\left(T^x \frac{\partial h}{\partial x}\right)_{i+1/2,j} - \left(T^x \frac{\partial h}{\partial x}\right)_{i-1/2,j}}{\Delta x} + \frac{\left(T^y \frac{\partial h}{\partial y}\right)_{i,j+1/2} - \left(T^y \frac{\partial h}{\partial y}\right)_{i,j-1/2}}{\Delta y} - Q_{i,j} = 0$$

(4.5.20)

Applying a finite difference approximation to the remaining derivatives results in

$$\frac{\left(T^x_{i+1/2,j} \frac{h_{i+1,j} - h_{i,j}}{\Delta x}\right) - \left(T^x_{i-1/2,j} \frac{h_{i,j} - h_{i-1,j}}{\Delta x}\right)}{\Delta x}$$

$$+ \frac{\left(T^y_{i,j+1/2} \frac{h_{i,j+1} - h_{i,j}}{\Delta y}\right) - \left(T^y_{i,j-1/2} \frac{h_{i,j} - h_{i,j-1}}{\Delta y}\right)}{\Delta y} - Q_{i,j} = 0$$

(4.5.21)

where

$$T^x_{i+1/2,j} = 2\frac{T^x_{i+1,j}T^x_{i,j}}{T^x_{i+1,j} + T^x_{i,j}} \quad \text{and} \quad T^y_{i,j+1/2} = 2\frac{T^y_{i,j+1}T^y_{i,j}}{T^y_{i,j+1} + T^y_{i,j}}$$

(4.5.22)

are the harmonic averages of the x- and y-direction transmissivities, respectively, between cells (i,j) and $(i+1,j)$ and between cells (i,j) and $(i,j+1)$. We can write this equation as

$$A_{i,j}(h_{i+1,j} - h_{i,j}) + B_{i,j}(h_{i,j} - h_{i-1,j}) + C_{i,j}(h_{i,j+1} - h_{i,j}) + D_{i,j}(h_{i,j} - h_{i,j-1}) - Q_{i,j} = 0$$

(4.5.23)

where

$$A_{i,j} = \frac{2}{\Delta x^2}\frac{T^x_{i+1,j}T^x_{i,j}}{T^x_{i+1,j} + T^x_{i,j}}, \quad B_{i,j} = \frac{-2}{\Delta x^2}\frac{T^x_{i-1,j}T^x_{i,j}}{T^x_{i-1,j} + T^x_{i,j}},$$

$$C_{i,j} = \frac{2}{\Delta y^2}\frac{T^y_{i,j+1}T^y_{i,j}}{T^y_{i,j+1} + T^y_{i,j}}, \quad D_{i,j} = \frac{-2}{\Delta y^2}\frac{T^y_{i,j11}T^y_{i,j}}{T^y_{i,j11} + T^y_{i,j}}$$

(4.5.24)

If no flow boundaries exist in the system, then appropriate modifications, as described in the previous section, must be made to this system of equations.

Example – Steady Flow in a 2-D Heterogeneous System

```
SETS
        I row index of model cells        / I0*I10 /
        J column index of model cells     / J0*J10 /

SCALARS
        DX       grid size x (m)                       / 10    /
        DY       grid size y (m)                       / 10    /
        HBW      boundary head west (m)                / 100   /
        HBE      boundary head east (m)                / 50    /
        HBN      boundary head north (m)               / 100  /
        HBS      boundary head south (m)               / 100  /
        DEM      water demand (1000 m3 per day) / 1000 /
        QMAX  max pumping rate (m3 per d per m2) / 1000    /
        QMIN  min pumping rate (m3 per d per m2) /    0   /
         ;

TABLE T(I,J) isotropic transmissivity (m2 per day)

      J0   J1   J2   J3   J4   J5   J6   J7   J8   J9  J10
I0   205  185  190  175  160  185  195  190  175  160  165
I1   205  185  190  175  160  185  195  190  175  160  165
I2   260  235  215  200  175  185  195  190  175  160  165
I3   255  250  230  205  150  185  195  190  175  160  165
I4   210  225  220  195  145  185  195  190  175  160  165
I5   185  195  190  175  160  185  195  190  175  160  165
```

```
I6  160 205 215 200 165 185 195 190 175 160 165
I7  155 200 210 215 150 185 195 190 175 160 165
I8  170 215 200 195 155 185 195 190 175 160 165
I9  170 215 200 195 155 185 195 190 175 160 165
I10 170 215 200 195 155 185 195 190 175 160 165
;

* finite-difference cell types
TABLE CELL(I,J) cell type

* -2 = north no-flow bndy
* -1 = south no-flow bndy
*  0 = interior cells
*  1 = west fixed potential
*  2 = east fixed potential
*  3 = north fixed potential
*  4 = south fixed potential

     J0   J1   J2   J3   J4   J5   J6   J7   J8   J9 J10
I0   1   -2   -2   -2   -2   -2   -2   -2   -2   -2    2
I1   1    0    0    0    0    0    0    0    0    0    2
I2   1    0    0    0    0    0    0    0    0    0    2
I3   1    0    0    0    0    0    0    0    0    0    2
I4   1    0    0    0    0    0    0    0    0    0    2
I5   1    0    0    0    0    0    0    0    0    0    2
I6   1    0    0    0    0    0    0    0    0    0    2
I7   1    0    0    0    0    0    0    0    0    0    2
I8   1    0    0    0    0    0    0    0    0    0    2
I9   1    0    0    0    0    0    0    0    0    0    2
I10  1   -1   -1   -1   -1   -1   -1   -1   -1   -1    2
;

* Use for fixed head all around
*I0   1    3    3    3    3    3    3    3    3    3    2
*I10  1    4    4    4    4    4    4    4    4    4    2
* Use for fixed head east and west
*I0   1   -2   -2   -2   -2   -2   -2   -2   -2   -2    2
*I10  1   -1   -1   -1   -1   -1   -1   -1   -1   -1    2

* finite-difference coefficients
PARAMETER
        A(I,J)    finite-difference coefficient
        B(I,J)    finite-difference coefficient
        C(I,J)    finite-difference coefficient
        D(I,J)    finite-difference coefficient
         ;

        A(I,J) = 0.0;
        B(I,J) = 0.0;
        C(I,J) = 0.0;
        D(I,J) = 0.0;
A(I,J) =  (2*T(I+1,J)*T(I,J)/(T(I+1,J)+T(I,J))/(DX*DX))$(ord(I) ne card(I))
      +(   T(I,J)/(DX*DX))$(ord(I) eq card(I));
B(I,J) = (-2*T(I-1,J)*T(I,J)/(T(I-1,J)+T(I,J))/(DX*DX))$(ord(I) ne 0)
      +(   T(I,J)/(DX*DX))$(ord(I) eq 0) ;
C(I,J) =  (2*T(I,J+1)*T(I,J)/(T(I,J+1)+T(I,J))/(DY*DY))$(ord(J) ne card(J))
      +(   T(I,J)/(DY*DY))$(ord(J) eq card(J));
D(I,J) = (-2*T(I,J-1)*T(I,J)/(T(I,J-1)+T(I,J))/(DY*DY))$(ord(J) ne 0)
      +(   T(I,J)/(DY*DY))$(ord(J) eq 0);

VARIABLES
        OBJ       total cost of pumping
```

```
             ;

POSITIVE VARIABLES
        Q(I,J)    pumping in cell ij [1000 m3 per day]
        H(I,J)    head in cell ij
          ;

*        boundary conditions
* Use for fixed head east and west
        H.fx(I,J)$(CELL(I,J) = 1) = HBW ;
        H.fx(I,J)$(CELL(I,J) = 2) = HBE ;
* Use for fixed head all around
        H.fx(I,J)$(CELL(I,J) = 3) = HBN ;
        H.fx(I,J)$(CELL(I,J) = 4) = HBS ;

* maximum/minimum pumping contrtraints
        Q.lo(I,J)$(CELL(I,J) le 0) = QMIN;
        Q.up(I,J)$(CELL(I,J) le 0) = QMAX;

EQUATIONS
        CONT(I,J)      finite-difference continuity eqn.
        DEMAND         demand constraint eqn.
        OBJECTIVE      objective function ;

* continuity equation (including no-flow boundaries)
CONT(I,J) ..
    ( A(I,J)*(H(I+1,J)-H(I,J))
     +B(I,J)*(H(I,J)  -H(I-1,J))
     +C(I,J)*(H(I,J+1)-H(I,J))
     +D(I,J)*(H(I,J)  -H(I,J-1)))$(CELL(I,J) eq 0)
   +( A(I,J)*(H(I+1,J)-H(I,J))
*     +B(I,J)*(H(I,J)  -H(I,J))
     +C(I,J)*(H(I,J+1)-H(I,J))
     +D(I,J)*(H(I,J)  -H(I,J-1)))$(CELL(I,J) eq -2)
   +(
*      A(I,J)*(H(I,J)  -H(I,J))
     +B(I,J)*(H(I,J)  -H(I-1,J))
     +C(I,J)*(H(I,J+1)-H(I,J))
     +D(I,J)*(H(I,J)  -H(I,J-1)))$(CELL(I,J) eq -1)
     -Q(I,J)
   =E= 0.0;

* demand constraint
DEMAND ..  SUM((I,J), Q(I,J)) =E= DEM ;

* objective function (sum of pumping or cost of pumping)
OBJECTIVE ..  OBJ =E= (SUM((I,J), H(I,J))) ;

MODEL PUMPING least-cost pumping model /ALL/ ;

SOLVE PUMPING USING LP MAXIMIZING OBJ ;

File GW2D /GW2D_KXY_output.txt/
Put GW2D;
PUT ' Objective value = '; Put OBJ.l;
Put /;
Put 'Pumping'/; Put '          ';
Loop(J, Put J.tl:6;) Put /;
Loop(I, Put I.tl:6; Loop(J, Put Q.L(I, J):6:1;); Put /;);
Put /;
Put 'Head'/; Put '           ';
```

```
       Loop(J,Put J.tl:6;) Put /;
       Loop(I, Put I.tl:6; Loop(J, Put H.L(I, J):6:1;); Put /;);
       Put /;
```

The results of running this model are:

```
Objective value =       7014.44
Pumping
         J0     J1     J2     J3     J4     J5     J6     J7     J8     J9     J10
I0      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I1      0.0  240.4    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I2      0.0  318.3    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I3      0.0  441.3    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I4      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I5      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I6      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I7      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I8      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I9      0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0
I10     0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0


Head
         J0     J1     J2     J3     J4     J5     J6     J7     J8     J9     J10
I0    100.0   50.8   37.5   36.3   38.6   41.5   43.9   45.8   47.4   48.7   50.0
I1    100.0   12.4   25.7   32.8   38.0   41.8   44.4   46.2   47.7   48.9   50.0
I2    100.0    0.0   20.3   31.8   38.6   43.0   45.5   47.2   48.3   49.2   50.0
I3    100.0    0.0   25.4   36.3   42.2   45.8   47.6   48.6   49.3   49.7   50.0
I4    100.0   54.0   47.1   47.4   48.9   49.9   50.4   50.5   50.5   50.3   50.0
I5    100.0   74.7   63.3   58.5   56.1   54.5   53.5   52.6   51.8   50.9   50.0
I6    100.0   83.3   73.2   66.8   62.1   58.7   56.4   54.7   53.1   51.5   50.0
I7    100.0   87.6   78.6   72.1   66.5   62.0   58.9   56.5   54.3   52.1   50.0
I8    100.0   90.1   82.1   75.5   69.6   64.5   60.8   57.8   55.2   52.6   50.0
I9    100.0   91.3   84.1   77.6   71.5   66.1   62.1   58.8   55.8   52.9   50.0
I10   100.0   91.8   84.9   78.6   72.4   66.9   62.7   59.3   56.1   53.0   50.0
```

4.6 Transient Problems

In transient problems, time is added as an independent variable. Therefore, we need to consider the rate of release or uptake of water from or to storage in an aquifer. We also need to specify initial head conditions in the aquifer in order to find a solution of a problem.

If we consider the transient, two-dimensional flow in a confined aquifer. The governing equation is

$$\frac{\partial}{\partial x}\left(T^x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T^y \frac{\partial h}{\partial y}\right) - Q = S\frac{\partial h}{\partial t} \tag{4.6.1}$$

where S is the aquifer storativity. When the transmissivity and storativity are heterogeneous, and we apply a finite-difference approximation to the second derivatives on the left-hand-side of this equation, we have

$$LHS_{i,j} = A_{i,j}(h_{i+1,j} - h_{i,j}) + B_{i,j}(h_{i,j} - h_{i-1,j}) + C_{i,j}(h_{i,j+1} - h_{i,j}) + D_{i,j}(h_{i,j} - h_{i,j-1}) - Q_{i,j} \tag{4.6.2}$$

Consider a backward finite difference approximation of the time difference on the right-hand-side

$$RHS_{i,j}^{t+1}=S_{i,j}\,\frac{h_{i,j}^{t+1}-h_{i,j}^{t}}{\Delta t}$$

(4.6.3)

where $h_{i,j}^{t}$ and $h_{i,j}^{t+1}$ are the heads in cell (i,j) at the time levels t, and $t+\Delta t$, respectively. Now, we have not specified the time level where we are evaluating the *LHS*. Using a weighted average of the LHS at the t and $t+\Delta t$ levels, we have

$$\theta LHS_{i,j}^{t+1}+(1-\theta)LHS_{i,j}^{t}=\frac{S_{i,j}}{\Delta t}\left(h_{i,j}^{t+1}-h_{i,j}^{t}\right)$$

(4.6.4)

where $0\le\theta\le1$.

Various time-stepping schemes result from different selections of θ: e.g., **Explicit Scheme:** $\theta=0$; and the **Crank-Nicolson Scheme:** $\theta=1/2$.

One of the mst common is the **Fully Implicit Scheme:** $\theta=1$

$$A_{i,j}(h_{i+1,j}^{t+1}-h_{i,j}^{t+1})+B_{i,j}(h_{i,j}^{t+1}-h_{i-1,j}^{t+1})$$
$$+C_{i,j}(h_{i,j+1}^{t+1}-h_{i,j}^{t+1})+D_{i,j}(h_{i,j}^{t+1}-h_{i,j-1}^{t+1})-Q_{i,j}^{t+1}$$
$$=F_{i,j}(h_{i,j}^{t+1}-h_{i,j}^{t})$$

(4.6.5)

where

$$F_{i,j}=\frac{S_{i,j}}{\Delta t}$$

(4.6.7)

Example: Transient Flow in 2-D Homogeneous System

```
SETS
    I          row index of model cells            /I0*I10 /
    J          column index of model cells         /J0*J10 /
    L          time index                          /L0*L12/

SCALARS
    DX         grid size x (m)                     /10/
    DY         grid size y (m)                     /10/
    DT         time step size t (s) a month        /2.628E6/
    S          storage coefficient                 /0.002/
    HBW        boundary head west (m)              / 100 /
    HBE        boundary head east (m)              / 50  /
```

```
      HBN        boundary head north (m)                / 100 /
      HBS        boundary head south (m)                / 100 /
      H0         initial head  (m)                      / 100 /
      QMAX       max pumping rate (m3 per d per m2) / 40 /
      QMIN       min pumping rate (m3 per d per m2) /   0 /
           ;


PARAMETER
     DEM(L)    water demand  (1000 m3 per day);
     DEM(L) = 100*ord(L);


TABLE T(I,J) isotropic transmissivity (m2 per day)

      J0   J1   J2   J3   J4   J5   J6   J7   J8   J9 J10
I0   205  185  190  175  160  185  195  190  175  160  165
I1   205  185  190  175  160  185  195  190  175  160  165
I2   260  235  215  200  175  185  195  190  175  160  165
I3   255  250  230  205  150  185  195  190  175  160  165
I4   210  225  220  195  145  185  195  190  175  160  165
I5   185  195  190  175  160  185  195  190  175  160  165
I6   160  205  215  200  165  185  195  190  175  160  165
I7   155  200  210  215  150  185  195  190  175  160  165
I8   170  215  200  195  155  185  195  190  175  160  165
I9   170  215  200  195  155  185  195  190  175  160  165
I10  170  215  200  195  155  185  195  190  175  160  165;


* finite-difference cell types
TABLE CELL(I,J) cell type

* -2 = north no-flow bndy
* -1 = south no-flow bndy
*  0 = interior cells
*  1 = west fixed potential
*  2 = east fixed potential
*  3 = north fixed potential
*  4 = south fixed potential

      J0   J1   J2   J3   J4   J5   J6   J7   J8   J9 J10
I0    1   -2   -2   -2   -2   -2   -2   -2   -2   -2    2
I1    1    0    0    0    0    0    0    0    0    0    2
I2    1    0    0    0    0    0    0    0    0    0    2
I3    1    0    0    0    0    0    0    0    0    0    2
I4    1    0    0    0    0    0    0    0    0    0    2
I5    1    0    0    0    0    0    0    0    0    0    2
I6    1    0    0    0    0    0    0    0    0    0    2
I7    1    0    0    0    0    0    0    0    0    0    2
I8    1    0    0    0    0    0    0    0    0    0    2
I9    1    0    0    0    0    0    0    0    0    0    2
I10   1   -1   -1   -1   -1   -1   -1   -1   -1   -1    2;


* Use for fixed head all around
*I0    1    3    3    3    3    3    3    3    3    3    2
*I10   1    4    4    4    4    4    4    4    4    4    2;
* Use for fixed head east and west
*I0    1   -2   -2   -2   -2   -2   -2   -2   -2   -2    2
*I10   1   -1   -1   -1   -1   -1   -1   -1   -1   -1    2;


* finite-difference coefficients
PARAMETER
        A(I,J)   finite-difference coefficient
        B(I,J)   finite-difference coefficient
```

```
          C(I,J)    finite-difference coefficient
          D(I,J)    finite-difference coefficient
           ;

          A(I,J) = 0.0;
          B(I,J) = 0.0;
          C(I,J) = 0.0;
          D(I,J) = 0.0;
A(I,J) =  (2*T(I+1,J)*T(I,J)/(T(I+1,J)+T(I,J))/(DX*DX))$(ord(I) ne card(I))
             +(    T(I,J)/(DX*DX))$(ord(I) eq card(I));
B(I,J) = (-2*T(I-1,J)*T(I,J)/(T(I-1,J)+T(I,J))/(DX*DX))$(ord(I) ne 0)
             +(    T(I,J)/(DX*DX))$(ord(I) eq 0) ;
C(I,J) =  (2*T(I,J+1)*T(I,J)/(T(I,J+1)+T(I,J))/(DY*DY))$(ord(J) ne card(J))
             +(    T(I,J)/(DY*DY))$(ord(J) eq card(J));
D(I,J) = (-2*T(I,J-1)*T(I,J)/(T(I,J-1)+T(I,J))/(DY*DY))$(ord(J) ne 0)
             +(    T(I,J)/(DY*DY))$(ord(J) eq 0);

SCALAR
          F          finite-difference coefficient;
          F = S/DT ;


VARIABLES
          OBJ      Objective value;

POSITIVE VARIABLES
          Q(I,J,L)  pumping  in  cell  ij  in  period  L  [1000  m3  per
day][decision]
          H(I,J,L) head in cell ij in period L
           ;

*         initial conditions
          H.fx(I,J,'L0')$(CELL(I,J) le 0) = H0 ;

*         boundary conditions
* Use for fixed head east and west
          H.fx(I,J,L)$(CELL(I,J) = 1) = HBW ;
          H.fx(I,J,L)$(CELL(I,J) = 2) = HBE ;
* Use for fixed head all around
          H.fx(I,J,L)$(CELL(I,J) = 3) = HBN ;
          H.fx(I,J,L)$(CELL(I,J) = 4) = HBS ;

* maximum/minimum pumping contrtraints
          Q.lo(I,J,L)$(CELL(I,J) le 0) = QMIN;
          Q.up(I,J,L)$(CELL(I,J) le 0) = QMAX;
          Q.lo(I,J,L)$(CELL(I,J) gt 0) = QMIN;
          Q.up(I,J,L)$(CELL(I,J) gt 0) = QMIN;

EQUATIONS
          CONT(I,J,L)     finite-difference continuity eqn. for period L
          DEMAND(L)       demand constraint eqn. for period L
          OBJECTIVE       objective function;

* continuity equation (including no-flow boundaries)
CONT(I,J,L) ..
     ( A(I,J)*(H(I+1,J,L+1)-H(I,J,L+1))
      +B(I,J)*(H(I,J,L+1)-H(I-1,J,L+1))
      +C(I,J)*(H(I,J+1,L+1)-H(I,J,L+1))
      +D(I,J)*(H(I,J,L+1)-H(I,J-1,L+1))
      -F*     (H(I,J,L+1))                   )$(CELL(I,J) eq 0)
    +( A(I,J)*(H(I+1,J,L+1)-H(I,J,L+1))
      +C(I,J)*(H(I,J+1,L+1)-H(I,J,L+1))
```

```
            +D(I,J)*(H(I,J,L+1)-H(I,J-1,L+1))
            -F*       (H(I,J,L+1))                )$(CELL(I,J) eq -2)
        +(  B(I,J)*(H(I,J,L+1)-H(I-1,J,L+1))
            +C(I,J)*(H(I,J+1,L+1)-H(I,J,L+1))
            +D(I,J)*(H(I,J,L+1)-H(I,J-1,L+1))
            -F        *(H(I,J,L+1))              )$(CELL(I,J) eq -1)
            -Q(I,J,L+1)
        =E= -F*       H(I,J,L);

    * demand constraint
    DEMAND(L) ..  SUM((I,J), Q(I,J,L)) =G= DEM(L);

    * objective function (sum of pumping or cost of pumping)
    OBJECTIVE ..  OBJ =E= (SUM((I,J,L), Q(I,J,L))) ;

    MODEL PUMPING /ALL/ ;
    SOLVE PUMPING USING LP MINIMIZING OBJ ;

    File GW2DT /GW2DT_output.txt/
    Put GW2DT;
    PUT ' Objective = '; Put OBJ.l; Put /;
    Put 'Pumping'/; Put '          ';
    Loop(L,PUT ' Time = '; Put L.tl:6; PUT ' Demand = '; Put DEM(L):6; Put /;
    PUT ' '; Loop(J, Put J.tl:6;) Put /;
    Loop(I, Put I.tl:6; Loop(J, Put Q.L(I, J,L):6:0);); Put /;););
    Put 'Head'/; Put '          ';
    Loop(J, Put J.tl:6;) Put /;
    Loop(L, PUT ' Time = '; Put L.tl:6; Put /;
    PUT ' '; Loop(J, Put J.tl:6;) Put /;
    Loop(I, Put I.tl:6; Loop(J, Put H.L(I, J, L):6:0); Put /;););
```

The results of running this model are:

```
    Objective =      9100.00
    Pumping
          Time = L0    Demand = 100.00
     J0    J1    J2    J3    J4    J5    J6    J7    J8    J9    J10
    I0      0    40    40    20     0     0     0     0     0     0     0
    I1      0     0     0     0     0     0     0     0     0     0     0
    I2      0     0     0     0     0     0     0     0     0     0     0
    I3      0     0     0     0     0     0     0     0     0     0     0
    I4      0     0     0     0     0     0     0     0     0     0     0
    I5      0     0     0     0     0     0     0     0     0     0     0
    I6      0     0     0     0     0     0     0     0     0     0     0
    I7      0     0     0     0     0     0     0     0     0     0     0
    I8      0     0     0     0     0     0     0     0     0     0     0
    I9      0     0     0     0     0     0     0     0     0     0     0
    I10     0     0     0     0     0     0     0     0     0     0     0
     Time = L1    Demand = 200.00
     J0    J1    J2    J3    J4    J5    J6    J7    J8    J9    J10
    I0      0     0     0     0    40    40     0     0     0     0     0
    I1      0     0     0     0     0     0     0    40     0     0     0
    I2      0     0     0     0     0     0     0     0     0     0     0
    I3      0     0     0     0    40     0     0     0     0     0     0
    I4      0     0     0     0     0     0     0     0     0     0     0
    I5      0     0     0     0     0     0     0     0     0     0     0
    I6      0     0     0     0     0     0     0     0     0     0     0
    I7      0     0     0     0     0     0     0     0     0     0     0
    I8      0     0     0     0     0     0     0     0     0     0     0
    I9      0     0     0     0     0     0     0     0     0     0     0
    I10     0     0     0    40     0     0     0     0     0     0     0
     Time = L2    Demand = 300.00
     J0    J1    J2    J3    J4    J5    J6    J7    J8    J9    J10
```

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	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	0	0	0	0	0	0	0	0	0	0	0
I1	0	0	0	0	0	0	0	0	0	0	0
I2	0	0	0	0	0	0	0	0	0	0	0
I3	0	0	0	0	0	0	0	0	0	0	0
I4	0	0	0	0	0	0	0	0	0	0	0
I5	0	0	0	0	0	20	0	0	0	0	0
I6	0	0	0	0	0	40	0	0	0	0	0
I7	0	0	0	40	0	0	0	40	0	0	0
I8	0	0	0	0	0	40	0	0	0	0	0
I9	0	0	0	0	0	0	0	0	0	0	0
I10	0	0	0	0	40	40	0	40	0	0	0

Time = L3   Demand = 400.00

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	0	0	0	40	0	40	0	0	0	0	0
I1	0	0	0	0	0	0	0	0	0	0	0
I2	0	0	0	0	0	40	0	0	0	0	0
I3	0	0	0	0	0	40	0	0	0	0	0
I4	0	0	0	0	40	0	0	40	0	0	0
I5	0	0	0	0	0	40	0	0	0	0	0
I6	0	0	0	0	0	40	0	0	0	0	0
I7	0	0	0	40	0	40	0	0	0	0	0
I8	0	0	0	0	0	0	0	0	0	0	0
I9	0	0	0	0	0	0	0	0	0	0	0
I10	0	0	0	0	0	0	0	0	0	0	0

Time = L12   Demand = 1.3E+3

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	0	0	0	40	15	0	0	28	0	0	0
I1	0	0	40	0	21	0	0	27	0	0	0
I2	0	0	40	40	15	0	0	25	0	0	0
I3	0	0	40	40	15	0	0	21	0	25	0
I4	0	0	40	0	23	0	0	17	0	40	0
I5	0	0	40	0	24	0	0	16	0	40	0
I6	0	0	40	0	24	0	0	15	0	40	0
I7	0	0	40	40	16	0	0	15	0	40	0
I8	0	0	40	0	23	0	0	14	0	40	0
I9	0	0	40	0	26	0	0	14	0	40	0
I10	0	0	40	0	27	0	0	14	0	40	0

Head

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10

Time = L0

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	100	100	100	100	100	100	100	100	100	100	50
I1	100	100	100	100	100	100	100	100	100	100	50
I2	100	100	100	100	100	100	100	100	100	100	50
I3	100	100	100	100	100	100	100	100	100	100	50
I4	100	100	100	100	100	100	100	100	100	100	50
I5	100	100	100	100	100	100	100	100	100	100	50
I6	100	100	100	100	100	100	100	100	100	100	50
I7	100	100	100	100	100	100	100	100	100	100	50
I8	100	100	100	100	100	100	100	100	100	100	50
I9	100	100	100	100	100	100	100	100	100	100	50
I10	100	100	100	100	100	100	100	100	100	100	50

Time = L1

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	100	88	76	61	40	34	38	39	43	47	50
I1	100	89	77	65	51	44	42	37	43	47	50
I2	100	90	79	68	56	50	47	45	46	48	50
I3	100	91	81	70	55	54	52	50	49	50	50
I4	100	91	82	73	64	59	55	53	52	51	50
I5	100	91	83	76	68	62	59	56	53	52	50
I6	100	91	84	77	70	65	61	57	55	52	50
I7	100	91	84	78	71	66	62	58	56	53	50
I8	100	91	84	77	71	66	62	59	56	53	50
I9	100	91	82	74	69	65	62	59	56	53	50
I10	100	90	80	67	67	65	62	59	56	53	50

Time = L2

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	100	93	86	79	72	66	62	58	55	52	50
I1	100	93	86	79	72	65	61	57	55	52	50
I2	100	93	85	77	70	64	59	56	53	52	50
I3	100	92	84	76	67	60	56	53	52	51	50
I4	100	90	81	72	63	54	51	50	49	49	50
I5	100	89	78	67	56	45	45	45	46	48	50
I6	100	86	74	61	50	37	38	39	42	46	50
I7	100	85	70	54	45	35	33	30	38	44	50
I8	100	84	70	55	41	28	30	31	36	43	50
I9	100	84	69	54	38	27	26	28	34	42	50
I10	100	83	68	51	27	18	21	20	31	41	50

Time = L3

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	100	83	65	44	36	24	30	34	39	44	50
I1	100	84	68	52	39	30	31	34	39	44	50
I2	100	85	70	55	40	25	30	33	38	44	50
I3	100	86	71	56	39	25	29	32	37	44	50
I4	100	85	71	55	34	28	29	27	36	44	50
I5	100	85	71	56	40	26	30	33	39	44	50
I6	100	85	71	56	43	28	33	37	41	45	50
I7	100	85	71	54	46	33	38	40	43	47	50
I8	100	86	74	62	53	45	44	44	45	48	50
I9	100	88	77	66	58	51	48	47	47	48	50
I10	100	88	78	68	60	53	50	48	48	49	50

Time = L12

	J0	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
I0	100	68	36	9	0	0	0	0	15	32	50
I1	100	67	31	12	0	0	0	0	15	31	50
I2	100	66	30	8	0	0	0	0	14	30	50
I3	100	66	30	9	0	0	0	0	11	23	50
I4	100	65	30	14	0	0	0	0	9	19	50
I5	100	64	29	14	0	0	0	0	9	17	50
I6	100	62	28	13	0	0	0	0	8	17	50
I7	100	61	27	9	0	0	0	0	8	17	50
I8	100	62	28	13	0	0	0	0	8	17	50
I9	100	63	29	15	0	0	0	0	8	17	50
I10	100	63	30	16	0	0	0	0	8	17	50

4.7 Exercises

1. (Adapted from Mays and Tung, 1992, prob. 8.4.3). Consider an undeveloped, homogeneous, isotropic, confined aquifer of large areal extent in which there are L potential production wells located at points $[(x_\ell, y_\ell), \ell = 1, ..., L]$ and K points at which water levels are to be observed $[(x_k, y_k), k = 1, ..., K]$. For steady-state flow in the aquifer, the Theim equation can be used to predict the drawdown s_k at the control points due to the pumping Q_ℓ at the production wells

$$s_k = \sum_{\ell=1}^{L} s_{k\ell} = \sum_{\ell=1}^{L} a_{k\ell} Q_\ell \qquad k = 1, \cdots, K$$

where $s_{k\ell}$ is the drawdown at point k due to pumping at well ℓ alone, and

$$a_{k\ell} = \frac{1}{2\pi T} \ln\left(\frac{R_\ell}{r_{k\ell}}\right) \qquad k = 1, \cdots, K; \quad \ell = 1, \cdots, L$$

T is the transmissivity of the aquifer, R_ℓ is the radius of influence of the ℓ-th well, and

$$r_{k\ell} = \sqrt{(x_k - x_\ell)^2 + (y_k - y_\ell)^2}$$

A management model with the objective of maximizing the total withdrawal from the aquifer while meeting specified upper bound constraints on the drawdown at the control points and upper and lower bounds on the pumping rates is

Maximize $\qquad \sum_{\ell=1}^{L} Q_\ell$

Subject to

$$\sum_{\ell=1}^{L} a_{k\ell} Q_\ell = s_k \le s_k^* \qquad\qquad k = 1, \cdots, K$$

$$Q_\ell^{low} \le Q_\ell \le Q_\ell^{up} \qquad\qquad \ell = 1, \cdots, L$$

Consider the situation (shown in the Figure) of three production wells ($L=3$) and five control points ($K=5$) a transmissivity of 5,000 gal/day/ft^2, and a radius of influence for each well of 700 ft. Additional data related to the problem are listed in the Table P4.1.1 and the response coefficients $a_{k\ell}$ are listed in the Table P4.1.2.
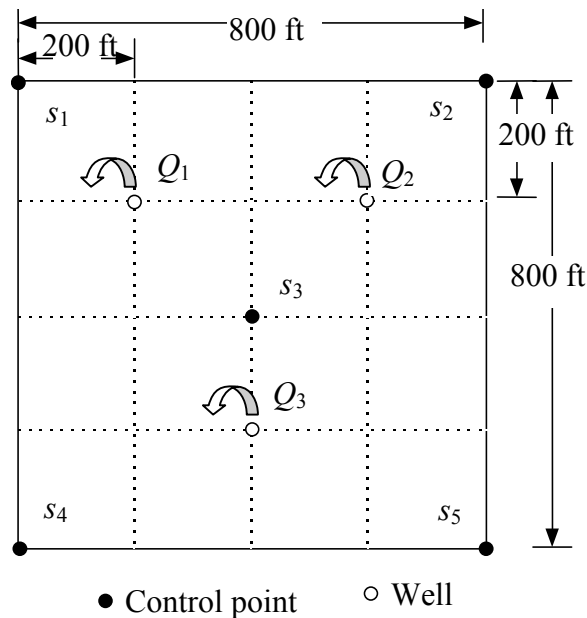
Figure P4.1.1. Aquifer for steady-state pumping optimization example.

Table P4.1.1. Distance (ft.) between well ℓ and control point k.

$r_{k,\ell}$ Well	Control $k = 1$	Point 2	3	4	5	Production Capacity, Q_ℓ^{up} (gal/day)
$\ell =$ 1	282.8	632.5	282.8	848.5	632.5	200,000
2	632.5	282.8	282.8	632.5	848.5	200,000
3	721.1	721.1	200	565.7	565.7	200,000
Max. Allowable Drawdown, s_k^* (ft)	7	7	15	7	7	

Table P4.1.2. Response matrix A ($\times 10^{-5}$) for 5 control points and 3 production wells.

$a_{k,\ell}$ Control Pt.	Well $\ell = 1$	2	3
$k =$ 1	2.885×10^{-5}	3.228×10^{-6}	0
2	3.228×10^{-6}	2.885×10^{-5}	0
3	2.885×10^{-5}	2.885×10^{-5}	3.988×10^{-5}
4	0	3.228×10^{-6}	6.781×10^{-6}
5	3.228×10^{-6}	0	6.781×10^{-6}

Formulate and solve this model in GAMS to find the optimal puming rates for each well.

2. One-dimensional problem

3. Two-dimensional, steady-state problem

4. Two-dimensional, transient problem

5. OPTIMAL SELECTION OF AGRICULTURAL CROPS

(Alex Meuraus)

THIS SECTION STILL NEEDS EDITING

Task:

The task of agricultural crop selection is a difficult one that can be Modeled in GAMS. Assume that the following is known:

1. Types of crops available for planting;
2. Modeling period (one year, using a time step of one month)
3. Irrigation water demands for each crop for each month

4. Stress coefficient for each crop for each month indicating the yield decrease of a given crop for a deficit of irrigation water in each time interval;
5. Price for each crop;
6. Yield for each crop;
7. Proposed area for planting crops; and
8. Available irrigation water hydrograph.

The model is to determine:

1. The amount of land to be allocated to each crop; and
2. The irrigation water required for each crop.

Objective: Maximize proceeds from the yield sales for each crop

The mathematical model is represented by the equations:

$$R = \sum_p C(p) * s(p) * Y(p) * \prod_t \left[\frac{Wr(p,t)}{Wd(p,t)} \right]^{l(p,t)}$$

$$\sum_p S(p) < So$$

$$\sum_p Wr(p,t) < Wo(t)$$

where

p	crop type
t	time index (month)
S(p)	area for crop p
$Wr(p,t)$	delivery of water for each crop in each time period;
$Wd(p,t)$	demand for water for each crop in each time period;
$l(p,t)$	stress coefficient for each crop in each time period;
So	total area of land available;
$Wo(t)$	hydrograph of available water;
$C(p)$	price for crop p;
$S(p)$	area planted with crop p; and
$Y(p)$	yield of crop p

The GAMS code for the model is:

```
SETS p  crops  / cotton, rice, wheat, maize, others /
     t  month  / jan,feb,mar,apr,may,jun,jul,aug,sep,oct,nov,dec /

TABLE wd(p,t) water demand (mill m3 per ha)
          jan  feb  mar  apr  may  jun  jul  aug  sep  oct  nov  dec
cotton                        1.2  1.2  1.2  1.2
```

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```
rice                                    2.9   2.8   2.9   2.8   2.6
wheat                                   0.8   0.8   0.9   0.8   0.7
maize                                   1.8   1.9   1.9   1.8   1.8
others                                  1.1   1.1   1.1   1.1   1.1


TABLE l(p,t) stress coefficients
          jan feb mar apr  may   jun   jul   aug   sep  oct  nov  dec
cotton                     1.0   1.0   1.0   0.9
rice                       0.9   0.8   0.9   0.8   0.6
wheat                      0.8   0.8   0.9   0.8   0.7
maize                      0.8   0.9   0.9   0.8   0.8
others                     1.0   1.0   1.0   1.0   1.0


PARAMETERS
   C(p)  (price $ per c)     / cotton 3, rice 5, wheat 7, maize
11,others 60/
   Y(p)  (yield c per ha)    / cotton 2, rice 3, wheat 4, maize  5,
others 3/
   wo(t) (water available mill m3 per month);
*          computed as a uniform number between numbers 900 - 1100.
   wo(t) = UNIFORM(900,1100);


SCALAR So land available / 1000 /;

VARIABLES  obj  revenue  ($);

POSITIVE VARIABLES
   S(p) cultivated land (ha)
   wr(p,t) water delivered (mill m3 per ha);

* Upper bound on Delivered water
wr.UP(p,t) = wd(p,t);

* Initial guesses for cultivated land and delivered water
S.L(p) = So/CARD(p);
wr.L(p,t)  = wo(t);


EQUATION   ben     Objective function
           land    land balance (ha)
           water(t) water balance (ha);

ben.. obj =E=
sum(p,C(p)*S(p)*Y(p)*prod(t$wd(p,t),(wr(p,t)/wd(p,t))**l(p,t)));

* Limit on cultivated land
land.. sum(p, S(p)) =L= So;

* Limit on irrigation water
water(t).. sum(p, S(p)*wr(p,t)) =L= wo(t);

MODEL crop / ALL /;

SOLVE crop USING NLP MAXIMiZING obj;

FILE res /Crop1.txt/;
PUT res
PUT "Revenue = ", obj.L//;
PUT "Crop             Land"/;
LOOP(p,PUT p.TL,  s.l(p)/;);
```

The results of running the model are:

```
    Revenue =    149626.12

    Crop             Land
    cotton          33.40
    rice             0.00
    wheat            0.00
    maize            0.00
    others         830.38
```

6. OPTIMAL DEVELOPMENT OF CANALS

<mark>THIS SECTION STILL NEEDS EDITING</mark>

```
*Grid points
set x /x1*x20/;
set y /y1*y20/;

SET INSIDE(X,Y);
INSIDE(X,Y)=YES$((ord(x)<card(x))and(ord(y)<card(y)));

* Topography of grid
TABLE zp(x,y)
        y1      y2      y3    y4    y5     y6      y7     y8    y9    y10    y11  y12  y13  y14  y15
y16  y17  y18  y19  y20
x1     1.1     1.1     1.1   1.1  -99.  -99.1  -99.1  -97.1  -89.  -99.  -90.  1.1  1.1  1.1  1.1
1.1  1.1  1.1  1.1  1.1
x2     1.2     1.2     1.2   1.2  -99.  -98.2  -99.2  -98.2  -69.  -89.  -90.  1.2  1.2  1.2  1.2
1.2  1.2  1.2  1.2  1.2
x3     1.3     1.3     1.3   1.3  -91.  -97.3  -99.3  -97.3  -49.  -79.  -8.3  8.3  1.3  8.3  1.3
1.3  1.3  1.3  1.3  1.3
x4     1.4     1.4     1.4   1.4  -91.  -67.4  -99.4  -67.4  -35.  -68.  -8.4  8.4  1.4  8.4  1.4
1.4  1.4  1.4  1.4  1.4
x5     1.5     1.5     8.5   1.5  -91.  -67.5  -99.5  -57.5  -25.  -57.  -8.5  8.5  1.5  8.5  1.5
1.5  1.5  1.5  1.5  1.5
x6     1.6     1.6     8.6   2.6   1.6  -37.6  -79.6  -47.6  -15.  -46.  -8.6  8.6  8.6  8.6  1.6
1.6  1.6  1.6  1.6  1.6
x7     -3      1.7     8.7   8.7   8.7  -23.7  -49.7  -33.7  -15.  -35.  -8.7  8.7  8.7  8.7  1.7
1.7  1.7  1.7  1.7  1.7
x8     -10     -9      8.8   8.8   8.8  -16.8  -22.8  -12.8  -10.  -14.  -0.8  1.8  0.8  8.8  8.8
1.8  1.8  1.8  1.8  1.8
x9     -39     -39     8.9   8.9   8.9  -10.9   -5.9   -7.9   -9.  -5.9  0.9  0.9  0.9  8.9  8.9
1.9  1.9  1.9  1.9  1.9
x10    -99     -79     -8.8  8.8   8.8  -10.8   -8.8   -8.8   -8.  -8.8  0.8  0.8  0.8  0.8  8.8
1.8  1.8  1.8  1.8  1.8
x11    -39     -39     1.7   8.7   8.7   -8.7   -8.7   -8.7   -8.  -8.7  0.7  0.7  0.7  0.7  8.7
1.7  1.7  1.7  1.7  1.7
x12    -9      9.6     1.6   1.6   8.6   -8.6    8.6    8.6   1.6  -9.6  0.6  0.6  0.6  0.6  8.6
3.6  1.6  1.6  1.6  1.6
x13    -3      9.5     1.5   1.5   8.5    8.5    8.5    8.5   1.5  -9.5  0.5  0.5  1.5  0.5  8.5
3.5  1.5  1.5  1.5  1.5
x14    1.4     1.4     1.4   1.4   8.4    1.4    1.4    8.4   1.4  -9.4  0.4  0.4  1.4  0.4  8.4
3.4  1.4  1.4  1.4  1.4
x15    1.3     1.3     1.3   1.3   8.3    1.3    1.3   -8.3   1.3  -9.3  -1.3  0.3  1.3  1.3  8.3
1.3  1.3  1.3  1.3  1.3
x16    1.2     1.2     1.2   1.2   8.2    8.2    8.2   -8.2   1.2  -9.2  -2.2  1.2  1.2  1.2  8.2
1.2  1.2  1.2  1.2  1.2
x17    1.1     1.1     1.1   1.1   8.1    8.1    8.1   -8.1  -1.1  -9.1  -3.1  1.1  1.1  1.1  1.1
1.1  1.1  1.1  1.1  1.1
x18    1.2     1.2     1.2   1.2   1.2    1.2    1.2   -6.2  -9.2  -9.2  -4.2  1.2  1.2  1.2  1.2
1.2  1.2  1.2  1.2  1.2
x19    1.3     1.3     1.3   1.3   1.3    1.3    1.3   -7.3  -9.3  -9.3  -5.3  1.3  1.3  1.3  1.3
1.3  1.3  1.3  1.3  1.3
x20    1.4     1.4     1.4   1.4   1.4    1.4    1.4   -8.4  -9.4  -9.4  -6.4  1.4  1.4  1.4  1.4
1.4  1.4  1.4  1.4  1.4
```

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```
;

PARAMETER START_END(X,Y);
          START_END(X,Y)=0;
          START_END('X4','Y3')= 1;
          START_END('X13','Y18')=-1;

VARIABLE Dx(x,y), Dy(x,y), obj;

EQUATION first(x,y), ben;

first(x,y)$(INSIDE(x,y)).. START_END(X,Y) =E= (dx(x+1,y)-dx(x,y))
                                              + (dy(x,y+1)-dy(x,y));

ben.. obj =e=  1000* sum((x,y)$(INSIDE(x,y)),
                        (30-zp(x,y)) * (sqrt(dy(x,y)*dy(x,y)+0.00001))
                      + (30-zp(x,y)) * (sqrt(dx(x,y)*dx(x,y)+0.00001)));

dx.up(x,y)=1;
dy.up(x,y)=1;

dx.lo(x,y)=-1;
dy.lo(x,y)=-1;

dx.fx('x1',y)=0;   dy.fx('x1',y)=0;
dx.fx(x,'y1')=0;   dy.fx(x,'y1')=0;
dx.fx('x20',y)=0;  dy.fx('x20',y)=0;
dx.fx(x,'y20')=0;  dy.fx(x,'y20')=0;

MODEL Canal /all/;

SOLVE Canal using   NLP MINIMIZING obj;

FILE res /Canal.txt/
PUT res;
PUT " Objective function= "; PUT obj.l
PUT /;

PARAMETER no_yes(x,y);
no_yes(x,y)=
              -START_END(X,Y)+
               SQRT(dx.l(x+1,y)*dx.l(x+1,y)+0.000001 )+
               SQRT(dx.l(x,y)  *dx.l(x,y)  +0.000001 )+
               SQRT(dy.l(x,y+1)*dy.l(x,y+1)+0.000001 )+
               SQRT(dy.l(x,y)  *dy.l(x,y)  +0.000001 )
                ;

PUT "Canal yes_no"; PUT /;
LOOP(X,
LOOP(Y, PUT no_yes(X,Y):5:1; ); PUT /;
    );

PUT "level of earth";PUT /;
LOOP(X,
LOOP(Y, PUT zp(X,Y):7:2; ); PUT /;
    );
PUT /;
```

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The results of the moel are:

```
Objective function=    664152.36
Canal yes_no
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   2.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   2.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   2.0   0.1   0.1   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   1.9   0.2   0.1   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   1.8   1.8   1.8   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.1   1.9   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.1   1.9   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   1.9   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   1.9   1.9   1.9   1.9   1.9   1.9   1.9   1.9   1.9   1.9   1.9   1.9   1.9   2.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
level of earth
   1.10    1.10    1.10    1.10 -99.00 -99.10 -99.10 -97.10 -89.00 -99.00 -90.00    1.10    1.10    1.10    1.10    1.10    1.10    1.10    1.10    1.10
   1.20    1.20    1.20    1.20 -99.00 -98.20 -99.20 -98.20 -69.00 -89.00 -90.00    1.20    1.20    1.20    1.20    1.20    1.20    1.20    1.20    1.20
   1.30    1.30    1.30    1.30 -91.00 -97.30 -99.30 -97.30 -49.00 -79.00  -8.30    8.30    1.30    8.30    1.30    1.30    1.30    1.30    1.30    1.30
   1.40    1.40    1.40    1.40 -91.00 -67.40 -99.40 -67.40 -35.00 -68.00  -8.40    8.40    1.40    8.40    1.40    1.40    1.40    1.40    1.40    1.40
   1.50    1.50    8.50    1.50 -91.00 -67.50 -99.50 -57.50 -25.00 -57.00  -8.50    8.50    1.50    8.50    1.50    1.50    1.50    1.50    1.50    1.50
   1.60    1.60    8.60    2.60    1.60 -37.60 -79.60 -47.60 -15.00 -46.00  -8.60    8.60    8.60    8.60    1.60    1.60    1.60    1.60    1.60    1.60
  -3.00    1.70    8.70    8.70    8.70 -23.70 -49.70 -33.70 -15.00 -35.00  -8.70    8.70    8.70    8.70    1.70    1.70    1.70    1.70    1.70    1.70
 -10.00   -9.00    8.80    8.80    8.80 -16.80 -22.80 -12.80 -10.00 -14.00  -0.80    1.80    0.80    8.80    8.80    1.80    1.80    1.80    1.80    1.80
 -39.00  -39.00    8.90    8.90    8.90 -10.90  -5.90  -7.90  -9.00  -5.90    0.90    0.90    0.90    8.90    8.90    1.90    1.90    1.90    1.90    1.90
 -99.00  -79.00   -8.80    8.80    8.80 -10.80  -8.80  -8.80  -8.00  -8.80    0.80    0.80    0.80    0.80    8.80    1.80    1.80    1.80    1.80    1.80
 -39.00  -39.00    1.70    8.70    8.70  -8.70  -8.70  -8.70  -8.00  -8.70    0.70    0.70    0.70    0.70    8.70    1.70    1.70    1.70    1.70    1.70
  -9.00    9.60    1.60    1.60    8.60  -8.60   8.60   8.60   1.60  -9.60    0.60    0.60    0.60    0.60    8.60    3.60    1.60    1.60    1.60    1.60
  -3.00    9.50    1.50    1.50    8.50   8.50   8.50   8.50   1.50  -9.50    0.50    0.50    1.50    0.50    8.50    3.50    1.50    1.50    1.50    1.50
   1.40    1.40    1.40    1.40    8.40   1.40   1.40   8.40   1.40  -9.40    0.40    0.40    1.40    0.40    8.40    3.40    1.40    1.40    1.40    1.40
   1.30    1.30    1.30    1.30    8.30   1.30   1.30  -8.30   1.30  -9.30   -1.30    0.30    1.30    1.30    8.30    1.30    1.30    1.30    1.30    1.30
   1.20    1.20    1.20    1.20    8.20   8.20   8.20  -8.20   1.20  -9.20   -2.20    1.20    1.20    1.20    8.20    1.20    1.20    1.20    1.20    1.20
   1.10    1.10    1.10    1.10    8.10   8.10   8.10  -8.10  -1.10  -9.10   -3.10    1.10    1.10    1.10    1.10    1.10    1.10    1.10    1.10    1.10
   1.20    1.20    1.20    1.20    1.20    1.20    1.20  -6.20  -9.20  -9.20   -4.20    1.20    1.20    1.20    1.20    1.20    1.20    1.20    1.20    1.20
   1.30    1.30    1.30    1.30    1.30    1.30    1.30  -7.30  -9.30  -9.30   -5.30    1.30    1.30    1.30    1.30    1.30    1.30    1.30    1.30    1.30
   1.40    1.40    1.40    1.40    1.40    1.40    1.40  -8.40  -9.40  -9.40   -6.40    1.40    1.40    1.40    1.40    1.40    1.40    1.40    1.40    1.40
```

7. PROBLEMS OF POWER NETWORKS

7.1 Problem of Power Generation, Distribution, and Consumption

Consider a power network comprised of two groups: the first group with power generation facilities and the second group for power consumption. The power transmission lines can be used to transfer power to consumers. Figure 7.1.1 shows a network with power generation and power consumption nodes.
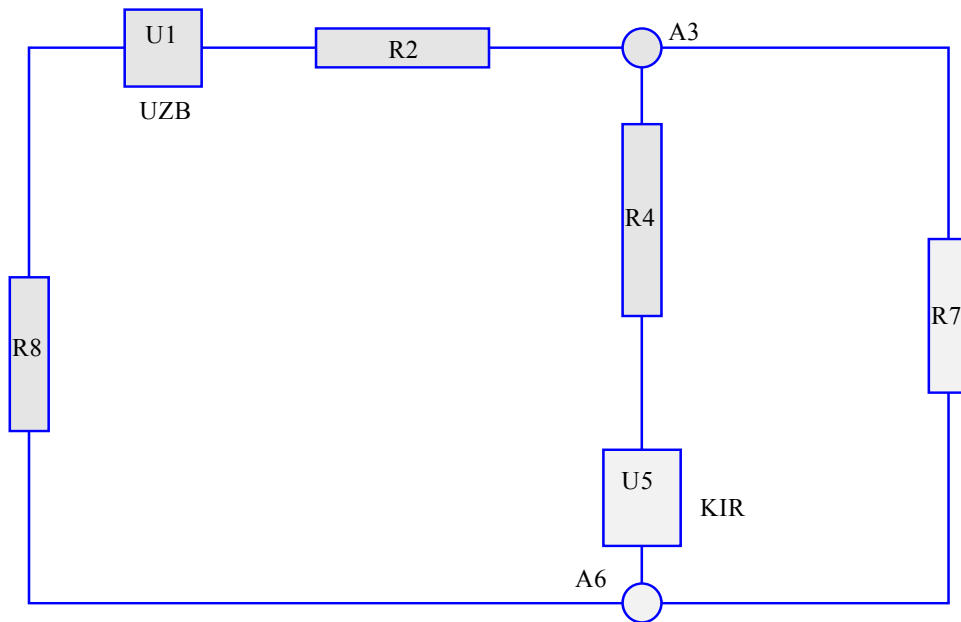
Fig. 7.1.1. Energy and Electricity Tasks Schematic

The power generation at node U5 (a5) is 100.
The power consumption at node R4 (a4) is 80.
The power consumption at node R8 (a8) is 30.
The power consumption at node R2 (a2) is 10.
The power transferred from node R8 (a8) to node a6 is 100.
The power transferred from node R2 (a2) to node a3 is 50.

The power transfer in all lines of the network and the power generation at the node U1 (a1) are to be determined. It is known that the power generation at this node is less than 900.

This problem is similar to the problem connected with power generation and power transfer between the nodes countries in the Central Asian network. The essence of the problem becomes clear assuming that "a1" refers to the set UZB (Uzbekistan), and "a5" to another set KIR (Kyrgyzstan).

Besides, there is a condition on specified power transfer through the transmission lines of the network.

Consider the laws characteristic for the model:

1) The net flow through a regular junction is 0:

$$\sum_{m\,incoming} i(m,n) = \sum_{m\,outgoing} i(n,m)$$

2) Flow from the first node in the direction of a second node of the circuit does not change. However, it has a "-" symbol for the first node, and "+" for the second node:

$$i(m,n) = -i(n,m)$$

3) Flow transferred through a power generating node is increased by the value of the generated power:

$$\sum_{m\,incoming} i(m,n) - \sum_{m\,outgoing} i(n,m) = e(n)$$

4) Flow transferred through a power consumption node is decreased by the value of the consumed power:

$$\sum_{m\,incoming} i(m,n) - \sum_{m\,outgoing} i(n,m) = -r(n)$$

In the equations:

$i(m,n)$	power flow from node m to node n,
$e(n)$	power generation at node n;
$r(n)$	power consumption at node n;

On the basis of these equations we can show that the total power generation is equal to the total power consumption.

```
SETS
m   /a1,a2,a3,a4,a5,a6,a7,a8    /

alias(m,m1);

SETS
mr(m)   consumers          /a2,a8,a4,a7/
me(m)   producers          /a1,A5/
m0(m)   junctions          /a3,a6/

m_and_m1(m,m1) all nodes and their connections /
a1.a2, a1.a8,
a2.a1, a2.a3,
```

```
a3.a2, a3.a4, a3.a7
a4.a3, a4.a5,
a5.a4, a5.a6,
a6.a5, a6.a8, a6.a7,
a7.a3, a7.a6,
a8.a1, a8.a6          /

VARIABLES
e(m),     power generation
r(m),     power consumption
i(m,m1),  power flow
obj;

equation
coni(m,m1),
conu0(m),
conue(m),
conur(m),
ben;

* All nodes
coni(m,m1)..         i(m,m1) =e= -i(m1,m)   ;
* Junction nodes
conu0(m)$(m0(m))..   sum(m1$(m_and_m1(m,m1)),i(m,m1)) =e=   0      ;
* Generation nodes
conue(m)$(me(m))..   sum(m1$(m_and_m1(m,m1)),i(m,m1)) =e=   e(m)   ;
* COnsumption nodes
conur(m)$(mr(m))..   sum(m1$(m_and_m1(m,m1)),i(m,m1))  =e=  -r(m)   ;

ben..           obj =E= 1;

e.up('a1')=          900;
e.lo('a1')=            0;

e.fx('a5')=          100;

i.fx('a8','a6')=100;
i.fx('a2','a3')= 50;

r.fx('a4')= 80;
r.fx('a8')= 30;
r.fx('a2')= 10;

MODEL VAN /ALL/;
SOLVE VAN USING NLP MinimaZING obj;

file res /odc1.txt/
put res;
put " I  Power Flow " /;
put "               ";
loop(m, put m.tl:10; );
        put /;
loop(m, put m.tl:10
        loop(m1, put i.l(m,m1):10:1; );
        put /;);
put /;
put /;
put " U  Generation " /;
put "               ";
loop(m, put m.tl:10; );
put /;
```

```
put "       ";
loop(m, put e.l(m):10:1; );
put /;
put /;
put " R  Consumption" /;
put "             ";
loop(m, put m.tl:10; );
put /;
put "        ";
loop(m, put r.l(m):10:1; );
put /;
```

The results are:

```
I  Power Flow
             a1       a2       a3       a4       a5       a6       a7       a8
a1          0.0     60.0      0.0      0.0      0.0      0.0      0.0    130.0
a2         -60.0      0.0     50.0      0.0      0.0      0.0      0.0      0.0
a3          0.0    -50.0      0.0    -30.6      0.0      0.0     80.6      0.0
a4          0.0      0.0     30.6      0.0   -110.6      0.0      0.0      0.0
a5          0.0      0.0      0.0    110.6      0.0    -10.6      0.0      0.0
a6          0.0      0.0      0.0      0.0     10.6      0.0     89.4   -100.0
a7          0.0      0.0    -80.6      0.0      0.0    -89.4      0.0      0.0
a8       -130.0      0.0      0.0      0.0      0.0    100.0      0.0      0.0

  U  Generation
          a1       a2       a3       a4       a5       a6       a7       a8
        190.0      0.0      0.0      0.0    100.0      0.0      0.0      0.0

  R  Consumption
          a1       a2       a3       a4       a5       a6       a7       a8
         0.0     10.0      0.0     80.0      0.0      0.0    170.0     30.0
```

This example served as the basis for the development of the model of water-energy market designed by Kyrgyzenergo (see A. Zyryanov and E. Antipova, Vol. 2, Section 2.1, "Optimization of the Syrdarya Water and Energy Uses under Current Conditions," McKinney, D.C. and A.K. Kenshimov (eds.), 2000).

7.2 Defining Voltage and Strength of Current in Direct-Current Circuit

The problem considered here is to determine the unknown variables of a direct current circuit of any configuration comprised of power sources and resistances. Kirkoff's and Ohm's laws are used in the model:

- **Kirkoff's law:** the sum of currents in each of the nodes is equal to zero.
- **Ohm's law:** the decrease in voltage across a resistor is proportional to the resistance multiplied by the current.

Consider the scheme shown in Figure 7.2.1.

```
SETS
m   /a1,a2,a3,a4,a5,a6,a7,a8   /
alias(m,m1);
```

```
alias(m,m2);

sets
mr(m)    consumers     /a2,a8,a4,a7/
me(m)    generators    /a1,a5/
m0(m)    junctions     /a3,a6/

m_and_m1(m,m1)          connection
/
a1.a2, a1.a8,
a2.a1, a2.a3,
a3.a2, a3.a4, a3.a7
a4.a3, a4.a5,
a5.a4, a5.a6,
a6.a5, a6.a8, a6.a7,
a7.a3, a7.a6,
a8.a1, a8.a6    /

m1_m_m2(m1,m,m2);
m1_m_m2(m1,m,m2)= yes$(m_and_m1(m1,m) and m_and_m1(m,m2));
m1_m_m2(m1,m,m1)$m1_m_m2(m1,m,m1)= no;

* If there is a connection between the "m", "m1" and "m2",
* then there is a connection between "m", "i1" and "m2".
* However, if "m1" coincides with "m2",
* it is not a double connection
* but a connection based on the principle "TO- FROM".
* It is excluded from the whole set of triple connections.

VARIABLES
* Current source
e(m),

* Resistors
r(m),

* Voltages between nodes
u(m,m1),

* Currents between nodes
i(m,m1),

obj;

EQUATION
* Calculate currents
coni(m,m1),

* Calculate voltages
conu(m,m1),

*  Kirchhoff 's Law
ii(m),

*  Ohm's Law for junction nodes
uu_0(m1,m,m2),

*  Ohm's Law for supply nodes
uu_e(m1,m,m2),
```

```
*  Ohm's Law for resistor nodes
uu_r(m1,m,m2),

ben;

*  Kirchhoff 's Law
ii(m)..            sum(m1$m_and_m1(m,m1),i(m,m1)) =e= 0;

*  Calculate currents
coni(m,m1)..       i(m,m1) =e= -i(m1,m)  ;

*  Calculate voltages
conu(m,m1)..       u(m,m1) =e=  u(m1,m)  ;

*  Ohm's Law for junction nodes
uu_0(m1,m,m2)$m0(m)..
    0  =e=  (u(m,m1)-u(m,m2))$m1_m_m2(m1,m,m2);

*  Ohm's Law for supply nodes
uu_e(m1,m,m2)$(me(m)$(m1_m_m2(m1,m,m2)$( ord(m1) gt  ord(m2))))..
    u(m,m1)-u(m,m2) =e= e(m);

*  Ohm's Law for resistor nodes
uu_r(m1,m,m2)$(mr(m)$m1_m_m2(m1,m,m2))..
    u(m,m1)-u(m,m2) =e= -r(m)*i(m,m1);

ben..            obj =E=i('a1','a8')*i('a1','a8');

e.up(m)=           100;
e.lo(m)=          -100;
r.up(m)=           100;
r.lo(m)=          -100;
i.l(m,m1)$m_and_m1(m,m1)=    -1 ;
i.up(m,m1)=          100;
i.lo(m,m1)=         -100;
u.up(m,m1)=          100;
u.lo(m,m1)=         -100;

r.fx('a2')= 10;
r.fx('a4')= 20;
r.fx('a7')= 40;

e.fx('a1')=20;
e.fx('a5')=40;

i.fx('a1','a2')=    0.125;

MODEL VAN /ALL/;
SOLVE VAN USING NLP MINIMIZING obj;

file res /odc2.txt/
put res;
put " m_AND_m1" /;
loop(m,  put m.tl:6;
        loop(m1, put M_and_M1(M,M1):7; );
put /;);
put /;
put " m_m_M1" /;
loop(m1,
     loop(m,
```

```
          loop(m2, put M1_m_M2(m1,M,M2):7; );
put /;);
put /;);
put " I amper   " /;
put "           ";
loop(m, put m.tl:7; );
put /;
loop(m,  put m.tl:6;
         loop(m1, put i.l(M,m1):7:3; );
put /;);
put /;
put " U volt " /;
put "           ";
loop(m, put m.tl:7; );
put /;
loop(m,  put m.tl:6;
         loop(m1, put u.l(M,m1):7:1; );
put /;);
put /;
put " E volt on supplies " /;
put "           ";
loop(m, put m.tl:6; );put /;
put "       ";
loop(m, put e.l(m):6:2; );put /;
put /;
put " R - resisters  " /;
put "           ";
loop(m, put m.tl:6; );put /;
put "       ";
loop(m, put r.l(m):6:2; );put /;
```

The results are:

```
I amper
          a1      a2      a3      a4      a5      a6      a7      a8
a1      0.000   0.125   0.000   0.000   0.000   0.000   0.000  -0.125
a2     -0.125   0.000   0.125   0.000   0.000   0.000   0.000   0.000
a3      0.000  -0.125   0.000   0.750   0.000   0.000  -0.625   0.000
a4      0.000   0.000  -0.750   0.000   0.750   0.000   0.000   0.000
a5      0.000   0.000   0.000  -0.750   0.000   0.750   0.000   0.000
a6      0.000   0.000   0.000   0.000  -0.750   0.000   0.625   0.125
a7      0.000   0.000   0.625   0.000   0.000  -0.625   0.000   0.000
a8      0.125   0.000   0.000   0.000   0.000  -0.125   0.000   0.000


 U volt
          a1      a2      a3      a4      a5      a6      a7      a8
a1       0.0   -10.4    0.0     0.0     0.0     0.0     0.0     9.6
a2     -10.4     0.0  -11.6     0.0     0.0     0.0     0.0     0.0
a3       0.0   -11.6    0.0   -11.6     0.0     0.0   -11.6     0.0
a4       0.0     0.0  -11.6     0.0   -26.6     0.0     0.0     0.0
a5       0.0     0.0    0.0   -26.6     0.0    13.4     0.0     0.0
a6       0.0     0.0    0.0     0.0    13.4     0.0    13.4    13.4
a7       0.0     0.0  -11.6     0.0     0.0    13.4     0.0     0.0
a8       9.6     0.0    0.0     0.0     0.0    13.4     0.0     0.0


 E volt on supplies
          a1    a2    a3    a4    a5    a6    a7    a8
       20.00  0.00  0.00  0.00 40.00  0.00  0.00  0.00


 R - resisters
```

```
        a1      a2      a3      a4      a5      a6      a7      a8
      0.00   10.00    0.00   20.00    0.00    0.00   40.00   30.00
```

This example served as the basis for the development of the models on power networks (see S. Zaitseva, Sh. Khisoriev, and A. Savitsky, Vol. 2, Section 3.2, "Optimization of Electric Mode of Energy Systems Operation," McKinney, D.C. and A.K. Kenshimov (eds.), 2000).

7.3 Passing a Complex Electrical Signal

In the previous electricity examples we considered direct and alternating current in complex electrical circuits. However, the alternating current and voltage in these examples only varied sinusoidally. Sometimes, it is necessary to deal with circuits in which the pulses of current are not sine waves. The solution of such tasks is complicated. In this section, we solve this problem using GAMS, in the hope that it is useful in construction of more complicated models consisting of capacity, inductivity and other electronic devices and possibly microchips. First we describe the governing equations which are necessary to solve the task.

Ohm's Law. The voltage drop across a resistor is proportional to the current:

$$V_i - V_j = I_{i,j} R_{i,j}$$

where
V_i, V_j Voltage at points i and j [volt];
$I_{i,j}$ Current flowing between points i and j [amper]; and
$R_{i,j}$ Resistance between points i and j [ohm]

Kirchoff's Law. The sum of currents into or out of any point is zero:

$$\sum_j I_{i,j} = 0 \qquad \forall i \quad (i\, not\, a\, source\, of\, current)$$

Current source. For a source of current we have a difference in potentials:

$$V_i - V_j = E_{i,j}$$

where $E_{i,j}$ voltage at points i and j on either side of the source of current.

Capacitor. Charge accumulation causes a voltage increases across a capacitor inversely proportional to the capacitance:

$$\partial V = \partial Q / C = I \partial t / C$$

where
∂ Differential,
Q Charge [coul],

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$$\begin{matrix} C & \text{Capacity [farad]}, \\ t & \text{Time [sec]}, \end{matrix}$$

or

$$\frac{\partial V}{\partial t} = \frac{I}{C}$$

Here, we deal with capacitors in networks and the accumulation of charge on one plate causes accumulationof on the opposite plate and the voltages on both sides are equal. As a result of the presence of two plates in the capacitor, we may substitute a factor of 2 in the right hand side of the previous equation. However if the electrical capacity device does not contain 2 plates then the factor of 2 is not required, e.g., if there is an situation with a body capable of accumulating charge on the surface and subsequently giving it back.

Inductor. Current change produces a potential difference proportional to inductivity:

$$\frac{\partial I}{\partial t} = V_j - V_j$$

where L is inductivity [henry].

Consider the following electrical circuit
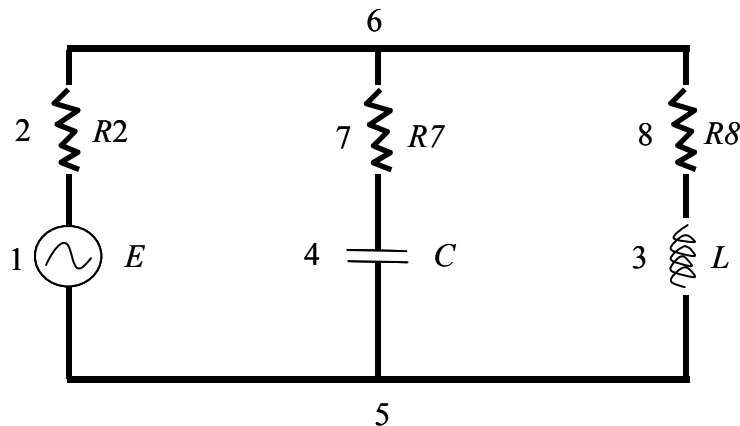
Figure 7.3.1. Simple circuit.

where

$$\begin{matrix} L & = 0.023 \text{ henry}, \\ C & = 0.000023 \text{ faraday}, \\ R_2 & = 5 \text{ ohm}, \\ R_7 & = 7 \text{ ohm}, \\ R_8 & = 11 \text{ ohm}, \\ \omega & = 18,840 \text{ hertz}, \\ E & = \sqrt{34} \sin(\omega t) \text{ volt} \end{matrix}$$

Or in complex form

$$E = 5 + 3j$$

The task can be solved by the method attributed to Gauss. We shall receive the system of equations

$$IR_2 + I_L(R_8 + j\omega L) = E$$

$$IR_2 + I_C(R_7 - \frac{j}{\omega C}) = E$$

$$I - I_C - I_L = 0$$

or in matrix form

$$\begin{bmatrix} 5 & 0 & 11 + 433.32j \\ 5 & 7 - 2307.7j & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} I \\ I_C \\ I_L \end{bmatrix} = \begin{bmatrix} 5 + 3j \\ 5 + 3j \\ 0 \end{bmatrix}$$

with solution

$$\begin{bmatrix} I \\ I_C \\ I_L \end{bmatrix} = \begin{bmatrix} 0.006 - 0.009126j \\ -0.001313 + 0.002157j \\ 0.00713 - 0.01128j \end{bmatrix}$$

or

$$\begin{bmatrix} |I| \\ |I_C| \\ |I_L| \end{bmatrix} = \begin{bmatrix} 0.01909 \\ 0.0025 \\ 0.0134 \end{bmatrix}$$

The GAMS model for this example is:

```
SETS
T / k1*k40 /
M / a1*a8  /

Parameters Volt(T), Cap(M), Ind(M);
Volt(T)  = SIN((ord(T)-1)/card(T) * 2 * 3.14)$(ord(T) gt 0);
Cap(M)   = -23/1000000000*18840*40/6.28 * 2;
Ind(M)   =  23/1000*18840*40/6.28;

Alias(M, M1);
Alias(M, M2);

Sets
ML(M)             inductivity       /a3/
MC(M)             capacity          /a4/
MR(M)             resisters         /a2, a7, a8/
ME(M)             electro supplies  /a1/
M0(M)             nodes             /a5, a6/
M_and_M1(M, M1) connection
                  /a1.a2, a1.a5, a2.a1, a2.a6, a3.a8, a3.a5,
```

```
                                a4.a7, a4.a5, a5.a4, a5.a3, a5.a1, a6.a2,
                                a6.a7, a6.a8, a7.a4, a7.a6, a8.a6, a8.a3 /
M1_M_M2(M1, M, M2);
M1_M_M2(M1, M, M2) = yes$(M_and_M1(M1, M) and M_and_M1(M, M2));
M1_M_M2(M1, M, M1)$M1_M_M2(M1, M, M1) = no;


Parameter E(M);
Parameter R(M);
E('a1') = sqrt(34);
R('a2') =   5;
R('a7') =   7;
R('a8') = 11;


VARIABLES
*           E(M),
*           R(M),
          V(M, M1, T),
          I(M, M1, T),
          OBJ;


* E.fx('a1') = sqrt(34);
* R.fx('a2') =   5;
* R.fx('a7') =   7;
* R.fx('a8') = 11;


EQUATION
          ConI(M, M1, T),
          ConU(M, M1, T),
          I_I(M, T),
          V_0(M1, M, M2, T),
          V_E(M1, M, M2, T),
          V_R(M1, M, M2, T),
          V_L(M, M1, M2, T),
          V_C(M, M1, T),
          V_C1(M1, M, M2, T),
          BEN;


I_I(M, T)..  sum(M1$M_and_M1(M, M1), I(M,M1,T)) =e= 0;
* Kirchoff law


ConI(M,M1,T)..  I(M,M1,T) =e= -I(M1,M,T);
* If there is a connection between units M and M1


ConU(M,M1,T).. V(M,M1,T) =e= V(M1,M,T);
* Potential on connection between units does not vary,
* Or, no resistance on wires connecting elements


V_E(M1,M,M2,T)$(ME(M)$(M1_M_M2(M1,M,M2)$(ord(M1) gt ord(M2))))..

             E(M)*Volt(T) =e= V(M,M1,T) - V(M,M2,T);
* Source units have a potential difference


V_0(M1,M,M2,T)$((M0(M)) and (M1_M_M2(M1, M, M2)))..

             V(M, M2, T) =e= V(M, M1, T);
* For simple units there is no power failure,
* Simple units are junctions


V_R(M1,M,M2,T)$((MR(M)) and (M1_M_M2(M1,M,M2))
                        and (ord(M2) gt ord(M1)))..
```

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```
                V(M1,M,T) - V(M2,M,T) =e= -I(M,M1,T)*R(M);
    * Ohm's Law: the power loss is proportional
    * to product of current and resistance

    V_C1(M1,M,M2,T)$((MC(M)) and (M1_M_M2(M1,M,M2)))..

                V(M1,M,T) =e= -V(M2,M,T);
    * the plates of the condenser always have
    * equal but opposite potential on the plates

    V_C(M,M1,T)$((MC(M)) and (M_and_M1(M, M1)))..

            V(M,M1,T) - V(M,M1, T -- 1) =e= + I(M,M1,T)/Cap(M);
    * Increase in potential on a plate
    * of the condenser is proportional to current

    V_L(M1,M,M2,T)$((Ml(M)) and (M1_M_M2(M1,M,M2))
                            and (ord(M2) gt ord(M1)))..

        V(M1,M,T) - V(M2,M,T) =e= -Ind(M)*(I(M,M1,T)-I(M,M1,T -- 1));
    * Differences in voltage across inductor is
    * proportional to the change in current,

    Ben.. OBJ =e= 1;

    MODEL IVAN / all /;
    SOLVE IVAN USING LP minimizing obj;
```

The results of solving this model are:

```
    Solution from GAUSS proceedure
     I Ic Il
       0.0109    0.0025    0.0134    5.3230
     Solution from GAMS
     I Ic Il
       0.0110    0.0026    0.0134    5.2353


     I amper
      a1     a2     a3     a4     a5     a6     a7     a8
    a1     0.000  0.010  0.000  0.000 -0.010  0.000  0.000  0.000
    a2    -0.010  0.000  0.000  0.000  0.000  0.010  0.000  0.000
    a3     0.000  0.000  0.000  0.000  0.013  0.000  0.000 -0.013
    a4     0.000  0.000  0.000  0.000 -0.003  0.000  0.003  0.000
    a5     0.010  0.000 -0.013  0.003  0.000  0.000  0.000  0.000
    a6     0.000 -0.010  0.000  0.000  0.000  0.000 -0.003  0.013
    a7     0.000  0.000  0.000 -0.003  0.000  0.003  0.000  0.000
    a8     0.000  0.000  0.013  0.000  0.000 -0.013  0.000  0.000


    U Volt
      a1     a2     a3     a4     a5     a6     a7     a8
    a1     0.000 -0.069  0.000  0.000  0.074  0.000  0.000  0.000
    a2    -0.069  0.000  0.000  0.000  0.000 -0.077  0.000  0.000
    a3     0.000  0.000  0.000  0.000  0.074  0.000  0.000 -0.099
    a4     0.000  0.000  0.000  0.000  0.074  0.000 -0.074  0.000
    a5     0.074  0.000  0.074  0.074  0.000  0.000  0.000  0.000
    a6     0.000 -0.077  0.000  0.000  0.000  0.000 -0.077 -0.077
    a7     0.000  0.000  0.000 -0.074  0.000 -0.077  0.000  0.000
    a8     0.000  0.000 -0.099  0.000  0.000 -0.077  0.000  0.000


    E Volt on supplies
```

```
a1     a2     a3     a4     a5     a6     a7     a8
   5.83   0.00   0.00   0.00   0.00   0.00   0.00   0.00


R - resisters
a1     a2     a3     a4     a5     a6     a7     a8
   0.00   5.00   0.00   0.00   0.00   0.00   7.00  11.00
```

Supply A1		Current A2-A6, A6-A7, A6-A8			Voltage V8-V3	V4-V7	V4-V5
Time	Voltage	Amp	Amp	Amp	Volt	Volt	Volt
1.00	0.0000	0.0108	-0.0026	0.0133	-0.1825	-0.0179	0.0179
2.00	0.9117	0.0104	-0.0025	0.0129	-0.6329	-0.4731	0.4731
3.00	1.8010	0.0098	-0.0024	0.0122	-1.0677	-0.9163	0.9163
4.00	2.6460	0.0089	-0.0023	0.0112	-1.4763	-1.3370	1.3370
5.00	3.4258	0.0078	-0.0021	0.0099	-1.8486	-1.7248	1.7248
6.00	4.1215	0.0064	-0.0019	0.0084	-2.1754	-2.0702	2.0702
7.00	4.7157	0.0050	-0.0016	0.0066	-2.4487	-2.3646	2.3646
8.00	5.1939	0.0034	-0.0013	0.0047	-2.6618	-2.6009	2.6009
9.00	5.5444	0.0017	-0.0010	0.0027	-2.8093	-2.7732	2.7732
10.00	5.7585	0.0000	-0.0006	0.0006	-2.8878	-2.8773	2.8773
11.00	5.8310	-0.0017	-0.0002	-0.0015	-2.8952	-2.9106	2.9106
12.00	5.7600	-0.0034	0.0002	-0.0036	-2.8315	-2.8723	2.8723
13.00	5.5473	-0.0050	0.0006	-0.0056	-2.6980	-2.7634	2.7634
14.00	5.1982	-0.0064	0.0010	-0.0074	-2.4982	-2.5864	2.5864
15.00	4.7212	-0.0077	0.0013	-0.0091	-2.2369	-2.3459	2.3459
16.00	4.1280	-0.0089	0.0016	-0.0105	-1.9206	-2.0476	2.0476
17.00	3.4334	-0.0098	0.0019	-0.0117	-1.5571	-1.6990	1.6990
18.00	2.6542	-0.0104	0.0022	-0.0126	-1.1553	-1.3086	1.3086
19.00	1.8098	-0.0108	0.0023	-0.0132	-0.7250	-0.8860	0.8860
20.00	0.9209	-0.0110	0.0025	-0.0134	-0.2769	-0.4416	0.4416
21.00	0.0093	-0.0108	0.0025	-0.0133	0.1780	0.0136	-0.0136
22.00	-0.9025	-0.0104	0.0025	-0.0129	0.6286	0.4686	-0.4686
23.00	-1.7921	-0.0098	0.0024	-0.0122	1.0636	0.9120	-0.9120
24.00	-2.6377	-0.0089	0.0023	-0.0112	1.4726	1.3329	-1.3329
25.00	-3.4183	-0.0078	0.0021	-0.0099	1.8453	1.7211	-1.7211
26.00	-4.1149	-0.0065	0.0019	-0.0084	2.1726	2.0670	-2.0670
27.00	-4.7102	-0.0050	0.0016	-0.0066	2.4464	2.3620	-2.3620
28.00	-5.1897	-0.0034	0.0013	-0.0047	2.6601	2.5989	-2.5989
29.00	-5.5415	-0.0018	0.0010	-0.0027	2.8084	2.7718	-2.7718
30.00	-5.7570	-0.0001	0.0006	-0.0006	2.8876	2.8766	-2.8766
31.00	-5.8309	0.0017	0.0002	0.0015	2.8957	2.9107	-2.9107
32.00	-5.7614	0.0033	-0.0002	0.0035	2.8327	2.8731	-2.8731
33.00	-5.5501	0.0049	-0.0006	0.0055	2.6999	2.7648	-2.7648
34.00	-5.2024	0.0064	-0.0010	0.0074	2.5008	2.5886	-2.5886
35.00	-4.7266	0.0077	-0.0013	0.0090	2.2401	2.3487	-2.3487
36.00	-4.1346	0.0088	-0.0016	0.0105	1.9243	2.0510	-2.0510
37.00	-3.4409	0.0097	-0.0019	0.0117	1.5612	1.7028	-1.7028
38.00	-2.6625	0.0104	-0.0022	0.0125	1.1597	1.3128	-1.3128
39.00	-1.8186	0.0108	-0.0023	0.0131	0.7297	0.8905	-0.8905
40.00	-0.9300	0.0109	-0.0025	0.0134	0.2817	0.4462	-0.4462
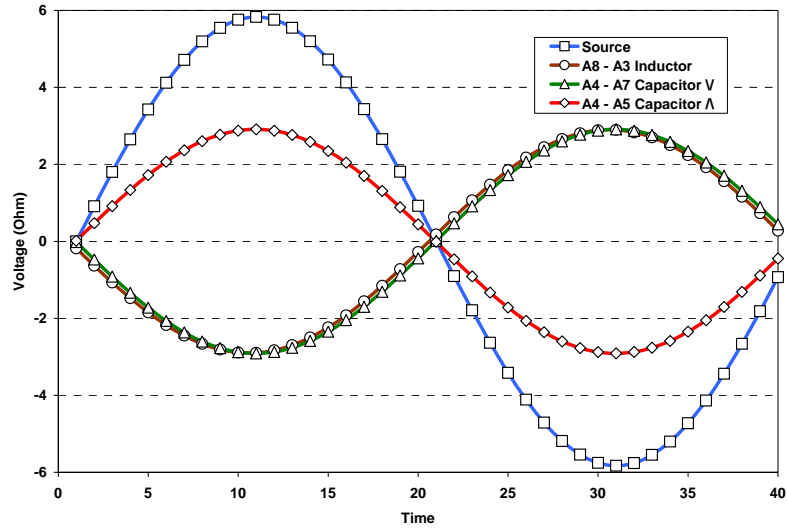
(a)
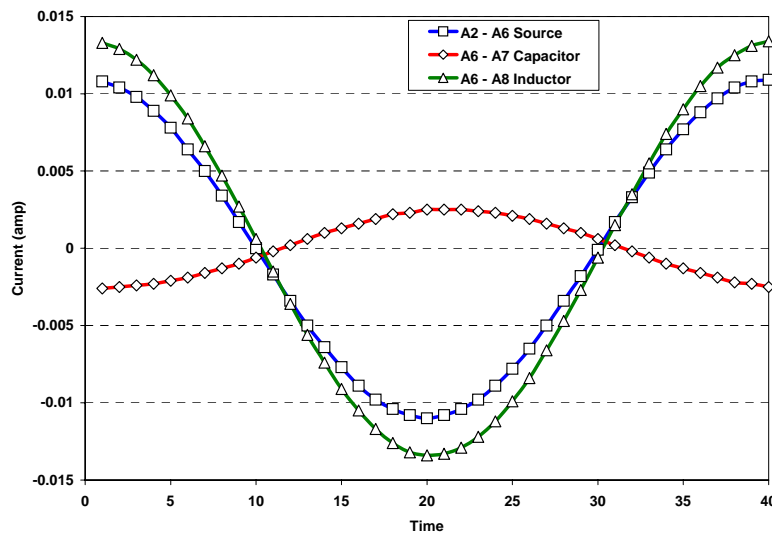
(b)

Figure 7.3.2. Results of simple circuit example: (a) voltage, (b) current.

All characteristics of the circuit can enter the model as variables. That is, the sizes and caracteristics of resistors and capacitors can be decision variables. This example is solved for a sine function input:

```
Volt(t) = SIN((ord(t)-1)/card(t)*2*3.14)$(ord(t) gt 0);
```

Actually, the model can handle a pulse of any form. Readers can change

```
(ord(t) gt 0) to (ord(t) gt 25)
```

and see how the pulse of current behaves in the circuit.

In the following example we shall show that the elementary task opens a whole area of tasks related

to electrical circuits. The equations of the resistor, the capacitor and inductor are the elementary converters of the input signal, but it is possible to include in the circuit diodes, transistors and microcircuits. For example, we shall illustrate this by adding a diode to the circuit. Diodes have the property that for current in one direction there is a resistance distinct from resistance to current in the opposite direction. As a rule these resistances differ by orders of magnitude.

To test this model we consider a "bridge" problem (Figure 3) where a sinusoidal signal is changed into a positive electricity flow.
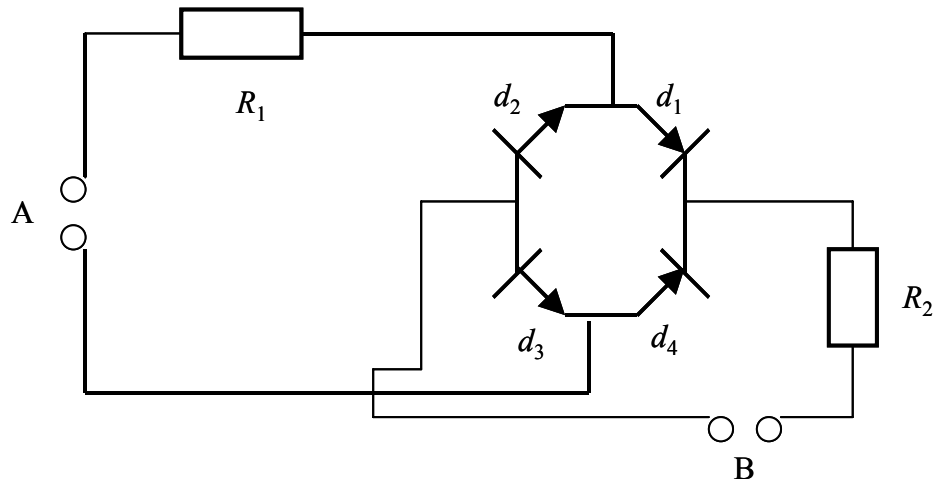

Figure 7.3.3. Bridge circuit.

From a source of alternating current, a sinusoudal wave of voltage is produced, causing a sine wave current in the circuit. Current flowing from left to right through resistor R_1 will go through diode d_1 $\rightarrow R_2 \rightarrow$ d3 and come back to the source at point A. If the current moves along the bottom from the source, it will pass through $d_4 \rightarrow R_2 \rightarrow d_1$. In the model, we must enter a new set determining the direction of current passing through diodes in the circuit. We name it (see Figure 4)

```
direct(M,M1) / a3.a8, a7.a10, a11.a8, a4.a6/
```

We must also add the equation responsible for directing the current flow in the diodes:

```
V_D_V(M1,M,M2,t)$((MD(M))and(M1_M_M2(M1,M,M2))and(direct(M,M1)))

.. V(M1,M,T) - V(M2,M,T)
   =e= 2000*(sqrt(I(M,M1,T)*I(M,M1,T)) - I(M,M1,T))/2;
```
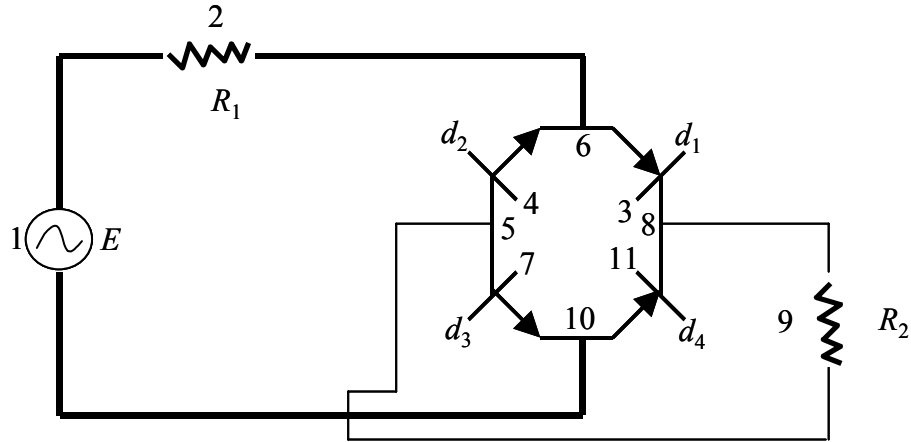
Figure 7.3.4. New circuit with diodes as bridge.

This equation has three conditions:

1. MD(M)   Condition of belonging to the group of diodes;
2. M1_M_M2(M1,M,M2)   Condition of existence of a connection between units M1 and M2
                              through unit M; and
3. direct(M,M1)   Condition that M → M1 is the direction of the diode.

The left side of the equation indicates the voltage drop across the unit. The right part is zero for positive current and double its size for negative current.

In table # the currents going through the bridge through first second branch and resistance are shown. The complete agreement of known values with the decision received is visible in GAMS. From the table it is seen that the current goes and on "closed" branch of diodes reducing efficiency the straightening "bridge"-network. By other computing methods the so thin phenomenon is difficult to compute. If the user enters something other than a sine wave signal other methods will not be able to solve the problem.

In fig. # is shown sine of a source of a current and currents in branhes of the "bridge" circuit. The reader can add to the resistor of the bridge capacity and inductance thus increasing the quality of the straightened signal. In the objective function it is possible to enter a parameter of quality of the output of the signal and to find capacity and inductance needed to achieve this quality. Tasks of thia type are very difficult to solve in other languages. The model presented here provides a great opportunity for solving these problems. Once again, we emphasize that from a number of simple examples the creation of large computing systems practical experiments is possible.

7.4 Alternating Current Circuits

The elementary model of circuits with a constant current was discussed above. It is possible to consider circuits with a sine wave current. This example was elaborate by Ivan Savitsky.

For this task we use the laws Ohm and Kirchoff. The difference is that they are written in the complex form and there are imaginary parts for current, voltage and resistance. To the usual, or active, resistance is added an implicit, or reactive, resistance. The resistance is the reaction of the circuit to the varying characteristics of the current (voltage and current).

Capacitor. The implicit resistance of a capacitor is given by

$$R_C = \frac{1}{j\omega C}$$

Inductor. Implicit resistance of an inductor is given by

$$R_L = j\omega L$$

where

   ω Frequency of fluctuations of the variable power supply;
   L, C Capacitors and inductors; and
   j Imaginary unit, $\sqrt{-1}$.

Active resistance is the real part of the complex resistance, and implicit resistance is the imaginary part. Active resistance is only the real part, and the imaginary part of this type of resistance is zero. Capacitative and inductive resistance is formed only by the imaginary part of the complex numbers.

Ohm's Law:
$$V_i - V_{i-1} = I * Z(j\omega)$$

where

   V_i complex potential on one side of an object;
   V_{i-1} complex potential on the other side of an object;
   I complex current; and
   $Z(j\omega)$ complex resistance.

Kirchoff's Law:

The sum of complex currents flowing into a unit and leaving a unit must equal zero. That is, the sum of real currents and the sum of imaginary currents on each unit must equal zero. The law reflecting invariancy and interconnection of the network as one system is

$$I_{mn} = -I_{nm}$$

The current leaving unit *m* to point *n* is the current which comes into point *n* from unit *m*. The mark (+) indicates outflow from a unit, and (-) indicates inflow to unit. For a source of current the difference of voltage is equal to the voltage of the source. For resistors, the resistance is given by Ohm's law. The voltages are not equal on each side, but the current entering is equal to that leaving. For junctions, the

voltage of all connections are equal among themselves.
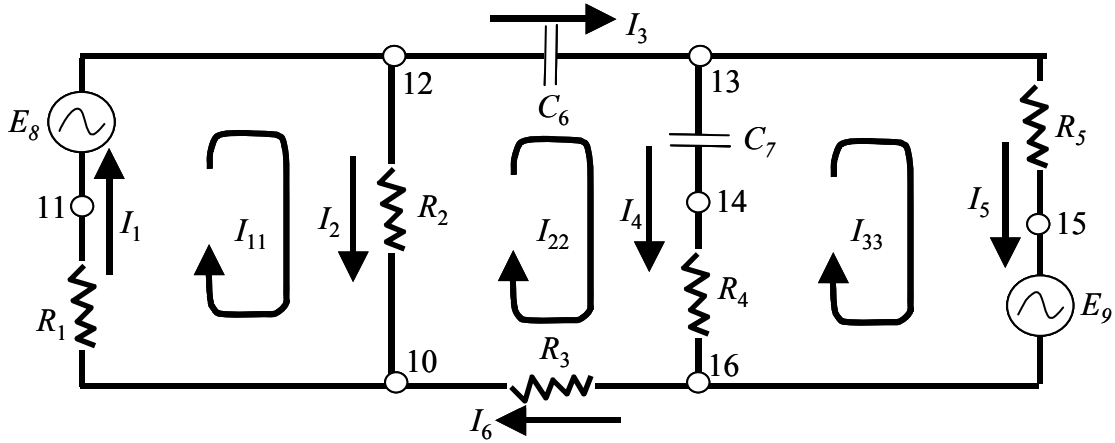
Consider the electrical circuit shown in Figure 7.4.1.

Figure 7.4.1. Electrical circuit.

In this circuit, we have the following characteristics of the components:

Sources: $E_8 = 20 + 30j$; $E_9 = 30 - 10j$ (amp);
Capacitors: $C_6 = 30$; $C_7 = 40$ (farad);
Resistors: $R_1 = 10$; $R_2 = 20$; $R_3 = 30$; $R_4 = 40$; and $R_5 = 50$ (ohm);

Let's make the equations on a method of planimetric currents for 1,2,3-circle lines:

$$I_{11}(R_1 + R_2) - I_{22}R_2 = E_8$$

$$I_{22}(R_2 + \frac{1}{j\omega C_6} + \frac{1}{j\omega C_7} + R_4 + R_3) - I_{11}R_2 - I_{33}(R_4 + \frac{1}{j\omega C_7}) = 0$$

$$I_{33}(R_4 + \frac{1}{j\omega C_7} + R_5) - I_{22}(\frac{1}{j\omega C_7} + R_4) = E_9$$

or

$$30I_{11} - 20I_{22} = 20 + 30j$$

$$-20I_{11} + (90 - 1166.66j)I_{22} - (40 - 500j)I_{33} = 0$$

$$0I_{11} - (40 - 500j)I_{22} + (90 - 500j)I_{33} = 30 - 10j$$

which can be written in the following form:

$$\begin{bmatrix} 30 & -20 & 0 \\ -20 & (90 - 1166.66j) & -(40 - 500j) \\ 0 & -(40 - 500j) & (90 - 500j) \end{bmatrix} \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} = \begin{bmatrix} 20 + 30j \\ 0 \\ 30 - 10j \end{bmatrix}$$

Solving the given system of the equations for the frequency of an alternating current with $\omega = 50$ (Hertz) gives:

$$\begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} = \begin{bmatrix} 0.6647885 + 1.04178\,jj \\ -0.002871 + 0.06267\,j \\ 0.03313 + 0.1164806\,j \end{bmatrix}$$

Then

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{11} - I_{22} \\ I_6 = I_{22} \\ I_{33} - I_{22} \\ I_{33} \end{bmatrix} \begin{bmatrix} 0.6647885 + 1.04178\,j \\ 0.6675 + 0.979\,j \\ -0.0028 + 0.0626\,j \\ 0.0359 + 0.0538\,j \\ 0.0331 + 0.1164\,j \end{bmatrix}$$

The GAMS model for this example is:

```
SETS
M /A1*A16/;

Alias(M,M1);
Alias(M,M2);

Sets
        MR(M)   resistors    /a1*a7/
        ME(M)   sources      /a8*a9/
        M0(M)   nodes        /a10*a16/
        M_to_M1(M1,M2)  connections near batery
                        /a12.a11
                         a16.a15/
        M_and_M1(M,M1)  connections
/
A1.A10, A1.A11, A2.A10, A2.A12, A3.A10, A3.A16, A4.A14, A4.A16, A5.A13
A5.A15, A6.A12, A6.A13, A7.A13, A7.A14, A8.A11, A8.A12, A9.A16, A9.A15
A10.A1, A10.A2, A10.A3, A11.A1, A11.A8, A12.A2, A12.A8, A12.A6, A13.A5
A13.A6, A13.A7, A14.A4, A14.A7, A15.A5, A15.A9, A16.A3, A16.A4, A16.A9
/

        M1_M_M2(M1,M,M2);
        M1_M_M2(M1,M,M2)  =yes$(M_and_M1(M1,M) and M_and_M1(M,M2));
        M1_M_M2(M1,M,M1)$M1_M_M2(M1,M,M1)  = no;

Variables
        E_d(M)     real supplies,
        E_k(M)     implicit supplies,
        R_d(M)     real resistance,
        R_k(M)     implicit resistance,
        V_d(M,M1)  real voltage,
        V_k(M,M1)  implicit voltage,
        I_d(M,M1)  real current,
        I_k(M,M1)  implicit current,
        OBJ;
```

```
I_d.up(M,M1)   =     1000;
I_d.lo(M,M1)   =    -1000;
V_d.up(M,M1)   =     1000;
V_d.lo(M,M1)   =    -1000;


R_d.FX('a1')= 10;
R_d.FX('a2')= 20;
R_d.FX('a3')= 30;
R_d.FX('a4')= 40;
R_d.FX('a5')= 50;
R_d.FX('a6')=  0;
R_d.FX ('a7')= 0;
R_k.FX('a1')=  0;
R_k.FX('a2')=  0;
R_k.FX('a3')=  0;
R_k.FX('a4')=  0;
R_k.FX('a5')=  0;
R_k.FX('a6')= -666.6666;
R_k.FX('a7')= -500;
E_d.FX('a8')=  20;
E_d.FX('a9')=  30;
E_k.FX('a8')=  30;
E_k.FX('a9')= -10;


Equation
        ConI_d(M,M1),
        ConV_d(M,M1),
        II_d(M),
        V_0_d(M1,M,M2),
        V_E_d(M1,M,M2),
        V_R_d(M1,M,M2),
        ConI_k(M,M1),
        ConV_k(M,M1),
        II_k(M),
        V_0_k(M1,M,M2),
        V_E_k(M1,M,M2),
        V_R_k(M1,M,M2),
        BEN;
*
II_d(M)..        sum(m1$m_and_m1(M,M1),I_d(M,M1)) =e= 0;
ConI_d(M,M1)..  I_d(M,M1) =e= -I_d(M1,M);
ConV_d(M,M1)..  V_d(M,M1) =e=  V_d(M1,M);

V_0_d(M1,M,M2)$m0(M)..

    0 =e= (V_d(M,M1)-V_d(M,M2))$M1_M_M2(M1,M,M2);

V_E_d(M1,M,M2)$(ME(M)$(M1_M_M2(M1,M,M2)$(M_to_M1(M1,M2))))..

    V_d(M,M1)-V_d(M,M2) =e= e_d(M);

V_R_d(M1,M,M2)$(MR(M)$M1_M_M2(M1,M,M2))..

    V_d(M,M1)-V_d(M,M2) =e= -(R_d(M)*I_d(M,M1)-R_k(M)*I_k(M,M1));

*
II_k(M)..        sum(m1$m_and_m1(M,M1),I_k(M,M1)) =e= 0;
ConI_k(M,M1)..  I_k(M,M1) =e= -I_k(M1,M)  ;
ConV_k(M,M1)..  V_k(M,M1) =e=  V_k(M1,M)  ;
```

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```
V_0_k(M1,M,M2)$M0(M)..

     0 =e= (V_k(M,M1)-V_k(M,M2)) $M1_M_M2(M1,M,M2);

V_E_k(M1,M,M2)$(ME(M)$(M1_M_M2(M1,M,M2)$(M_to_M1(M1,M2))))..

     V_k(M,M1)-V_k(M,M2) =e= e_k(M);

V_R_k(M1,M,M2)$(MR(M) $M1_M_M2(M1,M,M2))..

     V_k(M,M1)-V_k(M,M2) =e= -(R_d(M)*I_k(M,M1)+R_k(M)*I_d(M,M1));
*
BEN..      OBJ =E=1;

MODEL VAN /ALL/;
Option nlp = Minos5;
SOLVE VAN USING NLP maximizing OBJ;
```

The results given in Tables 1 and 2 show that the GAMS model produces the same result as the Gauss method solution.

Table 1. Current I (amp, real part)

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16
A1	0	0	0	0	0	0	0	0	0	-0.66	0.66	0	0	0	0	0
A2	0	0	0	0	0	0	0	0	0	0.67	0	-0.67	0	0	0	0
A3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A4	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0	-0.04
A5	0	0	0	0	0	0	0	0	0	0	0	0	-0.03	0	0.03	0
A6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A7	0	0	0	0	0	0	0	0	0	0	0	0	0.04	-0.04	0	0
A8	0	0	0	0	0	0	0	0	0	0	-0.66	0.66	0	0	0	0
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.03	0.03
A10	0.66	-0.67	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A11	-0.66	0	0	0	0	0	0	0.66	0	0	0	0	0	0	0	0
A12	0	0.67	0	0	0	0	0	-0.66	0	0	0	0	0	0	0	0
A13	0	0	0	0	0.03	0	-0.04	0	0	0	0	0	0	0	0	0
A14	0	0	0	-0.04	0	0	0.04	0	0	0	0	0	0	0	0	0
A15	0	0	0	0	-0.03	0	0	0	0.03	0	0	0	0	0	0	0
A16	0	0	0	0.04	0	0	0	0	-0.03	0	0	0	0	0	0	0

Current I (amp, implicit part)

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16
A1	0	0	0	0	0	0	0	0	0	-1.04	1.04	0	0	0	0	0
A2	0	0	0	0	0	0	0	0	0	0.98	0	-0.98	0	0	0	0
A3	0	0	0	0	0	0	0	0	0	0.06	0	0	0	0	0	-0.06
A4	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0	-0.05
A5	0	0	0	0	0	0	0	0	0	0	0	0	-0.12	0	0.12	0
A6	0	0	0	0	0	0	0	0	0	0	0	-0.06	0.06	0	0	0
A7	0	0	0	0	0	0	0	0	0	0	0	0	0.05	-0.05	0	0
A8	0	0	0	0	0	0	0	0	0	0	-1.04	1.04	0	0	0	0
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.12	0.12
A10	1.04	-0.98	-0.06	0	0	0	0	0	0	0	0	0	0	0	0	0
A11	-1.04	0	0	0	0	0	0	1.04	0	0	0	0	0	0	0	0
A12	0	0.98	0	0	0	0.06	0	-1.04	0	0	0	0	0	0	0	0
A13	0	0	0	0	0.12	-0.06	-0.05	0	0	0	0	0	0	0	0	0
A14	0	0	0	-0.05	0	0	0.05	0	0	0	0	0	0	0	0	0
A15	0	0	0	0	-0.12	0	0	0	0.12	0	0	0	0	0	0	0
A16	0	0	0.06	0.05	0	0	0	0	-0.12	0	0	0	0	0	0	0

Table 2. Voltage V (volt, real part)

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16
A1	0	0	0	0	0	0	0	0	0	0.08	-6.56	0	0	0	0	0
A2	0	0	0	0	0	0	0	0	0	0.08	0	13.44	0	0	0	0
A3	0	0	0	0	0	0	0	0	0	0.08	0	0	0	0	0	0
A4	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.44	0	0
A5	0	0	0	0	0	0	0	0	0	0	0	0	-28.34	0	-30	0
A6	0	0	0	0	0	0	0	0	0	0	0	13.44	-28.34	0	0	0
A7	0	0	0	0	0	0	0	0	0	0	0	0	-28.34	-1.44	0	0
A8	0	0	0	0	0	0	0	0	0	0	-6.56	13.44	0	0	0	0
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-30	0
A10	0.08	0.08	0.08	0	0	0	0	0	0	0	0	0	0	0	0	0
A11	-6.56	0	0	0	0	0	0	-6.56	0	0	0	0	0	0	0	0
A12	0	13.44	0	0	0	13.44	0	13.44	0	0	0	0	0	0	0	0
A13	0	0	0	0	-28.34	-28.34	-28.34	0	0	0	0	0	0	0	0	0
A14	0	0	0	-1.44	0	0	-1.44	0	0	0	0	0	0	0	0	0
A15	0	0	0	0	-30	0	0	0	-30	0	0	0	0	0	0	0
A16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Voltage V (volt, implicit part)

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15	A16
A1	0	0	0	0	0	0	0	0	0	-11.88	-22.3	0	0	0	0	0
A2	0	0	0	0	0	0	0	0	0	-11.88	0	7.7	0	0	0	0
A3	0	0	0	0	0	0	0	0	0	-11.88	0	0	0	0	0	-10
A4	0	0	0	0	0	0	0	0	0	0	0	0	0	-12.15	0	-10
A5	0	0	0	0	0	0	0	0	0	0	0	0	5.82	0	0	0
A6	0	0	0	0	0	0	0	0	0	0	0	7.7	5.82	0	0	0
A7	0	0	0	0	0	0	0	0	0	0	0	0	5.82	-12.15	0	0
A8	0	0	0	0	0	0	0	0	0	0	-22.3	7.7	0	0	0	0
A9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-10
A10	-11.88	-11.88	-11.88	0	0	0	0	0	0	0	0	0	0	0	0	0
A11	-22.3	0	0	0	0	0	0	-22.3	0	0	0	0	0	0	0	0
A12	0	7.7	0	0	0	7.7	0	7.7	0	0	0	0	0	0	0	0
A13	0	0	0	0	5.82	5.82	5.82	0	0	0	0	0	0	0	0	0
A14	0	0	0	-12.15	0	0	-12.15	0	0	0	0	0	0	0	0	0
A15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A16	0	0	-10	-10	0	0	0	0	-10	0	0	0	0	0	0	0

8. Optimal Solution of Heat Tranfer Problems

Many tasks, e.g., river and electrical networks, are obviously two-dimensional, but they were represented by one-dimensional systems as analogues in the equations. In this chapter we consider tasks which are not one-dimensional and some which are time dependent. For each example, along with the solution, we shall consider some additional options and operators of GAMS.

8.1 Background

The basic equation for the task is conservation of energy, as follows;

$$\frac{\partial(c\rho T)}{\partial t} = \nabla \cdot [V\nabla(c\rho T)] + I$$

where

t time (sec),
c specific heat capacity of substance (cal/(kg - degree),
p density of substance of (kg/m^3),
T temperature of substance (degree),
V thermal conductivity (m^2/sec),
I point source of heat (cal/(m^{3}sec).

Consider the task of solving this problem in a rectangular area in which there is a source of heat in the lower right-hand corner. On the boundary of the area the temperature is fixed

$$T = T_0(x, y) \qquad \text{for} \qquad (x, y)\, \text{on the boundary}$$

If the situation is steady state, then no characteristics of the problem change with time. In this example, the thermal conductivity V is heterogeneous in the solution domain (see Figure 8.1). There is a symmetric zone at the center of the solution area where V becomes larger than in the rest of the area. In view of these assumptions the governing equation becomes

$$\nabla \cdot [V\nabla(c\rho T)] + I = 0$$

If c and ρ do not vary spatially, this equation becomes

$$\nabla \cdot (V\nabla T) + \frac{I}{c\rho} = 0$$
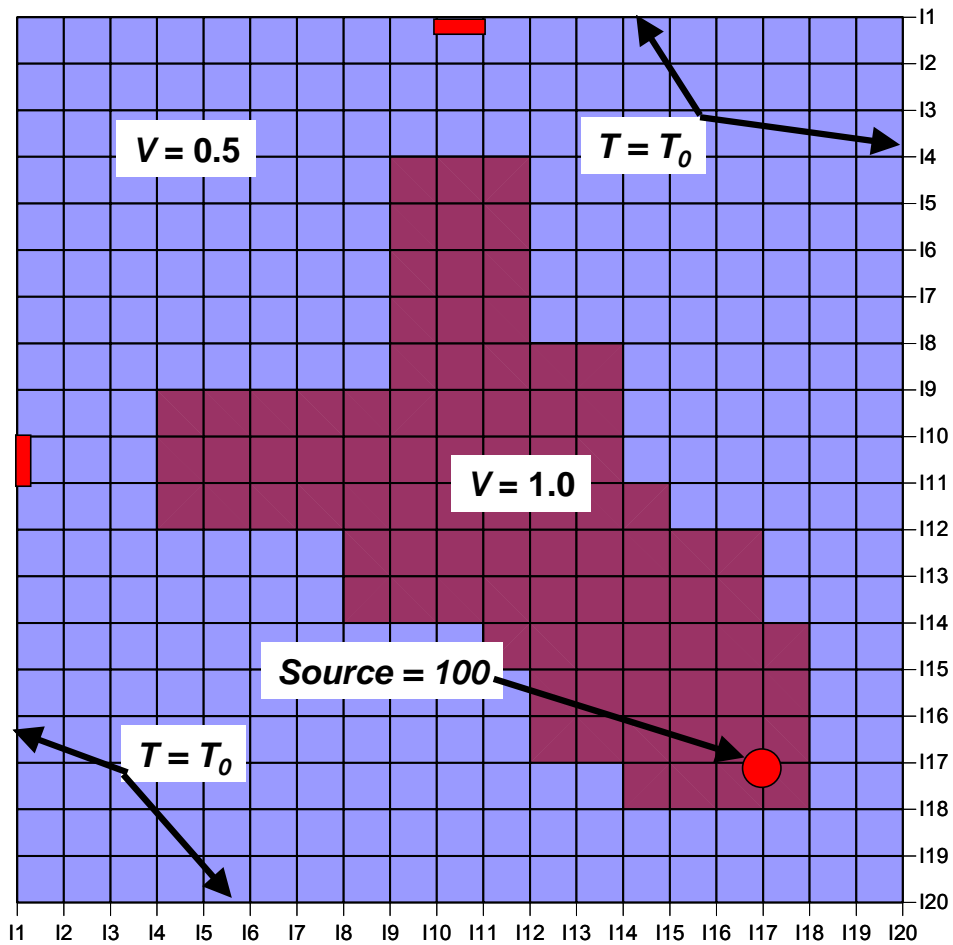
Figure 8.1.1. Solution domain for heat transfer problem.

This equation is difficult to solve due to the heterogeneity of the thermal conductivity (as opposed to $V\nabla \cdot \nabla T + I/c\rho = 0$). Also, the transient equation

$$\nabla \cdot (V\nabla T) + \frac{I}{c\rho} = \frac{\partial T}{\partial t}$$

is much easier and more commonly solved. A stationary solution to the transient equation is found by using a large number of iterations under stationary boundary conditions. After many iterations the influence of the initial conditions becomes very small and $\partial T/\partial t \to 0$. In any case, the differential operators must be approximated by algebraic analogues and a conservative scheme of solving the resulting system of algebraic equations must be used.

In this example we consider construction of the analog on a grid in more detail. In subsequent eamples we will not concentrate on this aspect as much. First, we subdivide the rectangular area into some number of small rectangles with sides parallel to the x and y axes. The result is a number of

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intersection points refered to as nodes (see Figure 1 where a system of 20x20 nodes is illustrated). This system of rectangles allows to us construct algebraic analogs for the differential operators $\nabla \cdot ()$ and $\nabla ()$.

The basic problem of constructing algebraic analogues of the differential operators is the preservation of invariancy. That is, the solution must not depend on the orientation of the coordinate axes, and symmetric areas under symmetric boundary conditions should result in symmetric solutions.

One of the variants of the differential operator in the previous equation can be derived using the well known formula from vector analysis

$$\nabla \cdot (a\nabla B) = a\nabla \cdot (\nabla B) + \nabla a \cdot \nabla B$$

which gives us the new equation:

$$V\nabla \cdot (\nabla T) + \nabla T \cdot \nabla V + \frac{I}{c\rho} = 0$$

Or using a finite difference approximation

$$V_{i,j} \frac{(T_{i+1,j} - 2T_{i,j} - T_{i-1,j})}{(\Delta x)^2} + V_{i,j} \frac{(T_{i,j+1} - 2T_{i,j} - T_{i,j-1})}{(\Delta y)^2}$$

$$\frac{(V_{i+1,j} - V_{i-1,j})}{2(\Delta x)} \frac{(T_{i+1,j} - T_{i-1,j})}{2(\Delta x)} + \frac{(V_{i,j+1} - V_{i,j-1})}{2(\Delta y)} \frac{(T_{i,j+1} - T_{i,j-1})}{2(\Delta y)}$$

$$+ \frac{I_{i,j}}{c\rho} = 0$$

8.2 Stationary temperature field in a rectangular area

Consider a stationary temperature field in a rectangular area with heterogeneous thermal conductivity and a point source of heat and fixed temperature on the boundaries of the solution domain.

```
SET X /I1*I20/;
SET Y /J1*J20/;

* boundary location determination
SET bound(X,Y);
    bound(X,Y)        =yes;
    bound('I1','J10') =no;
    bound('I20','J10')=no;
```

```
SET inside(X,Y);
    inside(X,Y)                    =yes;
    inside(X,Y)$(ord(X)=1)         =no;
    inside(X,Y)$(ord(X)=card(X)) =no;
    inside(X,Y)$(ord(Y)=1)         =no;
    inside(X,Y)$(ord(Y)=card(Y)) =no;


* temperature supply determination
PARAMETER SUPPLY(X,Y);
          SUPPLY(X,Y)           := 0;
          SUPPLY('I17','J17')  := 10000;


* parameters determination
SCALAR dx /0.1/;
SCALAR dy /0.1/;


TABLE V(X,Y)
     J1   J2   J3   J4   J5   J6   J7   J8   J9   J10  J11  J12  J13  J14  J15  J16  J17  J18  J19  J20
I1   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I2   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I3   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I4   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I5   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I6   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I7   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I8   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I9   0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I10  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I11  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I12  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5
I13  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5
I14  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I15  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5
I16  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I17  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I18  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I19  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I20  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5


*        MODEL   DESCRIPTION
VARIABLES
      obj
      T(X,Y);


******* Variables limits *********
T.lo(X,Y) =-100.0;
T.up(X,Y) = 200;
T.l(X,Y)  = 0;


******* Boundary Conditions ******
T.fx(X,Y)$(ord(X)=1)         = 0.0;
T.fx(X,Y)$(ord(X)=card(X)) = 0.0;
T.fx(X,Y)$(ord(Y)=1)         = 0.0;
T.fx(X,Y)$(ord(Y)=card(Y)) = 0.0;


T.fx('I1','J10')  = 100;
T.fx('I1','J11')  = 100;
T.fx('I10','J1')  = 100;
T.fx('I11','J1')  = 100;


EQUATION
      TEMP(X,Y)
      ben;

TEMP(X,Y)$(inside(X,Y)).. supply(X,Y)
        + V(X,Y)*( T(X-1,Y)-2*T(X,Y)+T(X+1,Y))/dx/dx
        +  V(X,Y)*( T(X,Y-1)-2*T(X,Y)+T(X,Y+1))/dy/dy
        + (V(X,Y+1)-V(X,Y-1))* (T(X,Y+1)-T(X,Y-1))/dy/dy/2.0
```

```
                + (V(X+1,Y)-V(X-1,Y))* (T(X+1,Y)-T(X-1,Y))/dx/dx/2.0
                =e= 0;

        ben.. obj =e= sum(X,T(X,'J2'));

        **********************************
        MODEL temper /all/;
        SOLVE temper using nlp minimazing obj;

        **************************************************************
        file res1 /heat_2.txt/
        file res2 /heat_2.dat/
        put res1;
        put obj.l; put /;
        loop(X,  put X.tl:6;loop(Y, put T.l(X,Y):6:2; );put /;);put /;
                 put " inside 20   " /;
         loop(X,  put X.tl:6;loop(Y, put inside(X,Y):6; );put /;);
                 put " inside 20   " /;
         loop(X,  put X.tl:6;loop(Y, put supply(X,Y):6; );put /;);
        put res2;
        loop(X,loop(Y, put T.l(X,Y):4:0; );put /;);put /;
```

The solution of the model is:

```
  0    0    0    0    0    0    0    0    0 100  100    0    0    0    0    0    0    0    0    0
  0    1    1    2    3    4    6   11   21   51   51   21   11    6    4    3    2    1    1    0
  0    1    3    4    6    8   11   15   22   32   32   22   15   10    7    5    4    2    1    0
  0    2    4    6    8   11   13   17   21   23   23   21   17   13   10    8    5    4    2    0
  0    3    6    8   11   13   15   18   21   22   22   21   18   15   12    9    7    5    2    0
  0    4    8   11   13   15   17   19   21   22   22   21   19   16   14   11    8    5    3    0
  0    6   11   13   15   17   19   20   22   22   22   22   20   18   15   12   10    6    3    0
  0   11   15   17   18   19   20   21   22   22   23   23   22   20   17   14   11    8    4    0
  0   21   22   21   21   21   22   22   23   23   23   23   23   22   19   16   13    9    5    0
100   51   32   23   22   22   22   22   23   23   24   24   24   24   22   19   15   11    5    0
100   51   32   23   22   22   22   23   23   24   25   25   26   27   25   22   18   13    7    0
  0   21   22   21   21   21   22   23   23   24   25   26   28   29   29   27   22   16    8    0
  0   11   15   17   18   19   20   22   23   24   26   28   29   31   33   33   28   21   11    0
  0    6   10   13   15   16   18   20   22   24   27   29   31   33   36   37   37   27   14    0
  0    4    7   10   12   14   15   17   19   22   25   29   33   36   39   42   44   37   18    0
  0    3    5    8    9   11   12   14   16   19   22   27   33   37   42   48   54   45   21    0
  0    2    4    5    7    8   10   11   13   15   18   22   28   37   44   54   78   52   22    0
  0    1    2    4    5    5    6    8    9   11   13   16   21   27   37   45   52   34   16    0
  0    1    1    2    2    3    3    4    5    5    7    8   11   14   18   21   22   16    8    0
  0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0    0
```

The symmetry of the problem and the boundary conditions results in the symmetric solution. The
solution of the model is written to the file "*.dat", which serves as an input file for the program
which graphically displays the solution (see Figure 8.2.1). In this figure all isolines of temperature
are symmetric, a necessary condition of the solution.
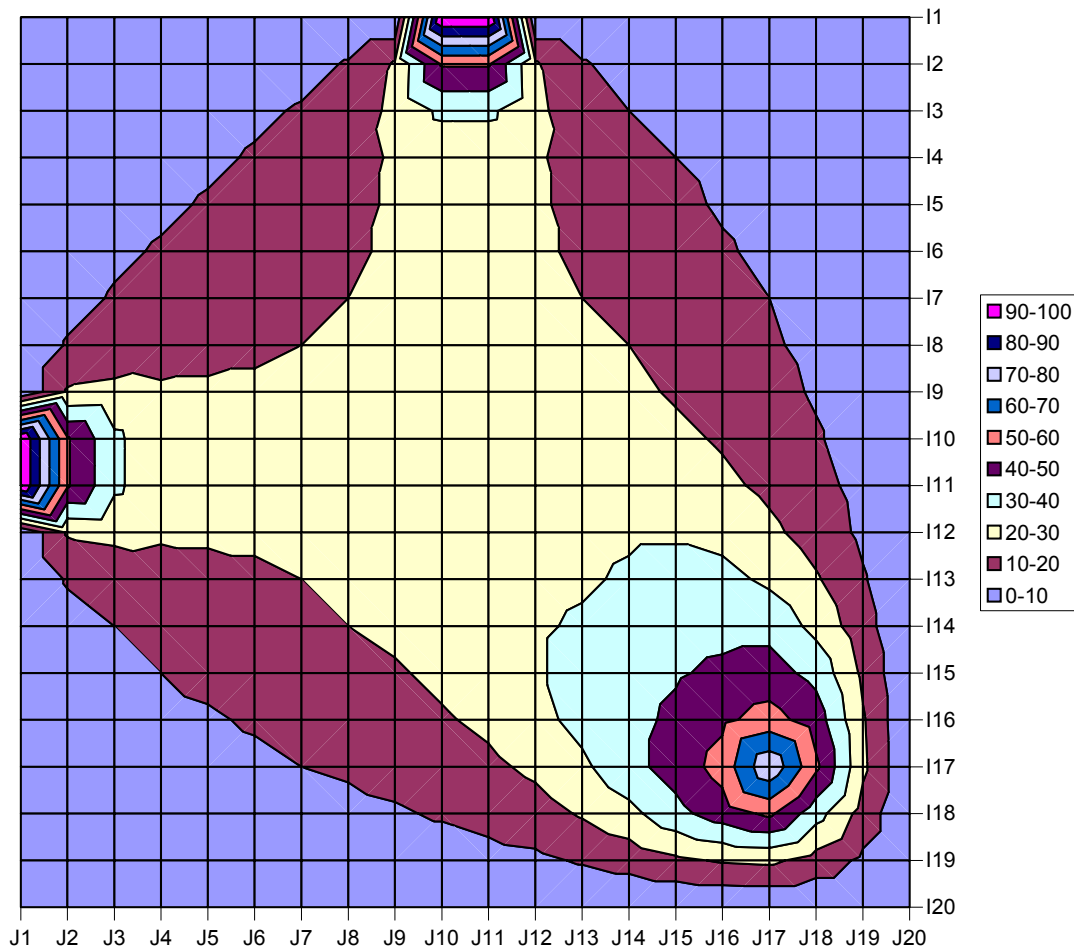
Figure 8.2.1. Solution to heat transport problem with fixed boundary conditions.

Next, we consider the possibility of finding a solution for other types of boundary conditions. Consider a problem where heat flow, not temperature, is given on the borders of the solution area, e.g., if the borders are made from insulating material. In this case

$$\frac{\partial T(x, y)}{\partial n} = 0 \quad on\, the\, boundary$$

where n is the unit normal vector to border of the solution area. The governing equation is

$$\nabla \cdot (V \nabla T) + \frac{I}{c\rho} = 0$$

Several difficulties present themselves immediately in this case:

1) The previous algebraic analogue of the differential equation is not conservative;

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2) The existence of insulating boundaries contradicts the heat conductivity equation; and
3) The internal source of heat should balance the flow of heat through the borders.

That is, a stationary solution can not exist within insulated borders around an area containing an internal source of heat. The temperature in the region should constantly grow and its stabilization will not occur. To solve this problem it is necessary:

1) To reconstruct algebraic analog of the differential operators;
2) To alter the objective function; and
3) To make a boundary condition which is more realistic.

The boundary condition is modified as

$$\frac{\partial T(x, y)}{\partial n} = s \quad on\,the\,boundary$$

where s is a given heat flux on the boundary.

Assume that outside the solution domain the environment has large thermal conductivity. As a result of this, at any point on the border there will be a larger temperature than that of the nearby external points, this will result in heat currents in the external area moving parallel to the borders. From the external area there will be sufficient hear capacity to maintain the temperature on the border.

Let's copy the algebraic approximation of the differential operators in the conservative form. The algebraic approximation of the conservative form allows no opportunity for sources or sinks of heat resulting from the apoximation of the differential operators. Thus, considering the governing equation

$$\nabla \cdot (V \nabla T) + \frac{I}{c\rho} = 0$$

or, in its algrbraic representation

$$\frac{(V_{i+1,j} + V_{i,j})(T_{i+1,j} - T_{i,j}) - (V_{i,j} + V_{i-1,j})(T_{i,j} - T_{i-1,j})}{(\Delta x)^2} +$$

$$\frac{(V_{i,j+1} + V_{i,j})(T_{i,j+1} - T_{i,j}) - (V_{i,j-1} + V_{i,j})(T_{i,j} - T_{i,j-1})}{(\Delta y)^2}$$

$$+ \frac{I_{i,j}}{c\rho} = 0$$

The given temperature should be minimized during the search for the solution

$$\text{Minimize } T_0(x, y)$$

To guarantee conservation of energy in the solution it is necessary to calculate heat flows through the borders of the solution domain and to compare them with the source of heat inside the domain. Conservative methods for solving the differential equation should conserve the source of heat inside the solution domain by heat flow through the borders of the area. In the model, inequalities enter the solution when the flow of heat is directed from the solution domain to the external environment.

8.3 Stationary Temperature Field in a Rectangular Area with Border Heat Flow

Consider a ttationary temperature field in a rectangular area with heterogeneous thermal conductivity and a point source of heat and heat flows through the borders of the solution domain.

```
SET X /I1*I20/;

ALIAS (X,Y);
****************************************************
*    Determine boundary location
*
*    Determine type of boundary conditions

SET bound(X,Y);
    bound(X,Y)      =yes;
    bound('I1','I10')=no;
    bound('I1','I11')=no;
    bound('I10','I1')=no;
    bound('I11','I1')=no;


*    Determine zone for heat transport equation
SET inside(X,Y);
 inside(X,Y)                   =yes;
 inside(X,Y)$(ord(X)=1)        =no;
 inside(X,Y)$(ord(X)=card(X)) =no;
 inside(X,Y)$(ord(Y)=1)        =no;
 inside(X,Y)$(ord(Y)=card(Y)) =no;


*    Determine parameters
SCALAR dx     step  in X direction  /0.1/
       dy     step  in Y direction  /0.1/ ;

*    Determine heat supply
PARAMETER SUPPLY(X,Y);
          SUPPLY(X,Y)        :=0;
          SUPPLY('I17','I17'):=100/dx/dy;
```

TABLE V(X,Y)   thermal conductivity distribution

	I1	I2	I3	I4	I5	I6	I7	I8	I9	I10	I11	I12	I13	I14	I15	I16	I17	I18	I19	I20
I1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I2	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I3	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I4	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1.0	1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I6	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1.0	1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I7	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1.0	1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I8	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1.0	1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
I9	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1.0	1.0	1.0	1.0	0.5	0.5	0.5	0.5	0.5	0.5	0.5

```
I10 0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I11 0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I12 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5
I13 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5  0.5  0.5
I14 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I15 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I16 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I17 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  1.0  1.0  1.0  0.5  0.5  0.5  0.5
I18 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I19 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I20 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5 ;

*******************************
*       MODEL   DESCRIPTION
*******************************


VARIABLES
        obj
        Q
        T(X,Y);

******* limitation *********
T.lo(X,Y)  =-100.0;
T.up(X,Y)  = 200;
T.l(X,Y)   = 0;
Q.lo  = 0;

******* boundary conditions ******
T.fx('I1','I10')  = 100;
T.fx('I1','I11')  = 100;
T.fx('I10','I1')  = 100;
T.fx('I11','I1')  = 100;

EQUATION
        TEMP(X,Y)   main equation

        f1(X,Y)     boundary condition dt:dn
        f2(X,Y)     boundary condition dt:dn
        f3(X,Y)     boundary condition dt:dn
        f4(X,Y)     boundary condition dt:dn

        fp1(X,Y)    boundary condition t
        fp2(X,Y)    boundary condition t
        fp3(X,Y)    boundary condition t
        fp4(X,Y)    boundary condition t

        ben         benefit equation
        ;

***************main equation********************************
TEMP(X,Y)$(inside(X,Y))..

          (( V(X+1,Y)+V(X,Y))*(T(X+1,Y)-T(X,Y))
          -(  V(X,Y)+V(X-1,Y))*(T(X,Y)-T(X-1,Y)))/dx/dx/2.0
          +(( V(X,Y+1)+V(X,Y))*(T(X,Y+1)-T(X,Y))
          -(  V(X,Y-1)+V(X,Y))*(T(X,Y)-T(X,Y-1)))/dy/dy/2.0
          +   supply(X,Y)
          =e= 0.0
           ;

***************equation which determines dT/dn********
f1(X,Y)$((ord(X)=1       $bound(X,Y))).. T(X+1,Y)-T(X,Y)   =G= 0;
f2(X,Y)$((ord(X)=card(X)$bound(X,Y))).. T(X,Y)  -T(X-1,Y) =L= 0;
f3(X,Y)$((ord(Y)=1       $bound(X,Y))).. T(X,Y+1)-T(X,Y)   =G= 0;
```

```
f4(X,Y)$((ord(Y)=card(Y)$bound(X,Y)))..  T(X,Y)  -T(X,Y-1) =L= 0;

*************equation which determines T on outside bndy******
fp1(X,Y)$((ord(X)=1       $bound(X,Y)))..  T(X,Y) =e= Q;
fp2(X,Y)$((ord(X)=card(X)$bound(X,Y)))..  T(X,Y) =e= Q;
fp3(X,Y)$((ord(Y)=1       $bound(X,Y)))..  T(X,Y) =e= Q;
fp4(X,Y)$((ord(Y)=card(Y)$bound(X,Y)))..  T(X,Y) =e= Q;

*************************************************************
ben.. obj=e=Q;

*********************************
MODEL heat4 /all/;
SOLVE heat4 using nlp minimizing obj;

*************************************************************
FILE res1 /heat_4.txt/
FILE res2 /heat_4.dat/

*************************************************************
*      You can see the flow through the boundaries
*      from equations in the model

PARAMETER
         x_left(X), x_right(X),
         y_top(Y),  y_bottom(Y),
         total,
         total_x1, total_x2,
         total_y1, total_y2,
         t_supply;

x_left(X)   =  (T.l('I2',X)  -T.l('I1',X)  )*(V('I1',X) +V('I2',X) )/dx/2;
x_right(X)  = -(T.l('I20',X) -T.l('I19',X))*(V('I20',X)+V('I19',X))/dx/2;
y_top(Y)    =  (T.l(Y,'I2')  -T.l(Y,'I1') )*(V(Y,'I1') +V(Y,'I2') )/dy/2;
y_bottom(Y) = -(T.l(Y,'I20') -T.l(Y,'I19'))*(V(Y,'I20')+V(Y,'I19'))/dy/2;

total_x1 = sum(X,x_left(X))*dy;
total_x2 = sum(X,x_right(X))*dy;
total_y1 = sum(Y,y_top(Y))*dx;
total_y2 = sum(Y,y_bottom(Y))*dx;
total    = sum(Y,y_top(Y))*dx  + sum(Y,y_bottom(Y))*dx
           +sum(X,x_left(X))*dy + sum(X,x_right(X))*dy;
t_supply = supply('I17','I17')*dx*dy;
```

The results are plotted in the following figure.
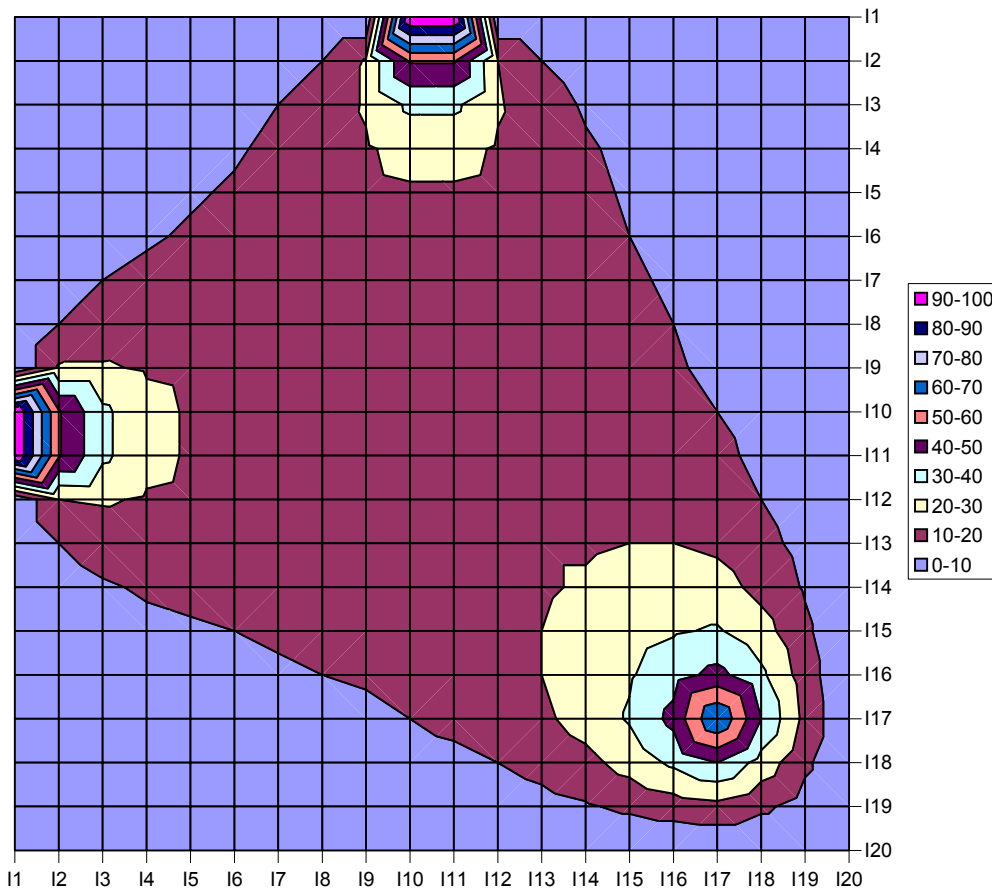
Figure 8.3.1. Solution of the heat transport problem minimizing the boundary temperature.

Part of the results file are

```
obj function                        =          0.00
 temperature on boundary            =          0.00
 inside heat supply                 =        100.00
 total heat transport               =        100.00
 total heat transport x1 boundary   =         -2.98
 total heat transport x2 boundary   =         52.98
 total heat transport y1 boundary   =         -2.98
 total heat transport y2 boundary   =         52.98
heat transport   left    direction
  0.0    3.3    6.6   10.0   14.1   19.9   29.9   50.9 102.9 -246.4 -246.6
102.2   49.8   28.4   18.0   12.0    8.1    5.0    2.4    0.0
heat transport    top     direction
  0.0    3.3    6.6   10.0   14.1   19.9   29.9   50.9 102.9 -246.4 -246.6
102.2   49.8   28.4   18.0   12.0    8.1    5.0    2.4    0.0
heat transport   right   direction
  0.0    2.4    4.7    6.6    8.3    9.8   11.2   12.9   14.9   17.6   21.3
 26.7   34.5   45.4   60.3   76.2   84.6   61.6   30.8    0.0
 heat transport  botton direction
  0.0    2.4    4.7    6.6    8.3    9.8   11.2   12.9   14.9   17.6   21.3
 26.7   34.5   45.4   60.3   76.2   84.6   61.6   30.8    0.0
```

The symmetry of the results, is due to the invariancy of the equation, its algebraic analogue, the

symmetry of the solution domain, the field of conductivity and the symmetry of the boundary conditions. The equality of the internal source of heat and the total flow of heat through borders of the area demonstrate the conservatism of the method.

If we change the problem from one of minimization to one of maximization, we will receive a new solution strongly distinguished from the previous one. Part of that solution file is reproduced below and the temperature solution is shown in Figure 8.3.2.

In this case the temperature of external environment has appeared equal 103.38. But in this case, through all borders the thermal flow is directed to the outside. In the previous case on two sides we saw two powerful concentrated flows of heat directed into the area and compensatory currents of heat on other sides outside. In both cases, the flow of heat is balanced with the internal source of heat.
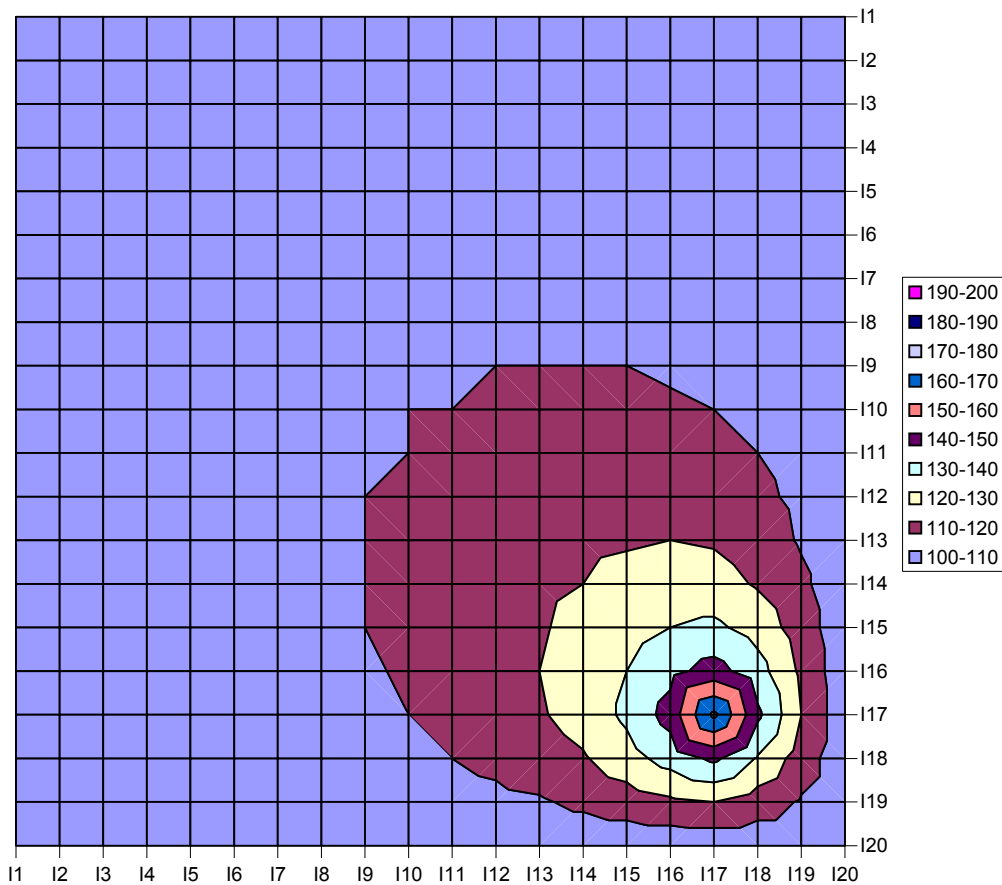
Figure 8.3.2. Solution of the heat transport problem maximizing the boundary temperature.

```
obj function                        =        103.38
temperature on boundary             =        103.38
inside heat supply                  =        100.00
total heat transport                =        100.00
total heat transport x1 boundary    =          4.53
total heat transport x2 boundary    =         45.47
```

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```
 total heat transport y1 boundary  =          4.53
 total heat transport y2 boundary  =         45.47
 heat transport left direction
  0.0   0.3   0.7   1.0   1.3   1.5   1.6   1.4   0.0
 12.1  12.3   0.4   2.0   2.4   2.4   2.2   1.8   1.2   0.6   0.0
 heat transport top direction
  0.0   0.3   0.7   1.0   1.3   1.5   1.6   1.4   0.0
 12.1  12.3   0.4   2.0   2.4   2.4   2.2   1.8   1.2   0.6   0.0
 heat transport right direction
  0.0   0.6   1.3   2.1   2.9   4.0   5.3   7.0   9.1
 12.1  16.2  21.9  29.9  41.3  56.4  72.8  82.0  59.9  29.9   0.0
 heat transport botton direction
  0.0   0.6   1.3   2.1   2.9   4.0   5.3   7.0   9.1  12.1
 16.2  21.9  29.9  41.3  56.4  72.8  82.0  59.9  29.9   0.0
```

8.4 Time Dependent Temperature Field

Consider a time dependent temperature field in a rectangular area with heterogeneous thermal conductivity and a point source of heat and heat flows through the borders of the solution domain.

In GAMS we can search for the solution in a cyclic, or time marching, process, in which each sulution is used as the beginning of the search for the following solution and strongly influences it. For this purpose the **LOOP** operator can be used. Inside the body of the **LOOP** operator the **SOLVE** command is executed repreatedly until a maximum number of cycles ahs been completed. In the following example, this method is applied to the problem of finding time dependent temperature fields.

Let a point source of heat begins to work in the solution domain at the initial time. Heat from the source is distributed over the solution domain according to the transient heat transport equation

$$\nabla \cdot (V \nabla T) + \frac{I}{c\rho} = \frac{\partial T}{\partial t}$$

The solution method consists of calculating changes in the temperature field inside the solution domain.

We shall compute the termperature solution for only one time step; however, it will be computed on the basis of the previous temperature field. After the current period temperature field is computed from the previous time period it is saved for use in the next time step.

```
SET   X              /I1*I20 /;
SET   time           /t1*t1 /;
Alias(X,Y);

* Determination of solution domain

SET      Inside(X,Y);
```

```
        Inside(X,Y)                     = yes;
        Inside(X,Y)$(ord(X) = 1)        = no;
        Inside(X,Y)$(ord(X) = card(X)) = no;
        Inside(X,Y)$(ord(Y) = 1)        = no;
        Inside(X,Y)$(ord(Y) = card(Y)) = no;


* Parameter determination
Scalar heat accumulation in one time step   /0.0/;
Scalar dx step for space in X directions     /0.1/;
Scalar dy step for space in Y directions     /0.1/;
Scalar dt step for time                      /0.1/;


* Temperature supply determination
Parameter past_T(X,Y);
        past_T(X,Y)        = 0;
Parameter SUPPLY(X,Y);
        SUPPLY(X,Y)        = 0;
        SUPPLY('I10','I11'):= 10/dx/dy;


* Thermal conductivity
Table V(X,Y) temperature conductivity distribution


     I1  I2  I3  I4  I5  I6  I7  I8  I9  I10  I11  I12  I13  I14  I15  I16  I17  I18  I19  I20
I1   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I2   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I3   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I4   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I5   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I6   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I7   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I8   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I9   0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I10  0.5 0.5 0.5 1.5 1.5 1.5 1.5 1.5 1.5  1.5  1.5  1.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I11  0.5 0.5 0.5 1.5 1.5 1.5 1.5 1.5 1.5  1.5  1.5  1.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I12  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I13  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I14  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I15  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I16  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I17  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I18  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I19  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5
I20  0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5  0.5;


* MODEL DESCRIPTION
VARIABLES
        T(X,Y) field of temperature
        Q      Temperature on boundaries
        Obj    Objective function;


EQUATION
        TEMP(X,Y) Main equation of heat transport
        F1(X,Y)  Boundary computation dt:dn
        F2(X,Y)  Boundary computation dt:dn
        F3(X,Y)  Boundary computation dt:dn
        F4(X,Y)  Boundary computation dt:dn
        Fp1(X,Y) Boundary computation t
        Fp2(X,Y) Boundary computation t
        Fp3(X,Y) Boundary computation t
        Fp4(X,Y) Boundary computation t
        Ben      Benefit equation;


* Main equation
TEMP(X,Y)$(inside(X,Y))..
        T(X,Y)-Past_T(X,Y) =e=
        dt*(Supply(X,Y)
            +( (V(X+1,Y)+V(X,Y))  *(T(X+1,Y)-T(X,Y))
            -(V(X,Y)  +V(X-1,Y))*(T(X,Y)  -T(X-1,Y)))
```

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```
                           /dx/dx/2.0
                       +(  (V(X,Y+1)+V(X,Y))*(T(X,Y+1)-T(X,Y))
                       -(V(X,Y-1)+V(X,Y))*(T(X,Y)   -T(X,Y-1)))
                        /dy/dy/2.0);

       * Equation which determines dT/dn
       F1(X,Y)$((ord(X) = 1))..         T(X+1,Y)-T(X,Y) =g= 0;
       F2(X,Y)$((ord(X) = card(X)))..   T(X,Y)-T(X-1,Y) =l= 0;
       F3(X,Y)$((ord(Y) = 1))..         T(X,Y+1)-T(X,Y) =g= 0;
       F4(X,Y)$((ord(Y) = card(Y)))..   T(X,Y)-T(X,Y-1) =l= 0;

       * Equation which determines temperature on boundary
       Fp1(X,Y)$((ord(X) = 1 ))..       T(X,Y) =e= Q;
       Fp2(X,Y)$((ord(X) = card(X)))..  T(X,Y) =e= Q;
       Fp3(X,Y)$((ord(Y) = 1 ))..       T(X,Y) =e= Q;
       Fp4(X,Y)$((ord(Y) = card(Y)))..  T(X,Y) =e= Q;

       * Objective function
       Ben.. Obj =e= Q;

       * Limit of temperature inside and on boundaries
       T.lo(X,Y) =   0.0;
       T.up(X,Y) = 1000;
       Q.l       =   0.0;

       ***************************************************************
       MODEL flow /all /;

       * Output file
       File res1 /flow.txt/
       Put res1;

       * Below is the cycle of computation
       ***************************************************************
       LOOP (time,
             SOLVE flow using nlp minimizing obj;

             PUT " Time interval = "; PUT time.tl:20; PUT /;
             LOOP (X,PUT X.tl:6;
                   LOOP (Y,PUT T.l(X,Y):6:1;);PUT /;
             ); PUT /;
             Heat = Heat + sum( (X,Y),(T.l(X,Y)-Past_T(X,Y)))*dx*dy;
             Past_T(X,Y) = T.l(X,Y);
             );
```

Let's consider more in detail the structure of the main **LOOP** over *time* in the model. Each pass through the loop executes the GAMS **SOLVER** defining a new temperature field for the moment *t*. The algorithm uses an implicit approach where all variables belonging to the current time period are unknown. Considering the model further, the temperature field *T(X,Y)* is determined based on the values from the previous time step *Past_T(X,Y)*. In the loop the current period temperature field is transferred to the previous temperature field. Then, the change in heat quantity in the domain is computed for the current time step. The amount of heat leaving the solution domain plus the amount of heat accumulated in the domain must be equal to the amount generated by the heat source.

Below is a fragment of the output file containing an account of the balance of heat for one time step:

```
Obj function                        = 0.00
Temperature on boundary             = 0.00
Heat storage                        = 0.93
 Inside heat supply                 = 1.00
 Total heat transport               = 0.068
 Total heat transport x1 boundary = 0.01696
 Total heat transport x2 boundary = 0.01277
 Total heat transport y1 boundary = 0.02165
 Total heat transport y2 boundary = 0.01766
```

The resulting temperature field for the third time period is illustrated in Figure. 8.4.1.
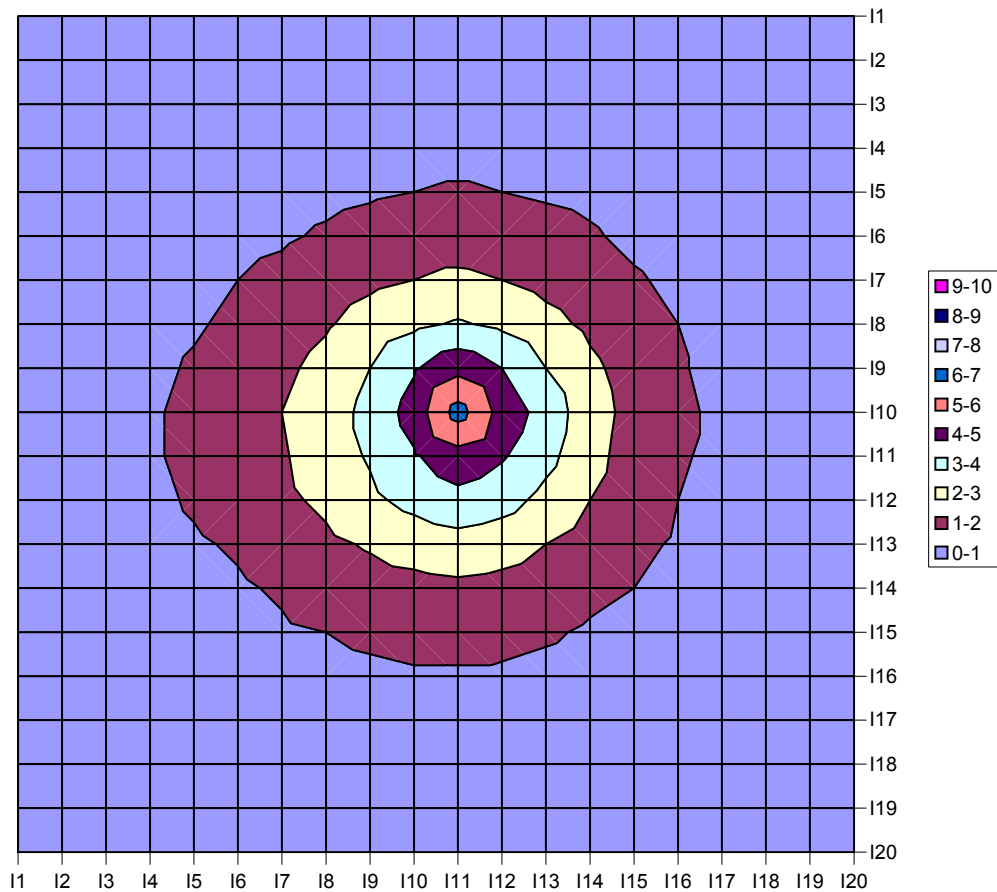
Figure 8.4.1. Solution of the transient heat transport problem after three time steps.

9. Optimal Solution of Fluid Flow Problems

9.1 Background

In this section, we consider solving some tasks of hydrodynamics. The governing equations for these examples are the conservation of linear momentum of a viscous, incompressible fluid in horizontal plane area

> *x*-direction component

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \mu \nabla^2 V_x$$

> *y*-direction component

$$\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \mu \nabla^2 V_y$$

In addition to these equations we need the equation of conservation of mass (continuity eqaution) for the fluid

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

where
- V_x *x* component of velocity [m/sec],
- V_y *y* component of velocity [m/sec],
- P Pressure [Pa],
- μ kinematic viscosity [m²/sec],
- ρ fluid density [kg/m³]

These equations apply to the case where the viscosity is constant and the fluid weight acts perpendicularly to the *x-y* plane, and, consequently, the gravitational force is absent from the equations. The liquid is considered incompressible, which complicates the search for the solution.

9.2 Stationary Flow of an Incompressible Fluid in a Rectangular Area

Consider a stationary flow of an incompressible fluid in a rectangular area with given inflow of fluid on the borders of solution domain.

Consider a rectangular area with zones of inflow and outflow of water on two opposite sides. In the zone of inflow of water, the boundary condition is:

$$V_x = 0.5 \ [m/\sec] \ \text{and} \ V_y = 0 \ [m/\sec]$$

On the other sides of the rectangle, we have the so-called "no-slip" condition:

$$V_x = 0 \ [m/\sec] \text{ and } V_y = 0 \ [m/\sec]$$

In water outflow zone, the boundary condition is

$$\frac{\partial V_x}{\partial x} = 0 \text{ and } V_y = 0$$

The given boundary conditions ensure a parallel output of fluid from the solution domain. This situation is illustrated in Figure 9.2.1.
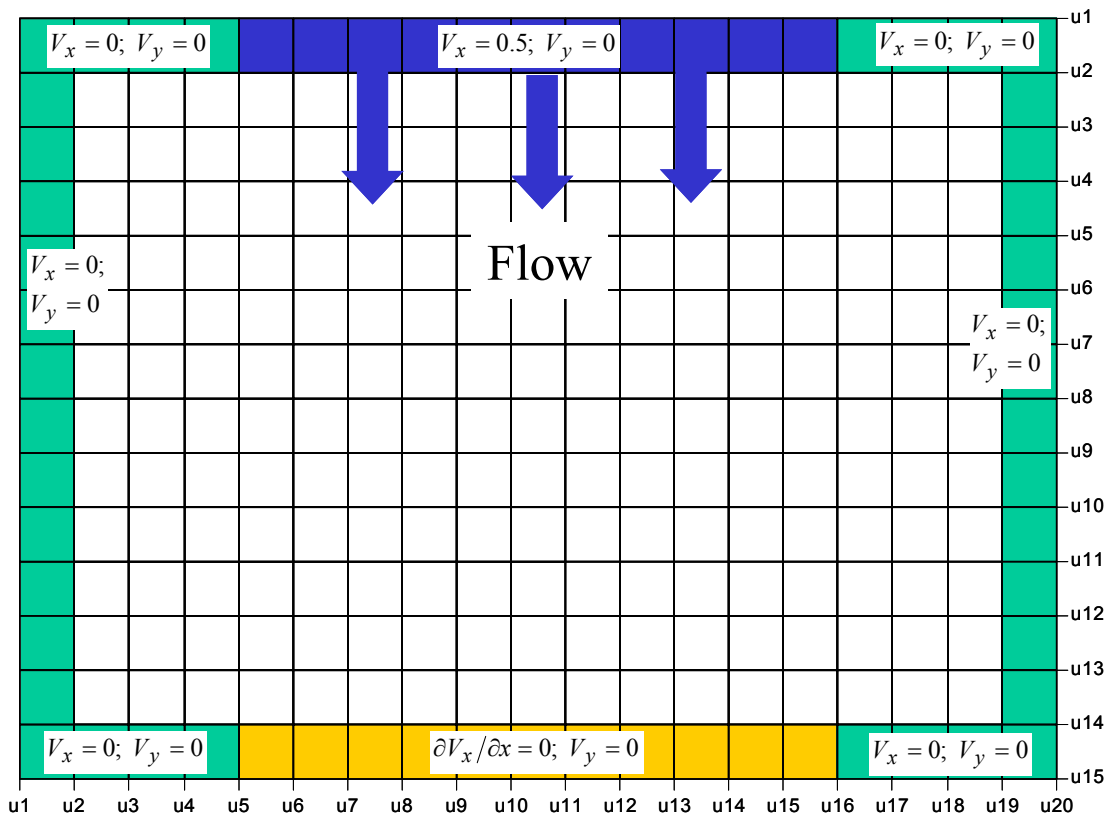
Figure 9.2.1. Flow domain for diffuser example.

Now, consider the finite difference analogs of the differential equations. We must divide the solution domain into small rectangles using a grid of parallel lines and give each intersection point appropriate indices (i,j). At each point we will compute fluid velocity. The third equation presents some difficulty, since it is not a time evolutionary form.

We compute velocity not at the points, but on the lines connecting the points. On lines parallel to the x axis we compute V_x, and on lines parallel to the y axis we calculate V_y.

Using this it is possible to make the following approximation for the continuity equation has the finite difference analog

$$\frac{V_{x,i,j} - V_{x,i-1,j}}{\Delta x} + \frac{V_{y,i,j} - V_{y,i,j-1}}{\Delta y} = 0$$

Pressure (P) is computed at the node points (i,j) of each line, where the pressure gradient is determined from the velocity. That is,

$$\left.\frac{\partial P}{\partial x}\right|_{i,j} \approx \frac{P_{i+1,j} - P_{i,j}}{\Delta x} \quad \text{and} \quad \left.\frac{\partial P}{\partial y}\right|_{i,j} \approx \frac{P_{i,j+1} - P_{i,j}}{\Delta y}$$

The diffusive terms are computed as

$$\mu \nabla^2 V_x \approx \mu\left(\frac{V_{x,i+1,j} - 2V_{x,i,j} + V_{x,i-1,j}}{(\Delta x)^2} + \frac{V_{x,i,j+1} - 2V_{x,i,j} + V_{x,i,j-1}}{(\Delta y)^2}\right)$$

$$\mu \nabla^2 V_y \approx \mu\left(\frac{V_{y,i+1,j} - 2V_{y,i,j} + V_{y,i-1,j}}{(\Delta x)^2} + \frac{V_{y,i,j+1} - 2V_{y,i,j} + V_{y,i,j-1}}{(\Delta y)^2}\right)$$

Details of the finite difference grid are illustrated in Figure 9.2.3.
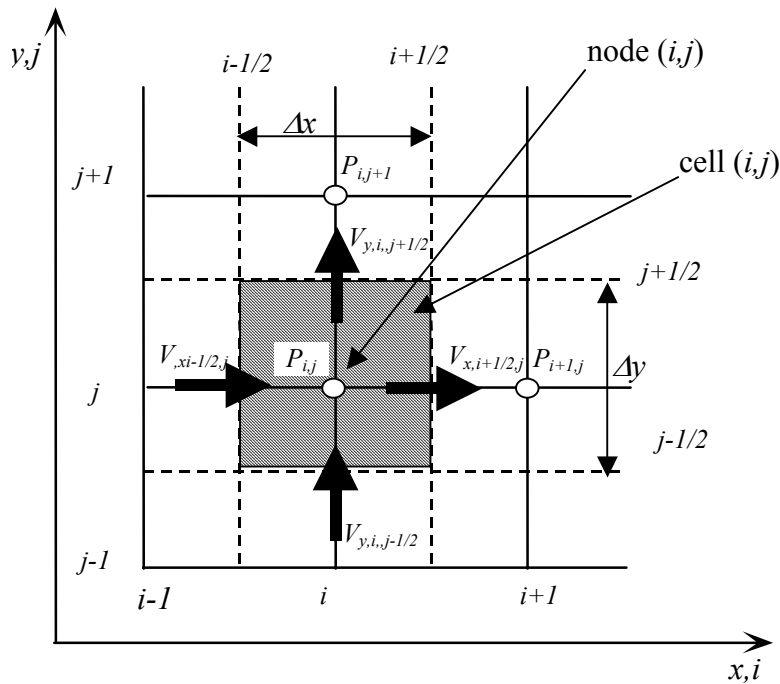
Figure 9.2.3. Finite-difference cell in two dimensional flow.

An attempt to solve the above system of the equations will be unsuccessful. It is not always possible to aproximate differentials by a finte difference, since there are errors introduced by the algebraic aproximations. Considering the initial system of equations, we notice that the first and second equations contain velocity components which are connected by ∇P. If P is not connected with boundary conditions, all apoximation errors will leave the solution domain through the borders. In the third equation, the velocity vector components are in strong contact with the boundary conditions. That is, in the calculation of the continuity equation we expect errors at each point.

In connection with this difficulty, we write the equation in a slightly different (relaxed) form.

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \delta$$

where δ is the error in the apoximation which must be minimized in the solution.

The final system of equations to solve is:

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} + \mu\nabla^2 V_x = 0$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} + \mu\nabla^2 V_y = 0$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \delta$$

with finite difference analogs:

$$\frac{1}{\rho}\frac{P_{i+1,j} - P_{i,j}}{\Delta x} = \mu \frac{V_{x,i-1,j} - 2V_{x,i,j} + V_{x,i+1,j}}{(\Delta x)^2}$$

$$\frac{1}{\rho}\frac{P_{i+1,j} - P_{i,j}}{\Delta y} = \mu \frac{V_{y,i-1,j} - 2V_{y,i,j} + V_{y,i+1,j}}{(\Delta y)^2}$$

$$\frac{V_{x,i,j} - V_{x,i-1,j}}{\Delta x} + \frac{V_{y,i,j} - V_{y,i,j-1}}{\Delta y} = \delta_{i,j}$$

The GAMS model is:

```
SET    X /u1*u15 /;
SET    Y /u1*u20 /;
SET    Yout(Y) /u5*u16 /;

* Determination of zone for water movement equation
SET Inside(X,Y);
    Inside(X,Y)                   = yes;
    Inside(X,Y)$(ord(X) = 1)      = no;
    Inside(X,Y)$(ord(X) = card(X)) = no;
    Inside(X,Y)$(ord(Y) = 1)      = no;
    Inside(X,Y)$(ord(Y) = card(Y)) = no;

* Determination of parameters
Scalar     dx  step space in X directions m        /1/;
Scalar     dy  step space in Y directions m        /1/;
Scalar     r   density of fluid kg per m3        /1000/;
Parameter  m(X,Y)  kinematic viscosity of fluid m2 per sec;
           m(X,Y):= 0.005;

Variables
        Obj            Objective
        D(X,Y)         Error
        P(X,Y)         Pressure
        Vx(X,Y)        x-direction velocity
        Vy(X,Y)        y-direction velocity
        Vdx(Y)         delta Vx at the outlet;

*   Variable Bounds
        Vx.up(X,Y)    =   1.5;
        Vx.lo(X,Y)    =  -1.5;
        Vx.l(X,Y)     =   0.5;
        Vy.up(X,Y)    =   1.0;
        Vy.lo(X,Y)    =  -1.0;
        Vy.l(X,Y)     =   0.0;
        D.lo(X,Y)     =   0.0;
```

```
        D.up(X,Y)      =  0.0000005;
        Vy.l(X,Y)      =  0.0;
        Vy.l(X,Y)$(inside(X,Y)) = 0.0000001;
        P.up(X,Y)      =  2000;
        P.lo(X,Y)      = -2000;
        P.l(X,Y)       =  0.000001;

*   Imposed Boundary conditions
        Vx.lo('u1',Y)       = 0.5;
        Vx.fx('u15',Y)      = 0.5;
        Vx.fx(X,'u1')       = 0;
        Vx.fx(X,'u20')      = 0;
        Vx.fx('u1','u2')    = 0;
        Vx.fx('u1','u3')    = 0;
        Vx.fx('u1','u4')    = 0;
        Vx.fx('u1','u17')   = 0;
        Vx.fx('u1','u18')   = 0;
        Vx.fx('u1','u19')   = 0;
        Vx.fx('u15','u2')   = 0;
        Vx.fx('u15','u3')   = 0;
        Vx.fx('u15','u4')   = 0;
        Vx.fx('u15','u17')  = 0;
        Vx.fx('u15','u18')  = 0;
        Vx.fx('u15','u19')  = 0;
        Vy.fx('u1',Y)       = 0;
        Vy.fx(X,'u1')       = 0;
        Vy.fx(X,'u20')      = 0;
        Vy.fx('u15',Y)      = 0;

Equation
        For_Vx(X,Y)    main equation 1
        For_Vy(X,Y)    main equation 2
        Div_Vxy(X,Y)   main equation 3 clear equation of balance
        Vx_Vx(Y)       outflow zone where determine outflow for X
        Ben            Objective function;

For_Vx(X,Y)$(Inside(X,Y))..
        (P(X+1,Y)-P(X,Y))/(r*dx) =e=
        m(X,Y)*((Vx(X+1,Y)-2*Vx(X,Y)+Vx(X-1,Y))/(dx*dx)
            +(Vx(X,Y+1)-2*Vx(X,Y)+Vx(X,Y-1))/(dy*dy));

For_Vy(X,Y)$(Inside(X,Y))..
        (P(X,Y+1)-p(X,Y))/(r*dy) =e=
        m(X,Y)*((Vy(X+1,Y)-2*Vy(X,Y)+Vy(X-1,Y))/(dx*dx)
            +(Vy(X,Y+1)-2*Vy(X,Y)+Vy(X,Y-1))/(dy*dy));

Div_Vxy(X,Y)$((ord(X) > 1)$(ord(Y) > 1))..
        (Vx(X-1,Y)-Vx(X,Y))/dx + (Vy(X,Y-1)-Vy(X,Y))/dy
            =e= D(X,Y);

Vx_Vx(Y).. Vdx(Y) =e= Vx('u15',Y)-Vx('u14',Y);

Ben..   Obj =e= SUM(Y$Yout(Y), Vdx(Y)*Vdx(Y))
            +SUM((X,Y),(D(X,Y)*D(X,Y)));

Model flow1 /ALL/;
Solve flow1 using nlp minimizing obj;
```

Figure 9.2.4 illustrates the velocity field in the solution domain. The points are the nodes of the

finite difference grid and the tails point in the direction of flow away from the point. Figure 9.2.5 shows three cross-sections of x-direction velocity, V_x.
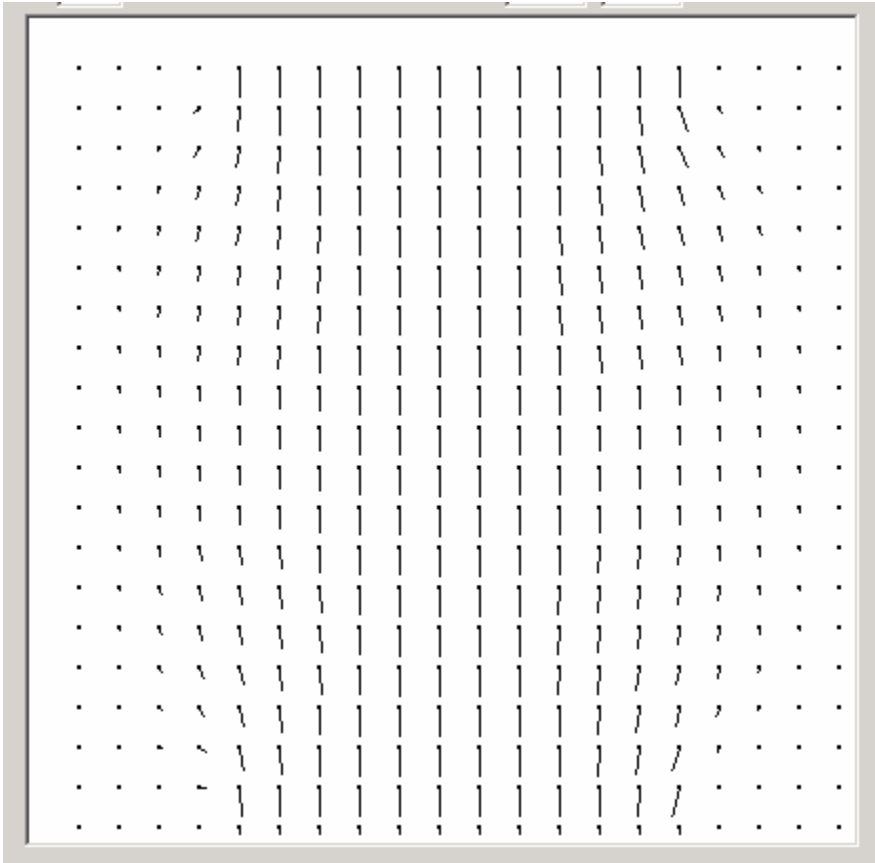
Figure 9.2.4. Flow field for flow from diffuser.
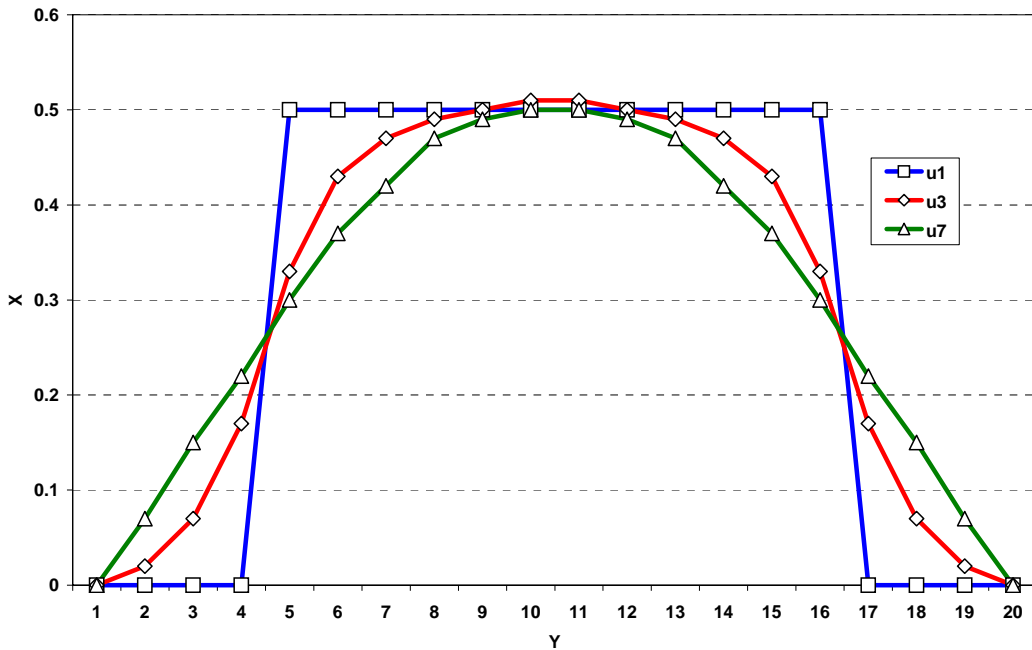
Figure 9.2.5. Three cross-sections of x-direction velocity, V_x.

This task has been solved using a numerical method and it can also be solved by potential thorey. Below we illustrate an example which can not be solved by analytical methods, however, the solution of the task by numerical methods is possible.

9.3 Stationary flow of water in a rectangular area in the presence of an obstacle

Consider a stationary flow of water in a rectangular area in the presence of an obstacle and complete account of hydrodynamic pressure and forces of inertia.

Let's consider a rectangular area. On the top edge of the area an inflow of water with a velocity of 0.5 m/sec occurs. The same flow leaves the area through the bottom edge of the region. The boundary conditions are the same as in the previous example. However, tn the middle of solution domain there is a symmetrically located obstacle, around which the fluid is compelled to flow. The equations governing flow in this case are the same as before and the only addition is the definition of the obstacle in the middle of the solution domain.

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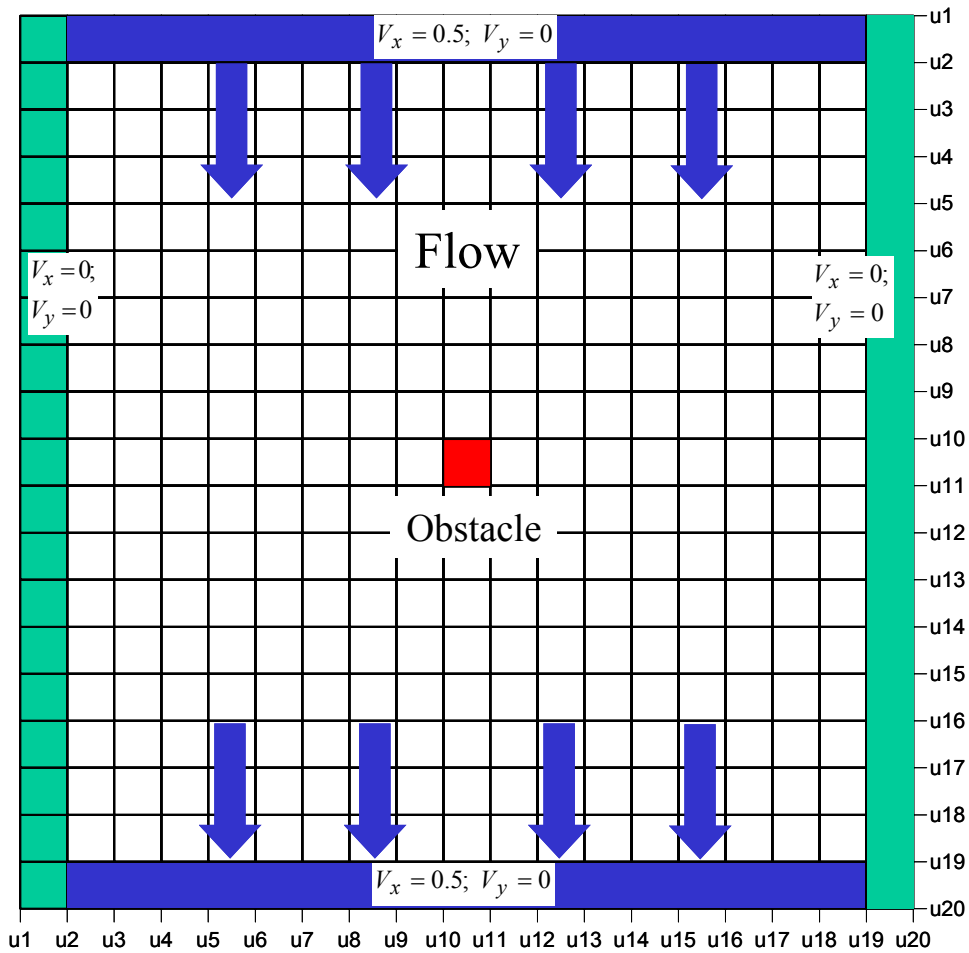
Figure 9.3.1. Flow domain for obstacle example.

The GAMS code for the model is:

```
SET X  /1*20/;
SET Y  /1*20/;

* Determination of zone for water movement
SET vyside(X,Y);
SET vxside(X,Y);
    Vxside(X,Y)                       = yes;
    Vxside(X,Y)$(ord(X) = 1)      = no;
    Vxside(X,Y)$(ord(X) = card(X)) = no;
    Vxside(X,Y)$(ord(Y) = 1)      = no;
    Vxside(X,Y)$(ord(Y) = card(Y)) = no;
    Vyside(X,Y) = Vxside(X,Y);
    Vxside('10','10')             = no;
    Vxside('10','11')             = no;

* Parameter determination
Scalar    dx  step space in X directions /1/;
Scalar    dy  step space in Y directions /1/;
Scalar    r   density of fluid kg per m3 /1000/;
```

```
Parameter m(X,Y) kinematic viscosity of fluid m2 per sec;
          m(X,Y):= 0.05;

* MODEL DESCRIPTION *
Variables
          Obj
          D(X,Y)          error
          P(X,Y)          Pressure
          Vx(X,Y)         x-direction velocity
          Vy(X,Y)         y-direction velocity
          ;

* Variable limits
D.lo(X,Y) = 0;
D.up(X,Y) = 7.50;
P.up(X,Y) = 20000;
P.lo(X,Y) =-20000;
P.l(X,Y) =  0.0;
Vx.l(X,Y) = 0.0;
Vy.l(X,Y) = 0.0;
Vy.l(X,Y)$(Vxside(X,Y)) = 0.0;

* Boundary conditions *
Vx.fx( '1',Y) = 0.5;
Vx.fx('20',Y) = 0.5;
Vx.fx(X, '1') = 0;
Vx.fx(X,'20') = 0;
Vy.fx( '1',Y) = 0;
Vy.fx('20',Y) = 0;
Vy.fx(X, '1') = 0;
Vy.fx(X,'20') = 0;

* Fence description *
Vx.fx('10','10') = 0;
Vx.fx('10','11') = 0;

Equation
          For_Vx(X,Y)     main equation 1
          For_Vy(X,Y)     main equation 2
          Div_Vxy(X,Y)    main equation 3 clear equation of balance
          Ben               benefit equation;

Div_Vxy(X,Y)$((ord(X) > 1)$(ord(Y) > 1))..

   (Vx(X,Y)-Vx(X-1,Y))/dx + (Vy(X,Y)-Vy(X,Y-1))/dy =e= D(X,Y);

For_Vx(X,Y)$(Vxside(X,Y))..
* Upwind scheme for inertial terms
*          Vx(X,Y)*(Vx(X+1,Y)-Vx(X-1,Y))/(2*dx)
*    +0.25*(Vy(X+1,Y-1)+Vy(X+1,Y)+Vy(X,Y-1)+Vy(X,Y))
*          *(Vx(X,Y+1)-Vx(X,Y-1))/(2*dy)
*    +(P(X+1,Y)-P(X,Y))/(r*dx)
*    =e=
*    m(X,Y)*((Vx(X+1,Y)-2*Vx(X,Y)+Vx(X-1,Y))/(dx*dx)
*          +(Vx(X,Y+1)-2*Vx(X,Y)+Vx(X,Y-1))/(dy*dy));

For_Vy(X,Y)$(Vyside(X,Y))..
* Upwind scheme for inertial terms
*    0.25*(Vx(X-1,Y+1)+Vx(X-1,Y)+Vx(X,Y+1)+Vx(X,Y))
```

```
*            *(Vy(X+1,Y)-Vy(X-1,Y))/(2*dy)
*              +Vy(X,Y)*(Vy(X,Y+1)-Vy(X,Y-1))/(2*dy)
*      +(P(X,Y+1)-P(X,Y))/(r*dy)
*      =e=
*      m(X,Y)*((Vy(X+1,Y)-2*Vy(X,Y)+Vy(X-1,Y))/(dx*dx)
*              +(Vy(X,Y+1)-2*Vy(X,Y)+Vy(X,Y-1))/(dy*dy));

Ben..
   Obj =e=  sum((X,Y),d(X,Y)*d(X,Y));

model flow2 /ALL/;
* Solve flow2 using nlp minimizing obj;
Option nlp = minos5;
*Option nlp = conopt2;
flow2.WORKSPACE=10;
Solve flow2 using nlp minimizing obj;
```

Figure 9.3.2 shows the flow field for this example. One can notice the flow of fluid around the obstacle and the high degree of recirculating flow behind the object. Figure 7 shows cross-sections of x-direction velocity before the obstacle, at the level of the obstacle and after the obstacle.
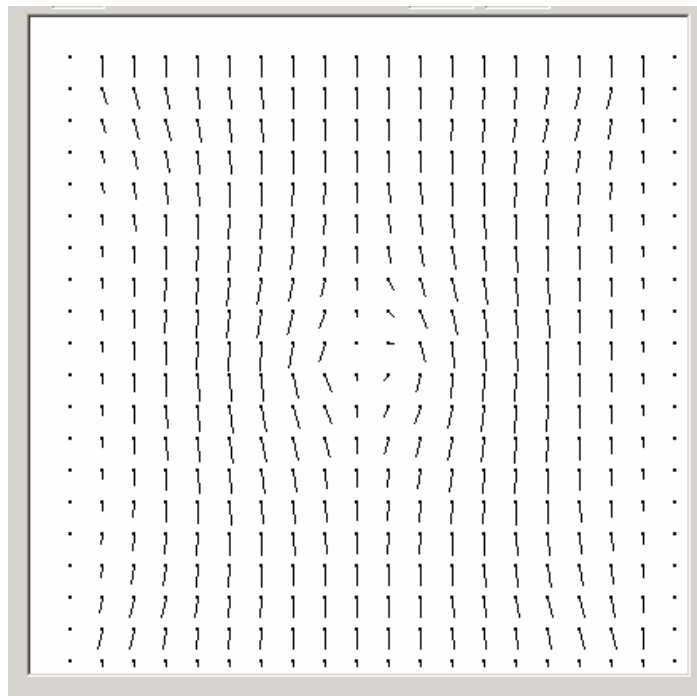
Figure 9.3.2. Flow field for fence example without initerial effects.
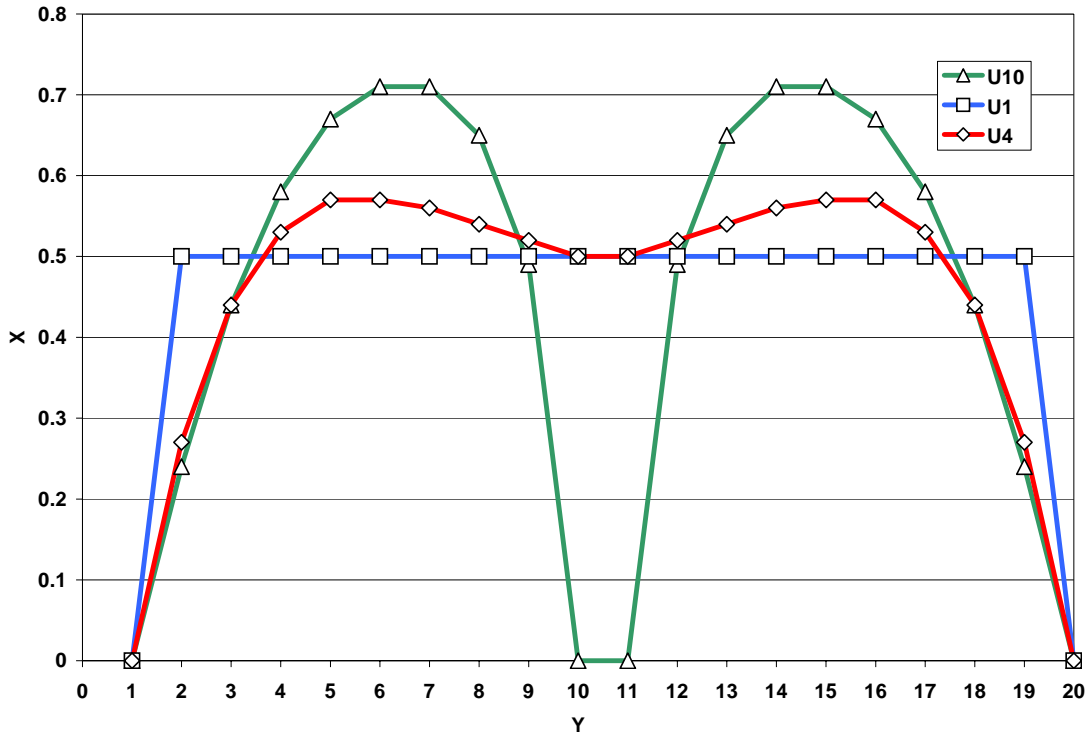
Figure 9.3.3. Three cross-sections of x-direction velocity, V_x.

The user will notice that the inertial terms of the equations are switched in the GAMS code. The user can include these terms in model by removing the asterisks from the code. The inertial terms are computed using the "marker and cell" method (*Peyret and Taylor*, 1985)

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y}$$

$$\approx V_{x,i,j} \frac{V_{x,i+1,j} - V_{x,i,j}}{2\Delta x}$$

$$+ \frac{V_{y,i+1,j} + V_{y,i,j} + V_{y,i,j-1} + V_{y,i+1,j-1}}{4} \frac{V_{x,i,j+1} - V_{x,i,j-1}}{2\Delta y}$$

$$V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y}$$

$$\approx \frac{V_{x,i,j} + V_{x,i,j+1} + V_{x,i-1,j+1} + V_{x,i-1,j}}{4} \frac{V_{y,i+1,j} - V_{y,i-1,j}}{2\Delta x}$$

$$+ V_{y,i,j} \frac{V_{y,i,j+1} - V_{y,i,j-1}}{2\Delta y}$$
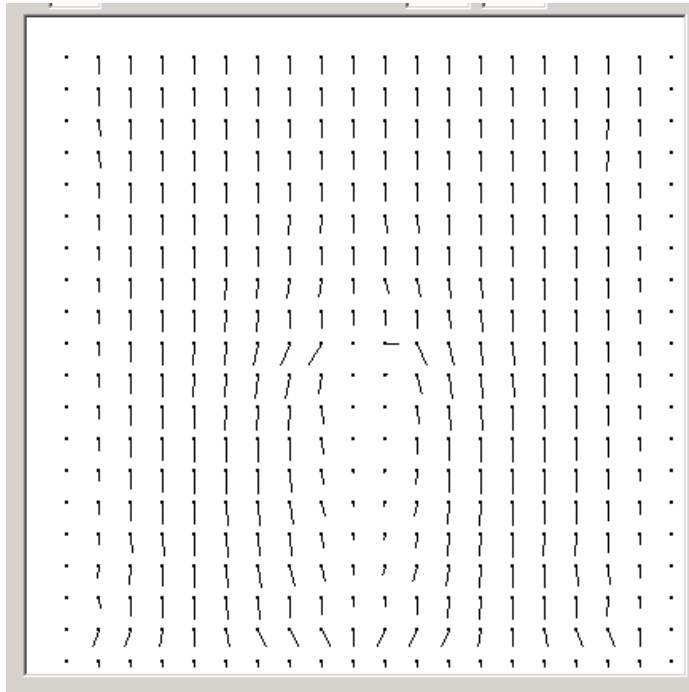
Figure 9.3.4. Flow field for fence example with initerial effects.
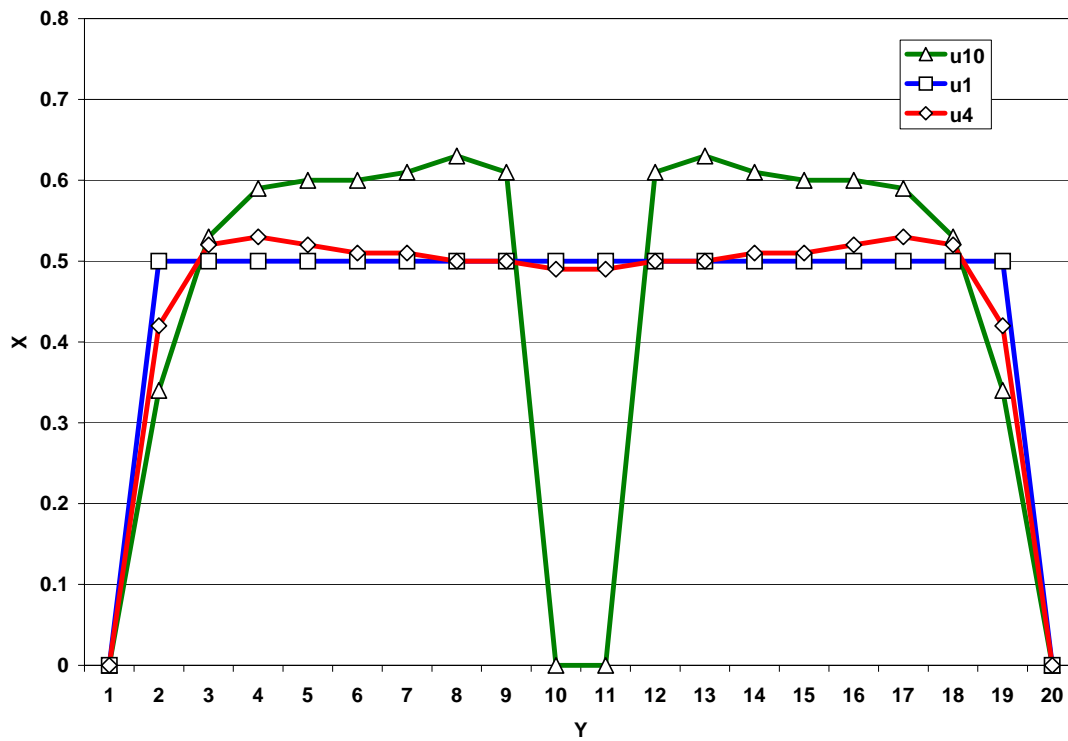
Figure 9.3.5. Three cross-sections of x-direction velocity, V_x, with initerial effects.

Comparing results of the models in which the inertial forces are taken into account indicates their influence. In one case return flow of the fluid is present (see Figure 8) and in the other it is not present (see Figure 6). There can also be a question about the asymmetry of the aproximations for the inertial terms. However, the velocities are determined not at cell centers, but on cell faces, and for these are symmetric. On the basis of the above elementary examples it is possible to construct a model of transient movement of fluids using the approach described in the secition on heat transfer.

PART 2: THE GAMS LANGUAGE

10. INTRODUCTION

The General Algebraic Modeling System (GAMS) is designed to facilitate the work of modelers, who create mathematical models of real processes around us. Like other languages of this type, GAMS relieves modelers of the necessity to design algorithms to solve models, in addition to developing the models themselves. Thus, the major functional objective of GAMS is to design algorithms for mathematical models. However, it does not mean that this system can be used only for mathematical models with algebraic equations. Those who are acquainted with methods of solving partial differential equations (PDEs) know that numerical methods for solving PDEs always result in systems of algebraic equations which must be solved according to some iterative algorithm. Thus, the applicability of GAMS is wide. In this Tutorial, models and programs using the GAMS language are presented that allow users to develop their own models. This Tutorial should be supplemented by the official GAMS Guide and Tutorials (Brooke et al., 1997).

10.1 Installation

Detailed instructions for how to install GAMS on your computer are contained in Appendix C. Periodic updates to these instructions can be found on the GAMS website (www.gams.com).

Installation requires a system capable of working in the Windows mode and its characteristics should not be less than those of a standard PC computer.

The user should be able to state a mathematical problem correctly. That is, formulation of all logical interrelations of the problem and features of its behavior, especially in extreme cases. GAMS cannot think for you. It can only help you to solve mathematical problems if they are stated correctly. If, for some reason, the model does not give correct results, generally, the mistake is not a GAMS mistake. Try to find the mistake in your own model formulation. If you do not manage to find the mistake at first, have a rest and try to find it again. Check your model for mistakes (the theoretical part plus formulation in the GAMS language). However, in some cases, and in the Tutorials there is such an example, GAMS can provide you with a decision which is not optimal. That is, you should be ready for many contingencies. Although in most cases, you will get what you seek.

You should know how to use a text editor in the Windows mode on your computer. You can use any other text editor in any other system, but the editor you use should be able to create text files. The Windows Notepad editor is very apt.

There are two sources on information GAMS and GAMS models:

1. Books and tutorials (the list is at the end of the Tutorials);
2. The internal library of the GAMS models supplied with the main GAMS package. It is in the subdirectory MODLIB. The library is created automatically as the package is installed on your computer. In the library, there are more than 150 models with necessary comments. These models are all operable. All models are text files, which can be printed by any editor.

10.2 Nonlinear Algebraic Equations

10.2.1 Solution of Algebraic Equations
(adapted from G. Kovalenko)

The problem of solving simultaneous equations (both linear and nonlinear) is often faced in science and engineering. GAMS is very well suited to solving systems of nonlinear algebraic equations $f(x) = 0$. The only difficulty is connected with defining all roots if there are multiple roots to an equation. For example, consider a system of two equations:

$$f_1(x, y)=3x^2 - y^2 =0 \tag{10.2.1.1}$$

$$f_2(x, y)=3xy^2 - x^3 - 1=0 \tag{10.2.1.2}$$

where we would like to find the values of x, such that $f_1(x) = 0$ and $f_2(x) = 0$.

To solve the problem, we can enter the equations as constraints in a GAMS model and set the objective function equal to any constant, say 1.0.

A simple GAMS model to solve this problem is

```
GAMS Model
VARIABLES x, y, obj;
EQUATIONS Eq1, Eq2, Objective;

Eq1.. 0 =E= 3*x*x - y*y;
Eq2.. 0 =E= 3*x*y*y - x*x*x - 1;
Objective.. obj =E= 0;

x.L = 0.01;
y.L = 0.01;

MODEL Eq /ALL/
SOLVE Eq USING NLP MINIMIZING obj;

FILE res /Eq1.txt/
PUT  res;
put  "Solution x =  ", put x.l, put /;
put  "Solution y =  ", put y.l, put /;

RESULTS
Solution x1 =          0.50
Solution x2 =          0.87
```

In this model 3 variables are defined, *x, y* and *obj*. All these variables are allowed to take on any vales, positive or negative. In addition, there are 3 equations defined in the model, *Eq1, Eq2* and *Objective*. *Eq1* is the same as $f_1(x)$ above and *Eq2* is the same as $f_2(x)$. The model is comprised of all the defined equations, as shown in the *model* statement. Since $f_2(x)$ is a nonlinear (quadratic) equation, the model is solved using a nonlinear programming solver (MINOS), as specified in the *solve* statement.

In this model a very simple output file is generated with only two lines of text. The output file is first defined as *Eq1.txt* using the *file* command and then the model output is directed to that file using the *put* commands. In GAMS, several output file can be defined if desired and output directed to each file.

Note that in the model the exponentiation operator (**) is avoided. This is because `x**n` is calculated inside GAMS as `exp[n*log(x)]`. This operation is not defined when `x` takes on negative values (which is will, see below), and an error will result. In this case, if the exponent is known to be an integer, a function call, `power(x,n)`, can be used instead.

Figure 10.2.1.1 shows a plot of the solution in this case. The first two lines are functions with points of intersection to be found. The third line is the distance between the functions depending on *x* which is a plot of the variable *obj*. It can be seen that this system has two roots: $x' = -4.162$ and $x'' = 2.162$.
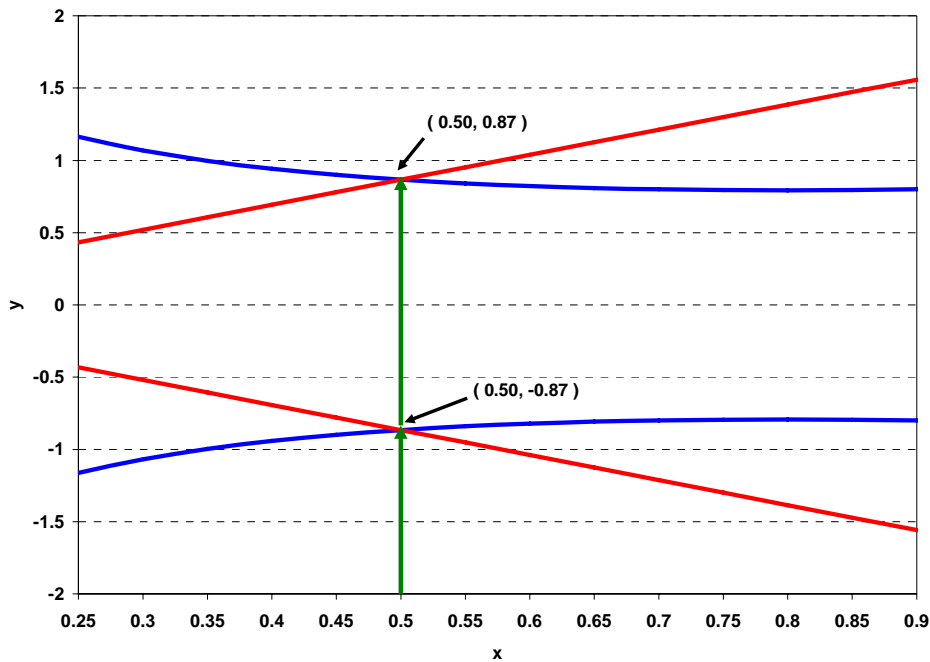
Fig. 10.2.1.1. Chart for the Task of Solution of Equations

When the above model is solved by GAMS the solution (x1, x2) = (0.50, 0.87) is obtained. In order to obtain the solution (x1, x2) = (0.50, -0.87) you must change the initial guess of the solution from (x1, x2) = (0.01, 0.01) to (x1, x2) = (0.01, -0.01).

The model described above can be generalized somewhat so that the data are more easily identified and available for editing. This is done using the GAMS notion of *scalars* and the coefficients of the equations are given names corresponding to those in Equations 10.2.1.1 and 10.2.1.2. Thus, the model is rewritten as

```
SCALAR A /3/, B /-1/, C /0/, D /3/, E /-1/, F /-1/, G /0/;

VARIABLES x, y, obj;
EQUATIONS Eq1, Eq2, Objective;

Eq1.. A*x*x + B*y*y =E= C;
Eq2.. D*x*y*y + E*x*x*x + F =E= G;

x.L = 0.01;
y.L = 0.01;

Objective.. obj =E= 0;

MODEL Eq /ALL/
SOLVE Eq USING NLP MINIMIZING obj;

FILE res /Eq1.txt/
PUT  res;
put  "Solution x =  ", put x.l, put /;
put  "Solution y =  ", put y.l, put /;
```

10.2.2 Method of Least Squares for Arbitrary Functions
(adapted from O.N. Tikhonova)

Suppose that three independent variables (x_1, x_2, x_3) are retated in a complex function with a fourth, dependent variable (y). The function relating these four variables has been defined on the basis of a physical phenomenon under study as

$$y = a(x_1)^2 - \frac{b}{x_3} - \frac{c}{x_2} + e^{-y^2} \tag{10.2.2.1}$$

where a, b, and c are unknown coefficients. Measurements of the four variables have been made at 8 different times. The difference (or residual) between the model value y (given in Equation 2.2.1 and which depends on the unknown parameter values a, b, and c) and the observed value $\hat{y}(t)$ can be written as

$$e(t) = y(t) - \hat{y}(t) \tag{10.2.2.2}$$

Numerical values of the coefficients can be determined by minimizing the squared residuals

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$$Minimize \ \sum_{t=1}^{8} [e(t)]^2 \tag{10.2.2.3}$$

In this case, suppose we have the following data

Table 10.2.2.1. Data for Least Squares Estimation.

	Time							
	1	2	3	4	5	6	7	8
x_1	2	3	3	3	5	5	6	7
x_2	30	60	70	60	80	90	100	100
x_3	1	6	7	3	5	9	8	17
$\hat{y}(t)$	10	20	20	20	40	50	60	70

In this case we will need to introduce the notion of *sets* and *parameters* into the GAMS model. Sets are a way of defining indices in GAMS and we need one for time in this problem. Consider the following model developed in the GAMS language:

```
SETS t  / 1, 2, 3, 4, 5, 6, 7, 8 /;

PARAMETER x1(t) /1 2, 2 3, 3 3, 4 3, 5 5, 6 5, 7 6, 8 7/;
PARAMETER x2(t) /1 30, 2 60, 3 70, 4 60, 5 80, 6 90, 7 100, 8 100/;
PARAMETER x3(t) /1 1, 2 6, 3 7, 4 3, 5 5, 6 9, 7 8, 8 17/;
PARAMETER y_hat(t) /1 10, 2 20, 3 30, 4 20, 5 40, 6 50, 7 60, 8 70/;

VARIABLES
a, b, c, y(t), e(t), obj;

EQUATION mod(t), residual(t), objective;

mod(t).. a*x1(t)*x1(t)-b/x3(t)-c/x2(t)+EXP(-y(t)*y(t)) =E= y(t);

residual(t).. e(t) =E= y(t)-y_hat(t);

objective.. obj=E=sum(t,power(e(t),2));

MODEL Leastsq / ALL /;

SOLVE Leastsq USING NLP MINIMIZING obj;

FILE res /Eq2.txt/
PUT res;
PUT "     t                    x(1,t)    x(2,t)       x(3,t)        y(t)
y_hat(t)"/;
LOOP((t),PUT t.TL, x1(t), x2(t), x3(t), y.L(t), y_hat(t)/;);
PUT /"          a              b             c"/;
PUT a.L, b.L, c.L;
```

In this model an output file is generated which prints a table of values in rows and columns. The model output is directed to that file using the *put* commands. The header for the table is printed first and then a *loop* cycles over each value of $x_1(t)$, $x_2(t)$, $x_3(t)$, $y(t)$, and $\hat{y}(t)$ printing one line

for each value of t. In GAMS, several output file can be defined if desired and output directed to each file.

The results are:

t	x1(t)	x2(t)	x3(t)	y(t)	y_hat(t)
1	2.0	30.0	1.0	10.2	10.0
2	3.0	60.0	6.0	26.5	20.0
3	3.0	70.0	7.0	24.5	30.0
4	3.0	60.0	3.0	20.5	20.0
5	5.0	80.0	5.0	41.5	40.0
6	5.0	90.0	9.0	43.0	50.0
7	6.0	100.0	8.0	55.8	60.0
8	7.0	100.0	17.0	75.6	70.0

a	b	c
1.33	36.27	-1234.14

10.2.3 Exercises

1. Add the following constraint:

$$y \geq 100 - 50x$$

to the model of section 2.1 and resolve. Verify that you still obtain the same solution. What has happened to the "feasible" region of possible solutions by adding this constraint?

2. Modify the output of the model of section 2.2 to print the percent error in the model calculations, where the error is defined as

$$[y\_hat(t) - y(t)]/y\_hat(t)*100$$

3. Solve the following set of nonlinear equations using the method of Section 1. Algebraic Equations

$$x^2 + y^2 = 4$$

$$e^x + y = 1$$

4. Consider the system of simultaneous, nonlinear equations

$$f_1(x_1, x_2) = 3x_1^2 - x_2^2 = 0$$

$$f_2(x_1, x_2) = 3x_1 x_2^2 - x_1^3 - 1 = 0$$

(a) In *very* general terms (don't plug any number in yet!), what is the formula for Newton's method for 2 nonlinear equations? That is $\mathbf{x}^{(k+1)} = \cdots$, and how do you get it.

(b) For the two equations given above, show the system of equations which must be solved in order to apply Newton's method (Don't plug in any numbers yet, just write down the equations using the functions from the Equation above.)

(c) Start with $k=0$, and $x_1^{(0)} = x_2^{(0)} = 1$, what is the system of linear equations that you must solve to obtain $x_1^{(1)} = x_2^{(1)}$? What is the solution to that system of linear equations?

(d) Now, knowing $\mathbf{x}^{(1)}$ you can calculate $\mathbf{x}^{(2)}$ for $k=2$ in a similar manner. What is the system of linear equations that you must solve to obtain $x_1^{(2)} = x_2^{(2)}$? What is the solution to that system of linear equations?

Continuing in this manner, the method will converge to the solution

$$x_1 = 0.5$$
$$x_2 = 0.5\sqrt{3}$$

(e) Draw a simple flowchart illustrating your algorithm for solving these equation.

5. Solve the following simultaneous set of nonlinear equations:

$$f(x,y,z) = 4x + 5\sin y + 0.1z - 5 = 0$$
$$g(x,y,z) = x^2 + 2y + \exp(-0.5z) - 5 = 0$$
$$h(x,y,z) = x + y + z^2 - 12 = 0$$

11. INTRODUCTORY INFORMATION

11.1 Model Structure

Each model in the GAMS language may have its own distinctions. It can be comprised of groups of alternating models, which are linked together only through a small number of common variables or parameters. However, most models have a structure in the form shown in Table 11.1.1.

Table 11.1.1.
Structure of a GAMS Model.

1.	SETS	Structures consisting of a complex of indices or names
		Declaration of fact of existence
		Determination of relations between individual structures
2.	DATA	PARAMETERS, TABLES, SCALARS
		Declaration of fact of existence
		Determination of values of input parameters
		Preliminary computations
3.	VARIABLES	Variables or arrays of variables
		Declaration with assigning a type of variable
		Declaration of limits for possible changes, initial level
4.	EQUATIONS	Equations or complexes and arrays of equations
		Declaration with assigning a name
		Recording of equations in the GAMS language
5.	MODEL, SOLVE	Model and methods of solution
6.	OUTPUT	Output of information into a separate file

During model execution, information is displayed on the screen for any solution circumstances. The compilation results of the model and computations are automatically sent to a special "*LIST*" file. This file has the same name as the file of the reference model, but a different extension – "*LST*". Whatever happens, whether the computation is right or wrong, the file "**.lst*" automatically appears in the same directory as the file "**.gms*". Depending on the situation, the contents of this file may differ from that given below, but the basis remains.

1. Reprinting of the model
2. Map of interrelations of parameters and variables
3. Information on equations
4. Information on the solution status
5. Information on the obtained solution

Before describing basic rules of the GAMS language, we will give some brief information about features of the language. No equation, variable or their interrelationship can be used until they have been declared in the zone of STATEMENTS (items 2 and 3 in Table 11.1.1). These statements can be formed in quite an arbitrary style. We will use this arbitrariness in the examples below. The GAMS language does not distinguish between capital and lowercase letters of the alphabet. In this case, it is useful to observe the GAMS rules. Capitals are used for function words, small letters are used to form the model's own meaning. GAMS accepts comments using only the English alphabet (in the Russian version of this document, comments in Russian appear in italics but should not appear in the model for execution).

11.2 Comments in Models

There are three main methods of introducing comments and elucidation's in the body of a GAMS

model.

1. If a line begins with the " * "symbol, the contents of the line are treated as a comment. For example,

```
*plants (plants)
```

2. A program section beginning with the line $ONTEXT and ending with the line $OFFTEXT is a comment. The above functional words begin with the first symbol of the line.

```
$ONTEXT;
- - - - - - - - - -
This guide is for a scientist who knows the Russian language.
+ + + + + + + + + +
$OFFTEXT;
```

The user can introduce comments in the bodies of GAMS declarations. In the Russian language version of this document, comments typed in a smaller font remind the user about the necessity to change them in the working program for comments in English or any other language using the Latin alphabet. For instance, the State Language in the Republic of Uzbekistan is based on the Latin alphabet and simplifies the use of comments in GAMS. On the other hand, the language of the Russian Federation uses the Cyrillic alphabet and this alphabet can not be used for comments in GAMS.

11.3 Declarations and Definitions

For most structures in the GAMS language, there are two types of statements:

1. DECLARATION – assignment of a name and possibly a value to a stated parameter, and
2. DEFINITION – calculation using previously defined parameters or structures

Parameters always require DECLARATIONS. Assignment of numerical values to parameters is not always compulsory. However, equations always require both declaration and definition. DEFINITION of equations means the construction of their mathematical structure, and for parameters it means their direct computation. While numerical interrelations are specified in the body of an equation, the order of equation definition does not matter. There is no such notion as "first equation", "second equation" and so on. For parameters and constants simultaneous declaration and definition in one operation is possible. Examples are given below.

For a series of GAMS structures, it is possible to use long names. However, it is better to limit yourself to 8 letters. All names of structures in the GAMS language must begin with a letter and can contain digits or a limited number of other symbols (maximum possible number is shown in the examples).

11.4 Terms, Symbols and Reserved Words

Every programming language, including GAMS, has keywords which have special meaning. Therefore, they cannot be used in the names of variables and parameters. For example, the name ASSIGN can not be used for any variable, because GAMS will read this as a keyword with special meaning, note it as improperly used, and show that an error has occurred. Table 11.4.1 is a list of GAMS keywords.

Table 11.4.1
Keywords in GAMS.

Abort	acronym	Acronyms	Alias	all	and
assign	binary	Card	Display	eps	eq
equation	equations	Ge	Gt	inf	integer
Le	loop	Lt	Maximizing	minimizing	model
models	na	Ne		negative	not
options	or	Ord	Parameter	parameters	positive
Prod	scalar	Scalars	Set	sets	smax
Smin	sos1	Sos2	Sum	system	table
Using	variable	Variables	Or	yes	repeat
Until	while	If	Then	else	semicont
semiint	file	Files	Putpage	puttl	free
No	option	Solve	For		

Table 11.2.2
Special Combinations of Symbols and Their Meaning.

=l=	Less	--	previous in cycle
=g=	More	++	next in cycle
=n=	not equal	**	involution
arithmetic	+,-,/,*		

Table 11.2.3
Symbols and Their Usage.

A to Z	Alphabet	-	minus
a to z	Alphabet	( )	parentheses
&	Ampersand	[ ]	brackets
*	Asterisk	{}	braces
@	at	%	percent
\	Slash	From 0 to 9	digits
:	Colon	#	number
,	Comma	?	question mark
$	Dollar	;	semicolon
.	Period	'	quotation marks, single
+	Plus	/	virgule
"	quotation marks, double_	Underlining	
=	Equals	!	exclamation point
<	is less than	^	diacritic mark (over a vowel)
>	is more than		

12. VARIABLES

12.1 Types and Declaration

Variables are the major object with which GAMS operates. To define a list of variables we use the functional word VARIABLE. There are five types of variables as indicated in Table 124.1.1.

Table 12.1.1
Type of Defining Variables in GAMS.

VARIABLE	*Variables* belong to the entire set of real numbers from "minus" infinity to "plus" infinity
POSITIVE VARIABLE	*Positive variables* belong to the entire set of positive real numbers from zero to "plus" infinity
NEGATIVE VARIABLES	*Negative variables* belong to the entire set of negative real numbers from zero to "minus" infinity
INTEGER VARIABLES	*Integer variables* belong to the entire set of positive integers
BINARY VARIABLES	*Binary variables* are Integer variables that can take the value of 1 or 0 only

VARIABLES can be zero-dimensional or multidimensional. The dimensions are determined by the availability and number of indices. The declaration of variables is based on the following principle.

For example, consider the variables from the nonlinear equation model in Section 2.1:

```
VARIABLES x, y1, y2, obj;
```

These are all aero-dimensional (scalar) variables. Variables are included in equations along with parameters and constants. Variables are always defined in the solution process. However, users and modelers are twice faced with variables when they make up a model. The first time, they face variables while declaring them and the second time in the equations.

12.2 Bounds of Variables, Initial and Fixed Values

In most models some variables have bounds on their values. For instance, if some variables are declared as positive, then "zero" is the lower boundary. The same boundary exists for negative variables, but "zero" is their maximum possible value. In addition, other boundaries can be defined with the help of specific suffixes on variables (see Table 12.2.1).

Table 12.2.1
Main Suffixes.

Suffix	Description
.L	Value (level)
.LO	Lower boundary
.UP	Upper boundary
.FX	Fixed value
.m	Dual value

GAMS begins the search of variables for a solution from a boundary of the allowable area of values, most often from zero. However, zero is a dangerous value which should be avoided if possible.

Examples show best how suffixes act.

```
S.lo=2.0;
```

defines the lower bound on instream flow in the water user model of Section 3.1.

```
b.up=100;
```

defines the upper bound of the coefficient b in the reservoir example of Section 3.2.

```
put  "Solution x =  ", put x.l, put /;
```

indicates that the value (level) of the variable x should be printed in the nonlinear equation example of Section 2.1.

In general, the more the user constricts the boundaries of variable values, the sooner a solution will

be obained. However, you should not constrict the boundaries of search too much, because in complicated models you may have the situation where there is no area left to search for solutions at all. Namely, there may be systems of equations in which one variable is within the boundaries allowed by the user, but the second variable is certain to be in an unallowed area. Usually, the consequences of calculating such systems are grave.

13. EQUATIONS

The block of equations in GAMS models consists of two parts:

1. Declaration of the names of equations
2. Definition of the equations

13.1 Declaration of Equations

Declaration of equation names is similar to the declaration of SETs or PARAMETERs. The similarity is in the fact that a list and comments are allowed and recommended. The syntax of declaring equation names is as follows:

```
EQUATION   name   comment ;
```

EQUATION is a keyword which must appear before the name of each equation if there is semicolon at the end of the line. It can be found only one time at the beginning of a list of equation names. Commas separate different names of equations in a row or one row is assigned for each name.

The *name* can include two parts:

- an *identifier* of the name of the equation, and
- *indices* in parentheses.

The identifier (name) of an equation may include no more than 10 symbols and must always begin with a letter. Indices can be either letters or digits representing SETS. Comments may not be more than 80 symbols long and should be on the row where the identifier is located. Consider the example of the equations from the model for solving nonlinear equations presented in Section 2.1

```
EQUATIONS Eq1, Eq2, Objective;
```

This declares three scalar equations, the first equation is named *Eq1*, the second *Eq2*, and the third *objective*.

13.2 Definition of Equations

The definition of equation structures is a mathematical peculiarity in the GAMS language. The syntax for defining equations in GAMS is as follows:

```
name..               expression // special relation character // expression;
```

First, we have *name* (the name of the equation) as shown in the list of declared equation names, then two periods ".." follow separating the name of the equation and the expression of the equation. The expression of the equation is an algebraic statement using allowed GAMS algebraic operations. The equation must contain a relation operator, such as:

```
=E=   right hand side is equal to the left hand side
=G=   right hand side is greater than or equal to the left hand side
=L=   right part is less than or equal to the left hand side
```

The user must carefully balance the number of open and closed brackets. An equation can occupy an unlimited number of rows. Spaces can be used for obvious and convenient comprehension. Equations, once defined, cannot be changed or redefined in a later portion of a model. If such necessity arises, it is essential to introduce a new equation with a new name.

However, it is possible to change the number of estimated elements of the equation by changing the data which the equation uses. In addition, you can exclude part of an equation by using logical conditions (\$ operator) in the name of an equation or in the computation part of an equation. Again, consider the scalar equations from the model in Section 2.1

```
EQUATIONS Eq1, Eq2, Objective;

Eq1.. y1 =E= 10*x*x+10*x+10;
Eq2.. y2 =E= -10*x+100;
Objective.. obj =E= (y1-y2)*(y1-y2);
```

Scalar equations are often encountered while making up optimization models. The examples written above are scalar equations. They contain only scalar variables. However, scalar equations can contain index variables and use index operators. For example, consider the equations form the model of water users presented in Section 3.1

```
EQUATIONS objective, cap;

objective..obj =E= SUM(i,a(i)*x(i)+b(i)*x(i)**2);

cap..sum(i,x(i))+s-r =E= 0.0;
```

In this case the names of the equations are scalar, and the equations use the index SET '*i*'. After summing, the index disappears and the equations are scalar. Correspondence between the dimensions of the name and left and right parts of equations is essential in constructing models.

Index (vector) equations differ from scalar equations only in the closer correspondence between the

dimensions of the equation's left and right parts. SETs which take a position in an equation to the left of the ".." and immediately after the name of an equation, can be considered as control indices, because the equation will be solved for all possible values of the indices. Consider the model of least squares estimation presented in Section 2.2

```
EQUATION mod(t), residual(t), objective;

mod(t).. a*x1(t)*x1(t)-b/x3(t)-c/x2(t)+EXP(-y(t)*y(t)) =E= y(t);

residual(t).. e(t) =E= y(t)-y_hat(t);

objective.. obj=E=sum(t,power(e(t),2));
```

The equation "*objective*" is a scalar equation, but the equations "*mod(t)*" and "*residual(t)*" are index equations indexed on the SET "*t*" and they will be solved for every value of the set.

Each GAMS model must contain at least one scalar variable (the value of an objective function being optimized) and one scalar equation (the objective function). GAMS can optimize only a scalar variable.

13.3 Using Symbols as Indices in Equations

Very often, it is necessary to use symbols (e.g., values of SETs or indices) in equations. This can be done using quotation marks. As an example, consider the establishment of upper bounds (capacity) on reservoir 1 storage in the model of river system management in Section 3.3

```
S.up('Res_1' ,t) = 1000;
```

The bound will be set only for reservoir '*Res_1*'. In this instance, SET *t* in the variable *s('Res_1',t)* is defined for all twelve months.

13.4 Assembling a Model

After all the SETS, PARAMETERS, VARIABLES, and EQUATIONS of a model have been declared a system (called a *MODEL*) can be formed from a collection of equations and given a name. When all the declared equations are included in the model, the syntax of declaring the model is as follows:

```
MODEL    name   comment /ALL/;
```

For example:

```
MODEL Lakes /all/;
```

The model is named "*Lakes*", and the keyword *ALL* means that the model includes all of the equations. Sometimes, the user may want to create numerous variants of equations and unite them in groups, forming different models with distinct names and they may have various optimization objectives.

13.5 Solving a Model

Once a model has been defined, the next step is to solve it. This is ordered by including a *SOLVE* statement in the model. Consider the example of the river system management model in Section 3.3

```
SOLVE Lakes USING NLP MINIMIZING obj;
```

The *SOLVE* statement contains 6 parts:

1. The word *SOLVE* (required);
2. The previously defined name of the model (*Lakes*) (required but the name is user defined);
3. The word *USING* (required);
4. The type of "*solver*" to use in the solution (*NLP* in this case, required but the name is user defined). See Table 13.5.1 for a list of available solvers;
5. The word *MINIMIZING* or *MAXIMIZING* (required); and
6. The name of a scalar variable (*obj* in this case) to be optimized (required but the name is user defined)

Table 13.5.1
Types of Solvers in GAMS (abbreviated list).

Type of Solver	Description
LP	Linear programming. The model cannot contain nonlinear or discrete (binary and integer) variables.
NLP	Nonlinear programming. In the model, nonlinear forms must be continuous functions and the model may not contain discrete variables.
MIP	Mixed integer programming. Similar to RMIP, but the requirements of discreteness of variables and equations are stringent. Discrete variables must take discrete values within boundaries.
MINLP	Mixed integer nonlinear programming. The same characteristics as for RMINLP, but the requirements of discreteness are very stringent.

14. OUTPUT TO TEXT FILES

14.1 Declaration of Output Files

GAMS allows writing information about the model and its solution to text files. In addition, a text output file "*model_name.LST*" will always be produced. This file is created automatically, but its format is not entirely useful for analysis purposes. For display of specific information in this file, the *DISPLAY* operator can be used. This operator is best appreciate by studying it in examples. This operator can be useful when testing programs, since it is possible to display information in the quickest way. However, there is an opportunity to create additional output files. The following is the full syntax for defining output text files

```
FILE pointer_name comments / external_filename /
```

where `FILE`   is a keyword used to define files;
     `pointer_name`   is a pointer used by GAMS model to direct output to the external text file *external_filename*;
     `external_filename`   is an external text file which receives the output from GAMS.

GAMS allows working with a set of output files, but you should read about it in the main GAMS User Guide.

Consider the model for solving a set of nonlinear equations in Section 2.1

```
FILE res /Eq1.txt/
```

where `res`   is a pointer to the external file named "*Eq1.txt*"; and
     `Eq1.txt`   is the text file that will receive the model output.

The pointer to the output file is named *res* in the model and it directs information into the file *Eq1.txt*.

14.2 Output of Information

The *PUT* operator is used for both assigning the name of the "*current_file*" to a declared text file and writing information into this file. The general syntax of the *PUT* operator is

```
put  Filename_1  elements;
```

Consider the model from Section 2.1 again as an example

```
PUT  res;
put  "Solution x =  ", put x.l, put /;
```

This example shows how the pointer *res* is associated with the external text file *Eq1.txt*. In the first line, the output is directed into the file associated with the pointer *res*. In the next line, some text and the value (level) of the variable *x* is inserted into the file.

Sometimes, it is necessary to output the value of parameters or tables. This usually requires the use fo the LOOP function. Consider the example from the least swuares model of Section 2.2

```
PUT " t   x(1,t)   x(2,t)     x(3,t)      y(t)      y_hat(t)"/;
LOOP((t),PUT t.TL, x1(t), x2(t), x3(t), y.L(t), y_hat(t)/;);
```

Where the first line prints a header at the top of a table and the second line loops through all the vales of the set *t* and prints the value of *t, x1(t), x2(t), x3(t), y.L(t),* and *y_hat(t)*. Note that the variable *y* needs the suffix ".*L*" after it, but the parameter *y_hat* does not. This is because *y_hat* is a defined data whose "level" is not determined by GAMS.

15. SETS

15.1 Set Naming and Declaration

SETS are a major item in the GAMS language. SETS can be used to construct connections between variables and equations in models. SETS are the equivalent of indices in a typical programming language. Usually, ordinals function as indices, for instance "first", "second", etc. In GAMS, indices have names, written through a combination of letters and digits, without spaces. On the one hand, these are ordinals, but on the other hand, they have names. A set name must begin with either a letter or a digit, but the next symbol can be a letter, digit or the marks "+" and "-". For example:

```
Jlobest     1999  1972-73
ROKETer     D6H83      Navruz-99
```

The next example illustrates using SETS in the GAMS language

```
SETS  i  plants         / Tashkent, Almaty/
      j  consumers      / Tokyo, London, Moscow/;
```

GAMS remembers the sequence of elements in a SET. For instance here, *Almaty* is second among *i*, and *Moscow* is third among *j*. The group of SETS begins with the word SETS and ends with the semicolon symbol ";". It is possible to declare the same information in the model through two individual groups:

```
SET       i  plants    / Tashkent, Almaty /;
SET       j  consumers / Tokyo, London , Moscow /;
```

Symbols "/" separate lists of SET elements. SET Element in a list are separated by commas. It is possible to write a comment between the SET name and the slashes, but all comments must fit on one row.

GAMS models can be very large and contain an enormous number of structures. One GAMS option

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that eases the definition of elements of a SET is shown in the example below

```
SET t  year    /1965, 1966, 1967, ...    ...   ... 1998, 1999/;
```

A different, and shorter, way to accomplish this in GAMS is

```
SET t  year    /1965*1999/;
```

Sometimes in models, we may want to define two identical SETS

```
SET   j  station  / Toktogul, Uchkurgan, Charvak, Chardara /;
SET   k  station  / Toktogul, Uchkurgan, Charvak, Chardara /;
```

Simultaneous definition of the same SET under different names is useful when the modeler wants to use the SET element interrelations in a model. To avoid repetition, two sets can be made identical by using the "ALIAS" command. For example

```
SET   j  station  / Toktogul, Uchkurgan, Charvak, Chardara /;
ALIAS(j,k);
```

That is, the set *j* was created and assigned members, and the set *k* was created and made identical to the set *j*.

In GAMS, it is possible to define subsets of sets. Consider the model of water management from Section 3.3 with the river network as shown in Figure 15.1.1 below.
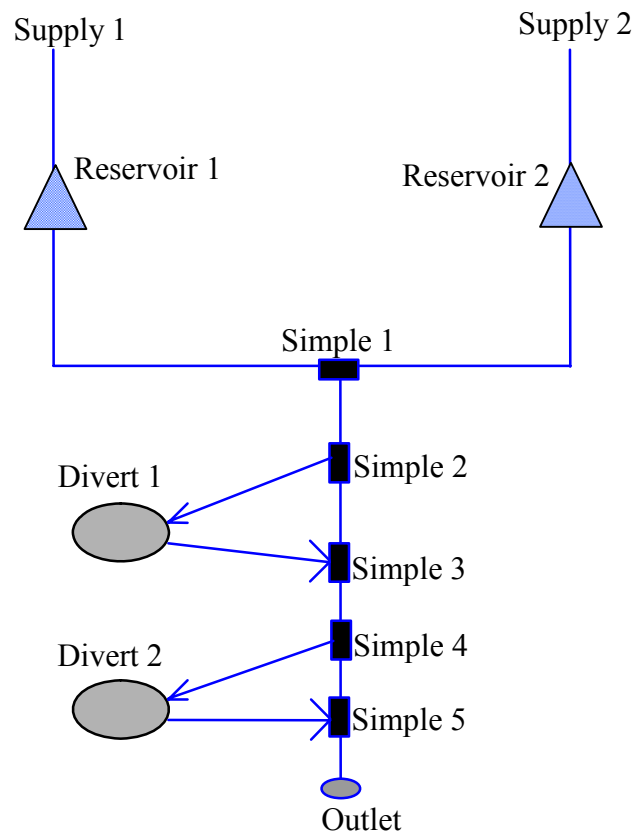
Figure 15.1.1. Structure of a River Network for the Water Management Task

The set of nodes in the river system is defined as

```
SET  n nodes
/
Supply_1, Supply_2,
Simple_1*Simple_5,
Divert_1, Divert_2,
Res_1, Res_2,
Outlet /;
```

and five subsets of this set are created which have various characteristics such as junctions, water sources, water users, irrigation zones, and reservoirs

```
SET
nn(n)     Simple nodes      /Simple_1*Simple_5/
ns(n)     Supply nodes      /Supply_1, Supply_2/
nr(n)     Water user nodes /Divert_1, Divert_2, Outlet/
nrr(n)    Irrigation nodes /Divert_1, Divert_2/
nl(n)     Reservoir nodes  /Res_1, Res_2/;
```

In this example, *n* is a collection of nodes in the modeling network, i.e., a list of objects to be used in the model. The subsets *nn*, *ns*, *nr*, *nrr*, and *nl* are subsets of the set *n* which have various geographical significance as specified in the name (which is a comment and GAMS ignores this).

Among the objects are simple (junction) nodes, *nn*, water sources (supplies) nodes, *ns*, water users, *nr*, irrigation nodes, *nrr*, and reservoir nodes, *nl*. *ns(n)* is a subset of flow sources, elements of which, *Supply_1,* and *Supply_2*. The same is true for the subsets of water users *nr(n)* and reservoirs *nl(n)*. Note, that the subset *nr( )* does not include all elements of set *n*, but only a part of them.

15.2 SET Operations

Operations can be performed on sets using the symbols

$$+, \quad -, \quad *, \quad \text{not}$$

Symbol " + " performs the set union operation

```
  s3(n) = s1(n) + s2(n);
```

The set *s3(n)* is the set equal to all elements in *s1(n)* and all elements in *s2(n*

Operator "*" performs the set intersection operation; only the elements included in both set A and set B belong to the intersection of the sets A and B

```
  s3(n)  = s1(n) * s2(n);
```

s3(n) is a set consisting of the elements which are both in *s1(n)* and in *s2(n)*.

The symbol "NOT" performs the set complement operation

```
  s3(n) = NOT s1(n);
```

The set *s3(n)* consists of elements included in *n*, but not in *s1(n)*. That is, all elements in *s1(n)* are deleted in *s3(n)*.

Operator "-" performs the operation of difference of sets. This set consists of elements, which belong to set A but not set B

```
  s3(n) = s1(n) - s2(n);
```

s3(n) is a set consisting of elements which are members of *s1(n)*, but not members of *s2(n)*.

15.3 Defining Multivariable Indices

Multidimensional SETS are often useful in modeling, especially when connections between set elements are important components of the system being considered. In a multidimensional set, the

indices of the set elements are separated by commas. Again, consider the model of water management from Section 3.3 with the river network as shown in Figure 12.1.1 above. A portion of the GAMS code which defines a two-dimensional set in this model is

```
ALIAS(n,n1);

SET
n_from_n(n,n1)  node n gets water from node n1 (any node)
/
Res_1.Supply_1,
Res_2.Supply_2,
Simple_1.Res_1,
Simple_1.Res_2,
Simple_2.Simple_1,
Divert_1.Simple_2,
Simple_3.Simple_2,
Simple_3.Divert_1,
Simple_4.Simple_3,
Divert_2.Simple_4,
Simple_5.Simple_4,
Simple_5.Divert_2,
Outlet.Simple_5
/;
```

Here, the set *n1*, which is identical to set *n*, is created using the "*ALIAS*" command. Then the two-dimensional set *n_from_n(n,n1)* is created which contains all of the downstream to upstream connections in the network. That is, it shows where each node in the system gets water from. The set elements are comprised of one element from the first set, *n*, and one element from the second set, *n1*, separate by a period. For example, the first element is *Res_1.Supply_1* which indicates that reservoir 1 gets water from water supply 1, and so on. This set is used in the model to determine the sources of water for any node in the system.

As another example, consider the set of main reservoirs on the Syrdarya River in Central Asia

```
SET  r  reservoirs  / Toktogul, Kairakum, Andijan, Charvak, Chardara /;
```

Each reservoir is located in one of the four countries (Kyrgyzstan, Uzbekistan, Tajikistan, or Kazakhstan) of the Syrdarya Basin. We can represent these countries as another set

```
SET  c  country    / Kyrgyzstan, Uzbekistan, Tajikistan, Kazakhstan /;
```

Then a two-dimentsional set can be created to associate each reservoir with its country, viz.,

```
SET  r_to_c(r,c)   reservoir_to_country
    / Toktogul.Kyrgyzstan,
      Kairakum.Tajikistan,
      Andijan.Uzbekistan,
      Charvak.Uzbekistan,
      Chardara.Kazakhstan /;
```

The previous example was an example of ONE-TO-ONE mapping. We may also have MANY-TO-MANY mappings in GAMS. Examples of this construction in the GAMS language can be found in

the Gams Language Guide.

15.4 ORD( ) and CARD( ) Operators

The GAMS compiler memorizes index names of a set as well as their sequence of definition. If the sequence of a set is arranged once, it will remain that way for other cases.

15.4.1 ORD( ) Operator

The *ORD* operator returns an ordinal number equal to the index position in a set. Here are some examples showing how the ORD operator is used.

```
SET t        /100*120/
PARAMETER   val(t);
val(t) = ORD(t);
```

The result of the operator will be the following:

```
val('100') = 1;
val('101') = 2,
…
val('120') = 21
```

15.4.2 CARD( ) Operator

The *CARD* operator returns an ordinal number equal to the number of elements in a set. For example:

```
SET  t      /45*68/
PARAMETER   s;
s = CARD(t);
```

The result will be that the *CARD( )* operator is assigned the value 24 (the number of values from 45 to 68, inclusive). The *CARD( )* operator is frequently used for initiating and canceling computation at the final stage. Consider the example

```
c.fx(t)$( ORD(t) = CARD(t) ) = demand(t);
```

which shows that when the value of the index *t* is equation to the number of items in the set, i.e., the last value, the value of the variable *c* is fixed equal to the parameter *demand(t)*.

15.5 LAG and LEAD Operators

Operators *LAG* and *LEAD* are used for correlating an element of a set with the next or preceding element of the set. GAMS has two variants of *LAG* and *LEAD* operators:

> 9. Linear LAG and LEAD operators (+, -)
> 10. Circular LAG and LEAD operators (++, --).

The difference between these two types of operators is in the method of processing at the initial and final points of the sequence. In circular operators, after the final element of the sequence, there comes the first, while in linear operators sets are broken.

15.5.1 Linear LAG and LEAD Operators

Linear *LAG* and *LEAD* operators (+, -) are very convenient for modeling time periods that are not cycled. A particular year is an example if the user does not want the initial values to be identical with the final ones. For example, it is unlikely that the initial storage of a long-term reservoir will be identical to its ending storage. GAMS can differentiate linear operators (+, -) from arithmetic operators according to the context.

Consider the following example from the reservoir operation model in Section 3.2.2 that illustrates application of linear *LAG* operator

```
balance(t).. (1+a(t))*S(t) =E= (1-a(t))*beg_S $(ord(t) EQ 1)
                             + (1-a(t))*S(t-1)$(ord(t) GT 1)
                             + Q(t) - R(t)- b(t);
```

In this case, when the index $t = 1$, the initial storage (*beg_s*) of the reservoir is used in the balance equation, but when the index $t > 1$ the variable storage is used in the calsulation.

15.5.2 Circular LAG and LEAD Operators

When circular *LAG* and *LEAD* operators are used the initial and final values of sets are adjacent, i.e. the "first-1" is "final", the "last +2" is the "first +1", etc. The circular LAG and LEAD operators (++, - -) are convenient for modeling cycled periods of time. A particular year is an example, if the user wants the initial values to be identical with the final ones. For example, often the initial storage of a seasonal reservoir is to be identical with its end storage. Naturally, January is the month following December despite changing years. This is illusteadte using the previous example, where it is not desired to specify the intial storage volume of the reservoir

```
balance(t).. (1+a(t))*S(t) =E= (1-a(t))*S(t--1) + Q(t) - R(t)- b(t);
```

That is, the storage value for time period $t = \text{card}(t)+1$ it is set equal to the value 1. This formulation of the equation will lead to a long-term, stationary solution for the model whch is independent of the initial storage condition of the reservoir (Loucks et al., 1981, Sec 2.8.1).

The operators *LEAD* and *LAG* can be used in assignments. Use of operators *LAG* and *LEAD* on the left side of the assignment is called an "index" while the use of it on the right side is called "assignment" and includes definition of the area of assignment.

The *LAG* and *LEAD* operators can be used in equations. Principles defined above for these operators are true for equations. The location of operators *LAG* and *LEAD* in the body of the equation is very important. *LAG* and *LEAD* operators located on the left side of the symbol ".." are modifications of the area for defining equations. They become indicators if located on the right side of the symbol "..". If their conditions are not met, they disappear.

16. SCALARS, PARAMETERS, AND TABLES

16.1 General Rules

The major working elements of the GAMS language are identifiers: names given to sets, parameters, variables, models, etc. Identifiers are required to start with a letter followed by up to 9 letters or digits (total length is 10 characters). Examples of inadmissible identifiers include

```
    15      $rhs        0ie        ewoifkoioiiio           ueru&wc
```

Digital data are contained in arrays (zero, scalar, or multidimensional matrices called *parameters* in GAMS). The SETs, described above, can play the role of indices for these arrays. To declare an array to contain data values, we use the functional words:

```
    SCALAR                  (zero-dimensional),
    PARAMETER          (one-dimensional), and
    TABLE              (multidimensional).
```

Parameters are declared in a free format with the following general structure

```
    Parameter a(i,j)  input-output matrix
```

where `Parameter`   data-type-keyword (required)
    `a`   identifier (required)
    `(i,j)`   domain list (optional)
    `input-output matrix`   text (optional)

Scalars are defined similarly but without the domain list. *Tables* are simply multi-dimensional versions of parameters.

The assignment of numerical values to parameters can be carried out in two ways:

1) Assignment, and
2) Computation.

16.2 Data Entry Through Assignment

As an example of how to enter numerical values in a GAMS model, consider the model for solving a set of nonlinear equations in Section 2.1. The coefficients of the set of equations are defined using SCALARS

```
SCALAR A /10/, B /10/, C /10/, M /-10/, N /100/;
```

Parameters *A, B, C, D, M,* and *N* are used in the model as zero dimensional parameters (scalars). The values are assigned simultaneously upon declaration (statement of fact of existence) of the scalars.

The sequence of declaration and definition of numerical values for one-dimensional arrays (*PARAMETERS*) is a bit different. Conside the model of least squares estimation from Section 2.2

```
SETS t  / 1, 2, 3, 4, 5, 6, 7, 8 /;

PARAMETER x1(t)  /1 2, 2 3, 3 3, 4 3, 5 5, 6 5, 7 6, 8 7/;
```

In this example, declaration and definition occur simultaneously. Each pair of values of the index *t* and the parameter *x1(t)* are separated by commas and the entire list is enclosed in slashes.

The following example from the model of river management in Section 3.3 illustrates data entry to a TABLE at the moment of definition:

```
SET n nodes
/ Supply_1, Supply_2,
  Simple_1*Simple_5,
  Divert_1, Divert_2,
  Res_1, Res_2,
  Outlet /;

SET t months / Jan,Feb,Mar,Apr,May,Jun,Jul,Aug,Sep,Oct,Nov,Dec /;

TABLE Supply(n,t) water supplies (m3 per sec)
          Jan   Feb   Mar   Apr   May   Jun   Jul   Aug   Sep   Oct   Nov   Dec
Supply_1  128   125   234   360   541   645   807   512   267   210   181   128
Supply_2   39    39    52   121   168   144   105    78    49    44    45    39
```

In this example, the water supply to each of two sources is defined as a table with two rows, one for each supply, and twelve columns, one for each month. If some cells are missing from a table, it means that the value of the array for the corresponding complex of indices is not defined. However, if this element of the array is used in further computations, it will be considered as equal to zero. Note that only two of the elements of the SET *n* are used in the table (the source nodes *Supply_1* and *Supply_2*), the rows of the table for the other elements are not defined, but we can still use the SET *n*

later in computations as if each node might have a source of water associated with it.

16.3 Data Entry Through Computation

After declaration, parameter values may be computed later in the model. Determination of data through direct computation or assignment following the initial definition is possible and sometimes useful. Consider the model of two-dimensional groundwater flow in Section 4.5 where the finite-difference coefficients are computed as parameters

```
PARAMETER
        A(I,J)    finite-difference coefficient
        B(I,J)    finite-difference coefficient
        C(I,J)    finite-difference coefficient
        D(I,J)    finite-difference coefficient;

        A(I,J) =  T/(DX*DX) ;
        B(I,J) = -T/(DX*DX) ;
        C(I,J) =  T/(DY*DY) ;
        D(I,J) = -T/(DY*DY) ;
```

Note the lack of any instructions as to which index values of I or J should be used in the calculation. GAMS carries out these computations for *all* indices I and J which are defined through the SET operator. If you need to assign or compute a specific value of one or a small group of elements of an array, you can do this as follows, for example:

```
c( 'Tashkent', 'Fergana' )  =  f * 0.001 ;
c( 'Tashkent', 'Chinaz' )   =  20 * 0.001 ;
```

To organize computations within some limits of the variables i and j, you can use subsets as discussed previously

A collection of standard logical and mathematical functions may be used in GAMS models for assignment or computation purposes. The reader is referred to the GAMS Users Manual for descriptions of these functions. Besides those standard functions, two special functions are available for the user:

```
SUM(..., ...)
PROD(..., ...)
```

An example of the SUM function is given in the objective function of the least squares estimation model in Section 2.2

```
objective.. obj=E=sum(t,power(e(t),2));
```

This means that the value of "*obj*" will be assigned the algebraic sum of all elements of the unidimensional array "$e(t)$" summed over the entire range of indices "t".

An example of the PROD function is

```
t(i) = PROD(j,b(i,j));
```

This means that in the unidimensional array "*t( )*" each element, indexed by "*i*", will be defined as a product of elements of the two-dimensional array "*b( )*" on "*j*" with the fixed related index "*i*".

17. CONDITIONAL OPERATOR ($)

17.1 General Statements

This section considers the conditional operator (*$*), which is one of the strongest features of the GAMS language. This operator provides the ability to guide calculations or definitions with logical conditions. The general format of the operator is

```
item$(condition)
```

and this can be read "fulfill the item if the condition is true", where the condition is of logical character capable of acquiring Boolean values (true or false, existence or nonexistence). Conditions may not contain variables. However, the use of suffixes on variables (e.g., *.L* or *.M*) is allowed in $ operations, but this transfers the variables into constant values.

The essence of the $ operator can be explained by considering the following example

```
if (b > 1.5), then a = 2;
```

which can be written in the GAMS language as follows

```
a$( b > 1.5 ) = 2;
```

If the condition in the parathenses '*b > 1.5*' is not fulfilled then the assignment '*= 2*' is not executed.

17.2 Embedded $ Conditions

Conditions can be embedded or nested in structures. The term *$(condition1$(condition2))* can be read as *$(condition1 and condition2)*. That is, the first and second conditions must be fulfilled simultaneously before the assignment will take place. When using embedded $ conditions all expressions following the $ sign must be inserted into a pair of parentheses.

Consider the following example:

```
d(i)$( w $q ) = v(i);
```

where *i*, *d*, and *w* are sets, while *q* and *v(i)* are parameters. The assignment will be executed for elements *d(i)* for which simultaneously the values of *w* and *q* have true values. The operator given above can be rewritten as follows:

```
d(i)$( w AND q ) = v(i);
```

17.3 Conditional Assignment

The example given above was connected with conditional assignment. In this example, the $ condition was on the left side of the assignment. The effect of the $ condition depends on its position with regard to the equal sign. The assignment is not fulfilled if the logical conditions are not satisfied. This means that values of parameters on the left will remain unchanged for those indices where conditions are not fulfilled.

If the parameters of the left side of the assignment are not numerically determined, their values will be equal to 0, since the logical conditions are not satisfied. The $ operator is very useful for cases when it is necessary to avoid dividing by zero. For example:

```
r(i)$( z(i) NE 0 ) = 1./z(i);
```

The *z(i)* parameter was defined in the model and the operator is used to calculate *r(i)* without making mistakes. If any value associated with *z(i)* is equal to 0, the assignment will not be executed and the preceding value for *r(i)* will persist. If *(i)* was not initialized before, then all indices with *z(i)* equaling 0, will be assigned 0 value. If *z(i)* was not defined for cases when it equaled 0, then the assignment above can be written as follows:

```
r(i)$z(i)   =   1./z(i);
```

17.4 Conditional Computation

Assignment is always executed when the assignment operator appears with a $ condition on the right side of the assignment operator (equal sign). If the logical condition is not fulfilled, then the parameter or the variable will be transformed into 0, even if it was equal to some number previously. For example

```
x = 20$(y > 7.5);
```

which means

```
if (y > 7.5)  then (x=20),  else (x=0)
```

Another example is from reservoir management model of Section 3.2

```
balance(t).. (1+a(t))*S(t) =E= (1-a(t))*beg_S $(ord(t) EQ 1)
                             + (1-a(t))*S(t-1)$(ord(t) GT 1)
                             + Q(t) - R(t)- b(t);
```

In this case, when the index *t* = 1, the initial storage (*beg_s*) of the reservoir is used in the balance equation, otherwise it is omitted. Similarly when the index *t* > 1 the variable *S(t-1)* is used in the calculation, otherwise it is omitted.

17.5 Conditional Indexing

Another useful application of the $ condition is the control of index operations. For example

```
    a  =  SUM( i$(n(i)), b(i));
```

The operator calculates the sum of *b(i)* for all *i* for which the parameter *n(i)* has a true value.

Another example of using $ operator is the correlation of SET-to-SET operations. The river management model of Section 3.3 uses the set *n_from_n(n,n1)* which defines the upstream to downstream connections in the flow network. When computing the flow balance for nodes in the network, it is necessary to know which upstream nodes contribute to the inflow of a node *n* under consideration. This is accomplished by summing over all nodes *n1* which are upstream from node *n* in the *n_from_n(n,n1)* set. That is

```
  * Simple node:  Inflow = Sum of Releases from Upstream - Sum of Diversions
  R_nn(n,t).. Q(n,t) =e= sum(n1$(n_from_n(n,n1)),R(n1,t))
                       - sum(n1$(n_to_nr(n,n1)),Divert(n1,t));
```

Similarly, in this equation, the sum of all water flowing from the node *n* to downstream nodes *n1* is accomplished by conditioning the sum on the set *n_to_nr(n,n1)*.

17.6 Conditional Equations

The $ operator is also used for managing the definition of equations. $ control of equation definition restricts the number of equations included in the model. Consider the following example from the river system management model in Section 3.3

```
  * Reservoir node:  Release = Mass Balance
  R_nl(n,t)$(nl(n)).. S(n,t) =e= beg_S(n)$(ord(t) EQ 1)
                              +  S(n,t-1)$(ord(t) GT 1)
                              + Q(n,t)-R(n,t);
```

The equation *R_nl(n,t)* is conditioned by the expression *$(nl(t))*. The equation is included for the indeces *n* and *t* if the calculation node, *n*, refers to a member of *nl(t)*, the set of reservoirs.

17.7 Special Constructions for Conditional Variables

One of the main difficulties of using $ operators is that they are prohibited from use with variables computed during the optimization process. Suppose we have to solve a problem with the equations:

$$y = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{otherwise} \end{cases}$$

where x, and y are variables. We have the following structure of computation

```
VARIABLES y, x;
EQUATION  first;
first.. y  =E=  (x*x)$(x < 0) + (x*x*x)$(x > 0);
MODEL andre /ALL/;
SOLVE andre USING NLP MINIMIZING y;
```

The GAMS compiler will inform you of an error made by using the $ operator with variables.

```
**** THE FOLLOWING NLP ERRORS WERE DETECTED IN MODEL ANDR:
**** 53 IN EQUATION FIRST      .. ENDOG $ OPERATION
**** 53 IN EQUATION FIRST      .. ENDOG $ OPERATION
```

However, using the feature of the computer to define a square root as positive value, the necessary result can be achieved by using the following construction

$$y = x^2 * \left(\frac{\sqrt{x^2} - x}{2\sqrt{x^2}} \right) + x^3 * \left(\frac{\sqrt{x^2} + x}{2\sqrt{x^2}} \right)$$

or in the GAMS language

```
first.. y =e= (x*x)*(0.5*(sqrt(x*x)-x)/(sqrt(x*x)+0.000001))
              + (x*x*x)*(0.5*(sqrt(x*x)+x)/(sqrt(x*x)+0.000001));
```

18. ADDITIONAL OPERATORS

GAMS has a number of other operators frequently used in other programming languages. This section discusses these operators and their use. There operators are:

1) LOOP   - cycle operator;
2) IF-ELSE - conditional operator;
3) FOR   - cycle operator; and
4) WHILE   - conditional cycle operator.

18.1 Loop Operator

The LOOP operator is often used with the PUT operator for printing out and displaying information on the screen or in files. The operator has the following syntax:

LOOP (control_ area [$(condition)] operator {; operator});

If *control_area* contains more than one set, then parentheses are needed. The LOOP operator provides performance of the operator within the cycle limits for each index of the set in turn. The sequence of operations is determined by the sequence of indices. LOOP is a general type of index operation. The cycle set can be controlled by the $ operator, but it must be static or embedded. The cycles can be controlled by more than one set. However, it is impossible to declare or solve equations within the LOOP operator. It is not allowed to transform control sets within the body of the cycle.

Consider the following hypothetical case of the integration model:

```
SET   x   / d15*d25 /
PARAMETER   nachalo(x)   / d15 423 /
            differ(x)    / d15  58.4, d16  47.7, d17  68.8
                           d18  46.5, d19  57.9,  d20  46.9
                           d21  64.5, d22  65.4,  d23  34.7
                           d24  76.4, d25  34.4/;
```

The LOOP operator can be used to compute increasing sums:

```
LOOP(x, nachalo(x+1)  =  nachalo(x)  +  differ(x));
```

Only one operator and one control set - x is presented here within the cycle operation area

```
nachalo(x+1) = nachalo(x) + differ(x)
```

Consider the following example used in the algorithm of the Newton method for extracting a square root. In practice, the imbedded function SQRT() is used. However, in this case, we show the functioning of LOOP operator using a program from the GMAS Language Guide.

The method is:

if x is an approximate square root of v, then $(x+v/x) - 2*x = 0$

The GAMS code is:

```
SET              i number of iterations is 100      /i-1*i-100/;
PARAMETER        x(i);
SCALARS
     v           number whose square root is sought /23.456/
```

```
        sqrtval      answer
        err          error /1.0/
        tol          error tolerance /1.0e-06/;

    x("i-1")  =  v/2 ;

    loop(i$(err > tol),
            x(i+1) = 0.5*(x(i) + v/x(i));
            sqrtval = x(i+1);
            err  = ABS (x(i+1) - x(i))/x(i+1)
         );

    ABORT$(err>tol) "square root not found"

    DISPLAY "square root found",sqrtval, x;
```

The result:

```
19 square root found
19 PARAMETER sqrtval                =         4.843  answer
19 PARAMETER x

        i-1 11.728,    i-2  6.864,    i-3  5.141,    i-4  4.852
        i-5  4.843,    i-6  4.843,    i-7  4.843
```

In this example the user can see the use of the ABORT operator, which is used for fail-safe canceling of computation.

18.2 IF-ELSE Operator

The IF-ELSE operator is useful for transferring from one operator to another. In some cases, it can be written down as a set of $ conditions. The IF operator can be used for making GAMS code more understandable.

The optional "ELSE" part allows formulating the traditional construction "IF-THEN-ELSE". The following syntax is for the "IF-THEN-ELSE" operator:

```
    if (condition,
           operators;
    {condition ELSEIF, operators}
           [operators ELSE;]
    );
```

Note, the braces and brackets are not required, but can be used. "Condition" means logical conditions described in the paragraph on conditional operations of GAMS.

Declaration or definition of equations can not be performed within the IF operator. Consider the following set of operators taken from the GAMS Language Guide:

```
    p(i)$(f <= 0) = -1;
    p(i)$(f >  0)  and (f < 1)) = p(i)**2;
    p(i)$(f >  1) = p(i)**3;
    q(j)$(f <= 0) = -1;
    q(j)$(f >  0)  and (f < 1)) = q(j)**2;
    q(j)$(f >  1) = q(j)**3;
```

the same can be expressed through the IF-ELSEIF-ELSE operators:

```
    IF (f <= 0,
        p(i) = -1;
        q(j) = -1;
    ELSEIF ((f > 0) AND (f < 1)),
        p(i) = p(i)**2;
        q(j) = q(j)**2;
    ELSE
        p(i) = p(i)**3;
        q(j) = q(j)**3;
    );
```

The body of the IF operator can contain the SOLVE operator. Consider the following example of the model from the GAMS Language Guide:

```
    IF  (ml.MODELSTAT EQ 4),
    *      the model ml is undefined;
    *      change the conditions on the boundaries of x and solve it again;
        x.up(j) = 2*x.up(j) ;
        SOLVE ml USING LP MINIMIZING  lop;
    ELSE IF (ml.MODELSTAT NE 1),
        ABORT "error occurred in solving the model ml";););
```

The user can see a new suffix ".MODELSTAT" here. Its meaning is quite transparent and its numerical value is relevant to situations raised in the process of solving the problem. A complete list of situations is given in the GAMS Language Guide.

18.3 WHILE Operator

WHILE operator is used for cycling the process of calculation before the execution of some logical condition. The syntax of the WHILE operator is the following:

    while ((condition)),
                operators;
            );

As with the FOR operator, declaration and definition of equations can not be done in the WHILE operator. However, the WHILE operator can be used for control of the SOLVE operator. Consider the following part of a GAMS program that is randomly looking for the global optimum of a non-convex function. The example is taken from the GAMS Language Guide:

```
SCALAR  count;
count = 1;
SCALAR  globmin;
globmin = INF;
OPTION  BRATIO = 1;

WHILE ((count  LE  1000),
        x.l(j) = UNIFORM(0,1);
        SOLVE ml  USING  lp  MINIMIZING  obj;
        IF  (obj.l  LE  globmin,
                globmin = obj.l;
                globinit(j)  = x.l(j);
            );
        count = count +1;
);
```

In this example, the non-convex model is solved for 1000 randomly taken numbers, while the global solution is found through constant comparison.

Model [PRIME] from the GAMS model library illustrates the use of the WHILE operator in an example with simple generation of numbers less than 200.

18.4 FOR Operator

The FOR operator is used to repeat a bloc of operators. It has the following syntax:

> FOR (i = initial TO final [BY step],
>   operators;
>);

Note that *i* is not a set, but a numerical parameter. The step defines the growth of *i* after each repetition of the FOR loop. The *initial*, *final* and *step* values do not have to be integer-valued. The *initial* and *final* values can be positive or negative real numbers. The *step* must be a positive real number.

The FOR operator can be used to control the SOLVE operator. Consider the following example from the GAMS program randomly studying a non-convex function for finding a global optimum. The example is taken from the GAMS Language Guide:

```
SCALAR i;
SCALAR globmin;
globmin = inf;
OPTION  BRATIO = 1;

FOR (i = 1 TO 1000,
        x.l(j)  =  UNIFORM(0,1);
        SOLVE ml  USING  NLP  MINIMIZING  obj ;
        IF  (obj.l  LE globmin,
```

```
              globmin = obj.L ;
              globinit(j)  =  x.L(j);
          );
      );
```

In this example, the non-convex function is examined for 1000 randomly taken points, while the global solution is found through comparison. The use of real numbers as initial and final values of the cycle and steps, is shown in the following example:

```
  FOR (s  =  -18.4  TO  10.3  BY 6.1, DISPLAY s; );
```

Resulting printout of the file will contain the following lines:

```
      1 PARAMETER  S           =   -18.400
      2 PARAMETER  S           =   -12.300
      3 PARAMETER  S           =   -6.200
      4 PARAMETER  S           =   -0.100
      5 PARAMETER  S           =   +6.000
      6 PARAMETER  S           =   -12.100
```

Note, that S was increased by the GAMS compiler 6.1 during each cycle until it exceeded 10.3.

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APPENDIX A. MATHEMATICAL PROGRAMMING: A BRIEF REVIEW

A.1 Introduction

An optimization problem (mathematical program) is to determine the values of a vector of decision variables

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

(1.1)

which optimize (maximize or minimize) the value of an objective function

$$f(x) = f(x_1, x_2, \cdots, x_n)$$

(1.2)

The decision variables must also satisfy a set of constraints X which describe the system being optimized and any restrictions on the decision variables. The decision vector x is feasible if

$$x \text{ is feasible if } x \in X$$

(1.3)

The **feasible region** X, an n-dimensional subset of R^n, is defined by the set of all feasible vectors. An **optimal solution** x^* has the properties (for minimization):

$$\mathbf{x}^* \in X$$

(1.4)

and

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \; \forall \mathbf{x} \neq \mathbf{x}^*$$

(1.5)

i.e., the solution is feasible and attains a value of the objective function which is less than or equal to the objective function value resulting from any other feasible vector.

A.2 General Mathematical Programming Problem

The **general mathematical programming problem** can be stated as

$$\text{Maximize } f(\boldsymbol{x})$$
$$\boldsymbol{x}$$
$$\text{subject to}$$
$$\boldsymbol{x} \in X \tag{2.1}$$

In words this says,

*maximize the objective function f(**x**) by choice of the decision variables **x** while ensuring that the optimal decision variables satisfy all of the constraints or restrictions of the problem*

The objective function of the math programming problem can be either a linear or nonlinear function of the decision variables.

Note that:

$$\text{Maximize } f(\boldsymbol{x}) \tag{2.2}$$

is equivalent to

$$\text{Maximize } a + bf(\boldsymbol{x}), \quad b > 0, \text{ or}$$

$$\text{Minimize } a + bf(\boldsymbol{x}), \quad b < 0 \tag{2.3}$$

That is, optimizing if a linear operator; multiplying by a scalar or adding a constant does not change the result and maximizing a negative is the same as minimizing a positive function.

A.3 Constraints

The **constraint set** X of the math program can consist of combinations of:

(1) Linear equalities:

$$\boldsymbol{Ax} = \boldsymbol{b} \tag{3.1}$$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad i = 1, \cdots m \tag{3.2}$$

where the a_{ij}, $i = 1 \cdots m$, $j = 1 \cdots n$ are the elements of the matrix $\boldsymbol{A}$, and $\boldsymbol{b}$ is a vector of right-hand-

side constants.

(2)   Linear inequalities:

$$Ax \le b \tag{3.3}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, \cdots m \tag{3.4}$$

(3)   Nonlinear inequalities:

$$g(x) \le 0 \tag{3.5}$$

$$g_j(x) \le 0 \qquad j = 1, \cdots r \tag{3.6}$$

where the functions $g(x)$ are nonlinear functions of the decision variables.

(4) Nonlinear equalities:

$$h(x) = 0 \tag{3.7}$$

$$h_i(x) \le 0 \qquad i = 1, \cdots m \tag{3.8}$$

where the functions $h(x)$ are nonlinear functions of the decision variables.

A.4 Typical Forms of Math Programming Problems

Different forms of the **objective function and constraints** give rise to different classes of mathematical programming problems:

A.4.1 Linear Programming

The objective function is linear and the constraints are linear equalities, inequalities, or both and non-negativity restrictions apply.

$$\begin{aligned}
&\text{Maximize } c \cdot x \\
&\quad x \\
&\text{subject to} \\
&\qquad Ax \le b \\
&\qquad x \ge 0
\end{aligned} \tag{4.1}$$

Example:

$$\text{Maximize } 2x_1 + 3x_2 + 5x_3$$

subject to

$$x_1 + x_2 - x_3 \geq -5$$
$$-6x_1 + 7x_2 - 9x_3 = -5$$
$$|19x_1 - 7x_2 + 5x_3| \leq 13$$

Of course this example has the difficulty of what to do with the absolute value. An inequality with an absolute value can be replaced by two inequalities, e.g.,

$$|g(x)| \leq b$$

can be replaced by replaced by

$$g(x) \leq b \text{ and } g(x) \geq -b$$

So our example can be converted to:

$$\text{Maximize } 2x_1 + 3x_2 + 5x_3$$

subject to

$$x_1 + x_2 - x_3 \geq -5$$
$$-6x_1 + 7x_2 - 9x_3 = -5$$
$$19x_1 - 7x_2 + 5x_3 \leq 13$$
$$19x_1 - 7x_2 + 5x_3 \geq -13$$

Note: An equation can be replaced by two inequalities of the opposite direction. For example an equation

$$g(x) = b$$

can be replaced by replaced by

$$g(x) \leq b \text{ and } g(x) \geq b$$

Often it is easier for programs to check the inequality condition rather than the strict equality condition.

A.4.2 Classical Programming

The objective function is nonlinear and the constraints are nonlinear equalities.

$$\text{Maximize } f(x)$$
$$x$$
subject to
$$h(x) = 0 \qquad\qquad (4.2)$$

Example:

$$\text{Minimize} \quad (x_1 - 1)^2 + (x_2 - 2)^2$$
subject to
$$x_1 - 2x_2 = 0$$

A.4.3 Nonlinear Programming

The objective function is linear or nonlinear and the constraints are linear or nonlinear equalities, inequalities, or both.

$$\text{Maximize } f(x)$$
$$x$$
subject to
$$h(x) = 0$$
$$g(x) \le 0 \qquad\qquad (4.3)$$

Example:

$$\text{Maximize} \quad \ln(x_1 + 1) + x_2$$
subject to
$$2x_1 + x_2 \le 3$$
$$x_1 \ge 0, x_2 \ge 0$$

A.5 Types of Solutions

Solutions to the general mathematical programming problem are classified as either *global* or *local* solutions depending upon certain characteristics of the solution. A solution x^* is a **global solution** (for maximization) if it is:

    1. Feasible; and

2. Yields a value of the objective function less than or equal to that obtained by any other feasible vector, or

$$x^* \in X, \text{ and}$$
$$f(x^*) \geq f(x) \quad \forall x \in X \tag{5.1}$$

A solution x^* is a **local solution** (for maximization) if it is:

1. Feasible; and
2. Yields a value of the objective function greater than or equal to that obtained by any feasible vector x sufficiently close to it, or

$$x^* \in X, \text{and}$$
$$f(x^*) \geq f(x) \quad \forall x \in (X \cap x \in N_\varepsilon(x^*)) \tag{5.2}$$

In this case multiple optima may exist for the math program and we have only established that x^* is the optimum within the neighborhood searched. Extensive investigation of the program to find additional optima may be necessary.

Sometimes we can establish the local or global nature of the solution to a math program. The following two theorems give some examples.

Weierstras Theorem: (Sufficient conditions for a global solution)

If X is non-empty and compact (closed and bounded) and $f(x)$ is continuous on X, then $f(x)$ has a global maximum either in the interior or on the boundary of X.

Local - Global Theorem: (Sufficient conditions for a local solution to be global)

If X is a non-empty, compact and <u>convex</u> and $f(x)$ is continuous on X and a <u>concave</u> (convex) function over X, then a local maximum is a global maximum.

The Figure 5.1 shows a non-convex function defined over a convex feasible region so we have no assurance that local maxima are global maxima. We must assume that they are local maxima. Figure 5.2 shows a concave (non-convex) function maximized over a convex constraint set, so we are assured that a local maximum is a global maximum (if we can find one) by the Local-Global Theorem.
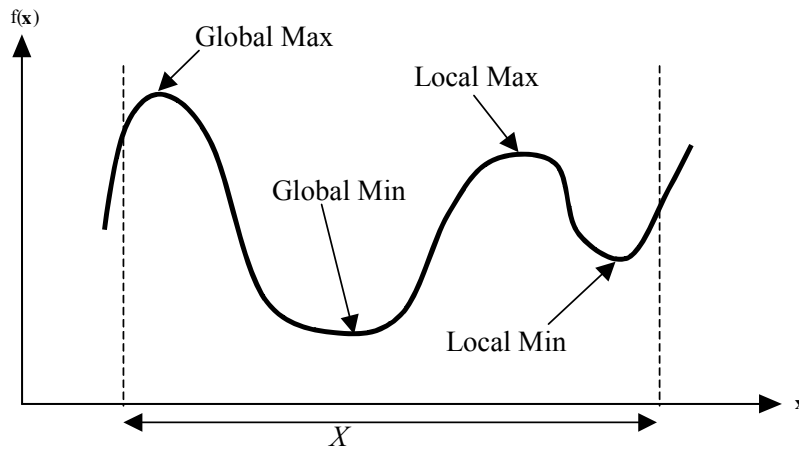
Figure 5.1. Illustration of global and local solutions.
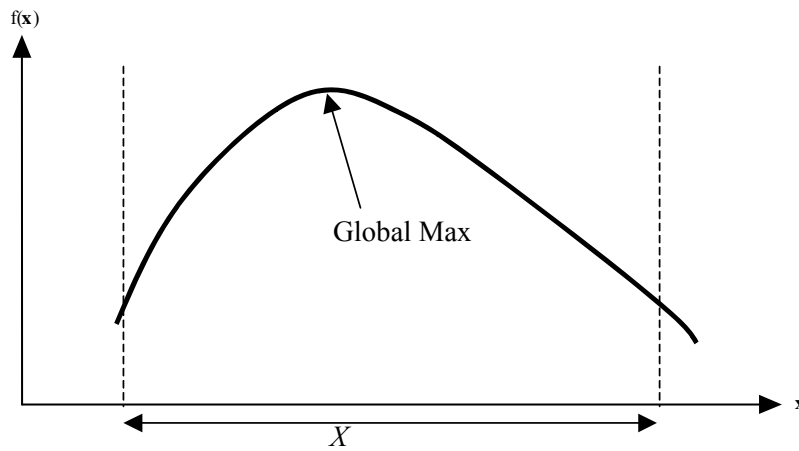
Figure 5.2. Concave function maximized over a convex constraint set

A.6 Classical Programming

$$\text{Maximize}_{x} \ f(x) \tag{6.1}$$

subject to

$$h(x) = 0 \tag{6.2}$$

A.6.1 Unconstrained Scalar Case

In this case, there are no constraints (Equation 6.2 is not present) and we consider only the objective function for a single decision variable, the scalar x

$$\text{Maximize } f(x) \atop x \qquad\qquad (6.1.1)$$

The **necessary conditions** for a local minimum are

$$\frac{df(x)}{dx} = 0 \quad \text{(first-order conditions)} \qquad\qquad (6.1.2)$$

and

$$\frac{d^2 f(x)}{dx^2} \leq 0 \quad \text{(second-order conditions)} \qquad\qquad (6.1.3)$$

The first-order conditions represent an equation which can be solved for $x*$ the optimal solution for the problem.

But what if the decision variable was constrained to be greater than or equal to zero (non-negativity restriction), e.g., $x \geq 0$? In this case there are two possibilities, either (1) the solution lies to the right of the origin and has an interior solution where the slope of the function is zero, or (2) or the solution lies on the boundary (at the origin) where $x = 0$ and the slope is negative. That is

$$\frac{df}{dx} \begin{cases} \leq 0 & \text{at } x = x* \quad \text{if } x* = 0 \\ = 0 & \text{at } x = x* \quad \text{if } x* > 0 \end{cases} \qquad\qquad (6.1.4)$$

This condition is often written as

$$\frac{df}{dx} \leq 0, \text{ and } x\frac{df}{dx} = 0 \text{ at } x = x* \qquad\qquad (6.1.5)$$

A.6.2 Unconstrained Vector Case

In this case, again we have no constraints and we consider only the objective function for a ***vector*** of decision variables $\boldsymbol{x}$

$$\text{Maximize } f(\boldsymbol{x}) \atop x \qquad\qquad (6.2.1)$$

The **necessary conditions** are

$$\nabla f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{0} \quad \text{(first-order conditions)} \qquad\qquad (6.2.3)$$

which is actually n simultaneous nonlinear equations

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f(\mathbf{x})}{\partial x_1} \\ \vdots \\ \dfrac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} = \mathbf{0}$$

(6.2.4)

The first-order conditions represent n - simultaneous equations which can be solved for x^* the optimal solution for the problem.

But what if the decision variable was constrained to be greater than or equal to zero (non-negativity restriction), e.g., $\mathbf{x} \geq \mathbf{0}$? In this case there are two possibilities, either (1) the solution lies to the right of the origin and has an interior solution where the slope of the function is zero, or (2) or the solution lies on the boundary (at the origin) where $\mathbf{x} = \mathbf{0}$ and the slope is negative. That is

$$\frac{\partial f}{\partial x_j} \begin{cases} \leq 0 & \text{at } \mathbf{x} = \mathbf{x}^* \ \text{ if } x_j^* = 0 \\ = 0 & \text{at } \mathbf{x} = \mathbf{x}^* \ \text{ if } x_j^* > 0 \end{cases}$$

(6.2.5)

for each j. This condition is often written as

$$\frac{\partial f}{\partial x_j} \leq 0, \ \text{ and } \ x_j \frac{\partial f}{\partial x_j} = 0 \ \ j = 1,...,n$$

(6.2.6)

A.6.3 Constrained Vector Case - Single Constraint

In this case, we consider the objective function for a vector of decision variables $\mathbf{x}$, and a single constraint, $h(\mathbf{x})=0$

$$\underset{x}{\text{Maximize }} f(\mathbf{x})$$
subject to
$$h(\mathbf{x}) = 0$$

(6.3.1)

We can multiply the constraint by a variable or multiplier and subtract the resulting expression from the objective function to form what is known as the "Lagrangian" function

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda[h(\mathbf{x})]$$

(6.3.2)

and then simply apply the methods of the previous case (unconstrained vector case). Note that for a

feasible vector the constraint must be satisfied, that is

$$h(\boldsymbol{x}) = 0 \qquad (6.3.3)$$

and

$$L(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) \qquad (6.3.4)$$

so we really have not changed the objective function as long as we remain feasible. The **necessary conditions** (first-order) are

$$\frac{\partial L}{\partial \boldsymbol{x}} = \frac{\partial\{f(\boldsymbol{x}) - \lambda[h(\boldsymbol{x})]\}}{\partial \boldsymbol{x}} = \frac{\partial f}{\partial \boldsymbol{x}} - \lambda \frac{\partial h}{\partial \boldsymbol{x}} = 0$$

$$\frac{\partial L}{\partial \lambda} = h(\boldsymbol{x}) = 0 \qquad (6.3.5)$$

The first-order conditions represent $n+1$ simultaneous equations

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0$$

$$h(\boldsymbol{x}) = 0 \qquad (6.3.6)$$

which must be solved for the optimal values of $\boldsymbol{x}^*$ and λ^*.

A.6.4 Example
(adapted from Loucks et al., 1981, Section 2.6, pp. 23-28)

Consider a situation where there is a total quantity of water R to be allocated to a number of different uses. Let the quantity of water to be allocated to each use be denoted by x_i, $i=1,\ldots, I$. The objective is to determine the quantity of water to be allocated to each use such that the total net benefits of all uses is maximized. We will consider an example with three uses $I = 3$.
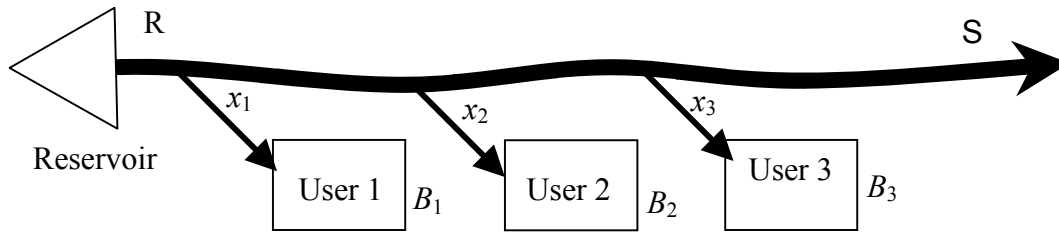
Figure 6.4.1. Reservoir release example.

The net-benefit resulting from an allocation of x_i to use i is given by

$$B_i(x_i) = a_i x_i - b_i x_i^2 \qquad\qquad i = 1,2,3$$

(6.4.1)

where a_i and b_i are given positive constants. These net-benefit (benefit minus cost) functions are of the form shown in Figure 6.4.2.
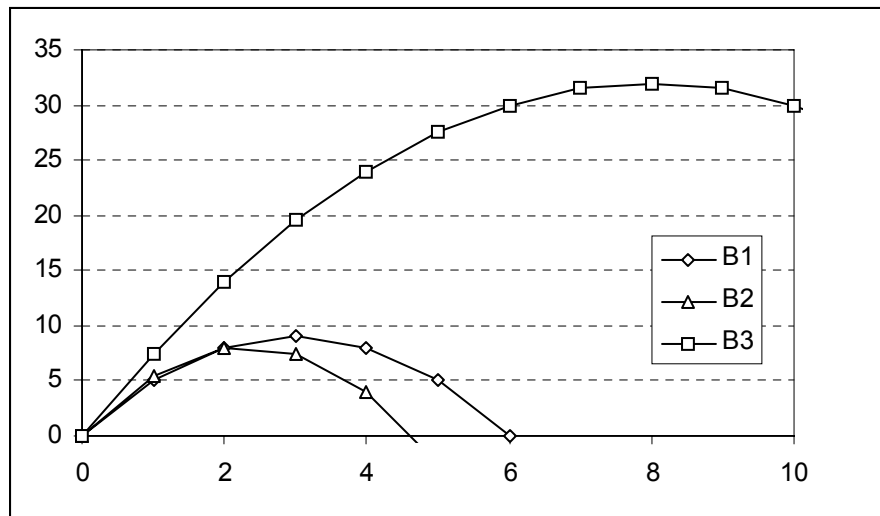
Figure 6.4.2. Net-benefit function for user i.

The three allocation variables x_i are unknown decision variables. The values that these variables can take on are restricted between 0 (negative allocation is meaningless) and values whose sum, $x_1 + x_2 + x_3$, does not exceed the available supply of water R minus the required downstream flow S. The optimization model to maximize net-benefits can be written as

$$\text{maximize}_{\mathbf{x}} \sum_{i=1}^{3}(a_i x_i - b_i x_i^2)$$

subject to

$$\sum_{i=1}^{3} x_i + S = R \tag{6.4.2}$$

The Lagrangian function is

$$L(\mathbf{x},\lambda) = \sum_{i=1}^{3}(a_i x_i - b_i x_i^2) - \lambda\left(\sum_{i=1}^{3} x_i + S - R\right) \tag{6.4.3}$$

There are now four unknowns in the problem, x_i, $i = 1,2,3$ and λ. Solution of the problem is obtained by applying the first-order conditions, setting the first partial derivatives of the Lagrangian function with respect to each of the variables equal to zero:

$$\frac{\partial L}{\partial x_1} = a_1 - 2b_1 x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = a_2 - 2b_2 x_2 - \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = a_3 - 2b_3 x_3 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 + S - R = 0 \tag{6.4.4}$$

These equations are the necessary conditions for a local maximum or minimum ignoring the nonnegativity conditions. Since the objective function involves the maximization of the sum of concave functions (functions whose slopes are decreasing), any local optima will also be the global maxima (by the Local-Global Theorem).

The optimal solution of this problem is found by solving for each x_i, $i = 1,2,3$ in terms of λ.

$$x_i = \frac{a_i - \lambda}{2b_i} \qquad i = 1,2,3 \tag{6.4.5}$$

Then solve for λ by substituting the x_i, $i = 1,2,3$ into the constraint

$$\sum_{i=1}^{3} x_i + S - R = 0 \tag{6.4.6}$$

$$\sum_{i=1}^{3} \frac{a_i - \lambda}{2b_i} + S - R = 0 \tag{6.4.7}$$

and solve for λ

$$\lambda = \frac{2\left(\sum_{i=1}^{3} \frac{a_i}{2b_i} + S - R\right)}{\sum_{i=1}^{3} \frac{1}{b_i}} \tag{6.4.8}$$

Hence knowing R, S, a_i and b_i this last equation can be solved for λ. Substitution of this value into the equation for the x_i, $i = 1, 2, 3$, we can solve for the optimal allocations, provided that all of the allocations are nonnegative.

A.6.5 Constrained Vector Case – Multiple Constraints

In this case, we consider the objective function for a vector of decision variables x, and a vector of constraints, $h(x)=0$

$$\underset{x}{\text{Maximize}} \ f(x)$$

subject to

$$h(x) = 0 \tag{6.5.1}$$

We can multiply the constraints by a vector of variables or multipliers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ or $\lambda \cdot h(x)$ and subtract the resulting expression from the objective function to form what is known as the "Lagrangian" function

$$L(x, \lambda) = f(x) - \lambda \cdot h(x) = f(x) - \sum_{i=1}^{m} \lambda_i h_i(x) \tag{6.5.2}$$

and then simply apply the methods of the previous case (unconstrained vector case). The **necessary conditions** (first-order) are

$$\nabla_x L(x, \lambda) = \nabla_x f(x) - \lambda \cdot \nabla_x h(x) = 0 \tag{6.5.3}$$

or

$$\frac{\partial L}{\partial x} = \frac{\partial [f(x) - \lambda \cdot h(x)]}{\partial x} = \frac{\partial f}{\partial x} - \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x} = 0 \tag{6.5.4}$$

and

$$\nabla_\lambda L(x, \lambda) = 0 \tag{6.5.5}$$

or

$$h(x) = 0 \tag{6.5.6}$$

The first-order conditions (Eq. 6.5.4 and Eq. 6.5.6) represent $n+m$ simultaneous equations must be solved for the optimal values of the **vectors** of decision variables and the Lagrange multipliers, x^* and λ^*.

Example (after Haith, 1982, Example 4-2):

Solve the following optimization problem using Lagrange multipliers.

Maximize $x_1^2 + 2x_1 - x_2^2$
subject to
$$5x_1 + 2x_2 \leq 10$$
$$x_1 \geq 0$$
$$x_2 \geq 0 \tag{6.5.7}$$

The last three constraints must be turned into equalities in order to use Classical Programming to solve the problem. Introduce three new variables, s_1, s_2, and s_3

$$5x_1 + 2x_2 + s_1^2 = 10$$
$$x_1 - s_2^2 = 0$$
$$x_2 - s_3^2 = 0 \tag{6.5.8}$$

These slack variables (difference between the left and right sides) are always introduced on the side of the inequality that the inequality sign points toward.

The "Lagrangian" function is

$$L(x, \lambda) = x_1^2 + 2x_1 - x_2^2 - \lambda_1(5x_1 + 2x_2 + s_1^2 - 10) - \lambda_2(x_1 - s_2^2) - \lambda_3(x_2 - s_3^2) \tag{6.5.9}$$

The first-order optimality conditions are

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2 - 5\lambda_1 - \lambda_2 x_1 = 0 \tag{6.10a}$$

$$\frac{\partial L}{\partial x_2} = -2x_2 - 2\lambda_1 x_2 - \lambda_3 x_2$$

(6.10b)

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0$$

(6.10c)

$$\frac{\partial L}{\partial s_2} = 2\lambda_2 s_2 = 0$$

(6.10d)

$$\frac{\partial L}{\partial s_3} = 2\lambda_3 s_3 = 0$$

(6.10e)

$$\frac{\partial L}{\partial \lambda_1} = 5x_1 + 2x_2 + s_1^2 - 10$$

(6.10f)

$$\frac{\partial L}{\partial \lambda_2} = x_1 - s_2^2 = 0$$

(6.10g)

$$\frac{\partial L}{\partial \lambda_3} = x_2 - s_3^2 = 0$$

(6.10h)

Equations 6.10c-e require that λ_i or s_i be equal to zero. There can be several solutions to the problem depending on whether one or another of the λ_i or s_i are equal to zero.

A.6.6 Nonlinear Programming and the Kuhn-Tucker Conditions

In this case, we consider the objective function for a vector of decision variables *x,* a vector of equality constraints, $h(x)=0$, and a vector of inequality constraints, $g(x)\leq 0$

Maximize $f(x)$
     x
subject to
       $h(x) = 0$
       $g(x) \leq 0$

(6.6.1)

We can multiply the constraints by vectors of variables or multipliers $\lambda = (\lambda_1, \lambda_2,..., \lambda_m)$ or $\lambda \cdot h(x)$ and $u = (u_1, u_2,..., u_r)$ or $u \cdot g(x)$ and subtract the resulting expression from the objective function to form what is known as the "Lagrangian" function

$$L(x, \lambda, u) = f(x) - \sum_{i=1}^{m} \lambda_i h_i(x) - \sum_{j=1}^{r} u_j g_j(x)$$

(6.6.2)

and then simply apply the methods of the previous case (unconstrained vector case). The **necessary**

conditions (first-order) are the **Kuhn-Tucker Conditions**

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x_k} - \displaystyle\sum_{i=1}^{m} \lambda_i \dfrac{\partial h_i}{\partial x_k} - \displaystyle\sum_{j=1}^{r} u_j \dfrac{\partial g_j}{\partial x_k} \le 0 \\[4mm] x_k^* \left[\dfrac{\partial f}{\partial x_k} - \displaystyle\sum_{i=1}^{m} \lambda_i \dfrac{\partial h_i}{\partial x_k} - \displaystyle\sum_{j=1}^{r} u_j \dfrac{\partial g_j}{\partial x_k} \right] = 0 \end{array}\right\} \text{ at } x = x^*, \text{ for } k = 1,...,n$$

(6.6.3)

$$\left.\begin{array}{l} g_j(x^*) \le 0 \\[3mm] u_j g_j(x^*) = 0 \end{array}\right\} \text{ for } j = 1,...,r$$

(6.6.4)

$$h_i(x^*) = 0, \text{ for } i = 1,...,m$$

(6.6.5)

$$\begin{array}{l} x_k^* \ge 0, \ k = 1,...,n \\ u_j \ge 0, \ j = 1,...,r \end{array}$$

(6.6.6)

A.7 Exercises

1. (after Mays and Chung, 1992, Exercise 3.4.5) Water is available at supply points 1, 2, and 3 in quantities 4, 8, and 12 thousand units, respectively. All of this water must be shipped to destinations A, B, C, D, and E, which have requirements of 1, 2, 3, 8, and 10 thousand units, respectively. The following table gives the cost of shipping one unit of water from the given supply point to the given destination. Find the shipping schedule which minimizes the total cost of transportation.

Source	Destination				
	A	B	C	D	E
1	7	10	5	4	12
2	3	2	0	9	1
3	8	13	11	6	14

Supply   Destination
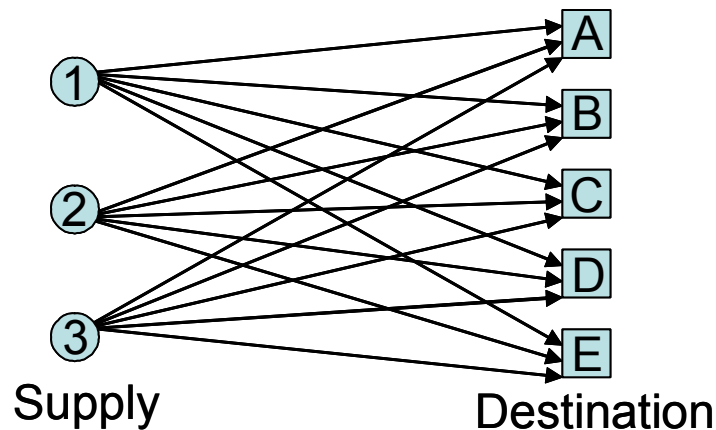
2. (adapted from Mays and Tung, 1992, Exercise 3.1.1) Solve the following Linear Program

$$\text{Maximize } 2x_1 + 3x_2 + 5x_3$$

subject to

$$x_1 + x_2 - x_3 \geq -5$$
$$-6x_1 + 7x_2 - 9x_3 = -5$$
$$|19x_1 - 7x_2 + 5x_3| \leq 13$$

3. (adapted from Mays and Tung, 1992, Exercise 3.2.1) Consider the following Linear Program

$$\text{Maximize } 3x_1 + 5x_2$$

subject to

$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$3x_1 + 2x_2 \leq 18$$

(a) Graph the feasible region for the problem.

(b) Solve the problem graphically.

(c) How much can the nonbinding constraints be reduced without changing the feasibility of the optimal solution?

(d) What is the range of the objective function coefficient of x2 so that the optimal solution remains feasible?

4. (after Haith, 1982, Example 5-1) 1000 ha of farmland surrounding a lake is available for two

crops. Each hectare of crop 1 loses 0.9 kg/yr of pesticide to the lake, and the corresponding loss from crop 2 is 0.5 kg/yr. Total pesticide losses are not allowed to exceed 632.5 kg/yr. Crop returns are $300 and $150/ha for crops 1 and 2, respectively. Costs for crops are estimated to be $160 and $50/ha for crops 1 and 2, respectively.

(a) Determine the cropping combination that maximizes farmer profits subject to a constraint on the pesticide losses into the lake.

(b) If crop returns decrease to $210/ha for crop 1, what is the optimal solution?

(c) If crop returns increase to $380/ha for crop 1, what is the optimal solution?

5. (after Haith, 1982, Exercise 5-1) A metal refining factory has a capacity of 10x 10^4 kg/week, produces waste at the rate of 3 kg/kg of product, contined in a wastewater at a concentration of 2 kg/m^3. The factory's waste treatment plant operates at a constant efficiency of 0.85 and has a capacity of 8x10^4 m^3/week. Wastewater is discharged into a river, and the effluent standard is 100,000 kg/week. There is also an effluent charge of $1000/$10^4$ kg discharged. Treatment costs are $1000/$10^4$ m^3, product sales price is $10,000/$10^4$ kg, and production costs are $6850/$10^4$ kg.

(a) Construct a linear program that can be used to solve this wastewater problem. Solve the model graphically.

(b) If the effluent charge is raised to $2000/$10^4$ kg, how much will the waste discharge be reduced?

6. (after Haith, 1982, Exercise 5-9) A standard of 1 kg/10^3m^3 has been set as the maximum allowable concentration for a substance in a river. Three major dischargers of the substance are located along the river as shown in the figure. The river has a flow of 500,000 m^3/day and an ambient concentration of the regulated substance of 0.2 kg/10^3m^3 upstream of the first discharger. The three waste sources presently discharge 100, 100m, and 1600 kg/day of the regulated substance, resulting in violations of the standard in the river. The substance is not conserved in the river, but decays at a rate of K=0.03 km^{-1}. Thus is C_1 and C_2 are the concentrations of the substance immediately after the discharge points 1 and 2, respectively, the concentrations at any point L km downstream of discharge 1 ($L < 10$) is C_1e^{-KL}. Similarly, the concentration L km downstream of discharge 2 ($L < 15$) is C_2e^{-KL}. The cost of removing the substance from the wastewater is $10X/1000$ m^3 where X is the fraction of the substance removed. Use LP to determine an optimal treatment program for the regulated substance.

10 km   15 km

300 10³ m³/day
0.2 kg/10³ m³ →   River

①   ②   ③

Flow   100 10³ m³/day   50 10³ m³/day   200 10³ m³/day
Discharge   10 kg/10³ m³   20 kg/10³ m³   8 kg/10³ m³
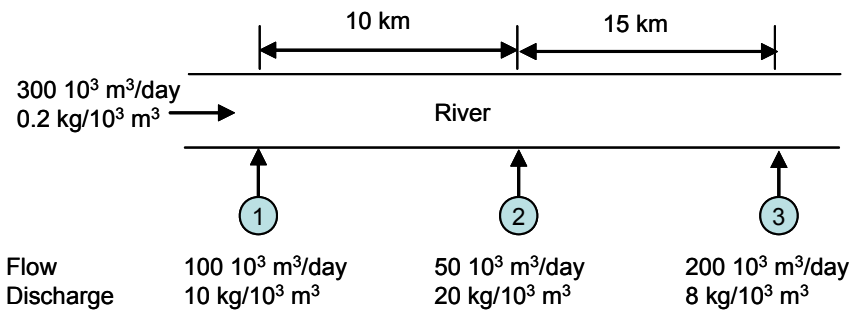
7. (after Haith, 1982, Example 4-1) Solve the following optimization problem using Lagrange multipliers.

$$\text{Maximize} \quad 0.5x_1^2 + 20x_2x_3 + 10x_3$$

subject to

$$x_1 - 3x_2 + 0.5x_3 = 6$$
$$x_2 + 2x_3 = 10$$

8. (after Haith, 1982, Exercise 4-1) Solve the following optimization problem using Lagrange multipliers (Classical programming)

$$\text{Maximize} \quad 4x_1^2 + x_2 + 6x_3^3$$

subject to

$$x_1 + 3x_2 + x_3 = 10$$
$$x_2 + 2x_3 = 4$$

9. (after Haith, 1982, Exercise 4-2) Solve the following optimization problem using Lagrange multipliers (Classical programming)

$$\text{Maximize} \quad 4e^{-x_1} - x_2^2$$

subject to

$$6x_1 - x_2 = 6$$
$$x_1 \geq 0$$

10a. (after Revelle et al., 1997) A vertical cylindrical steel tank of height h and diameter d is to be constructed. The tank is open at the top, and it is known that the bottom must be twice as thick as the side thickness, t. Determine the dimensions of the minimum cost tank of volume k, where the weight of the material represents the total cost of the tank.

10b. (after Willis, 2002) A waste storage facility consists of a right circular cylinder of radius 5 units and a conical cap. The volume of the storage facility is V. Determine H, the height of the

storage facility, and h, the height of the conical cap, such that the total surface area is minimized.

11. (after Taha, 1997) Find the maximum value of the following function

$$f(x_1, x_1, x_1) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$$

12. (after Taha, 1997) Consider the problem

$$\text{Minimize } f(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2$$
subject to
$$g_1(\boldsymbol{x}) = x_1 + x_2 + 3x_3 = 2$$
$$g_2(\boldsymbol{x}) = 5x_1 + 2x_2 + 3x_3 = 5$$

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Appendix B. Mathematics Review

B.1 Linear Algebra

B.1.1 Introduction

An important tool in many areas of scientific and engineering analysis and computation is matrix theory or linear algebra. A wide variety of problems lead ultimately to the need to solve a linear system of equations $Ax = b$. There are two general approaches to the solution of linear systems.

B.1.2 Matrix Notation

A matrix is an array of real numbers. Consider an (m x n) matrix $\vec{A}$ with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

(A.1.2.1)

The horizontal elements of the matrix are the *rows* and the vertical elements are the *columns*. The first subscript of an element designates the row, and the second subscript designates the column. A *row matrix* (or *row vector*) is a matrix with one row, i.e., the dimension $m = 1$. For example

$$r = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{pmatrix}$$

(A.1.2.2)

A *column vector* is a matrix with only one column, e.g.,

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

`

(A.1.2.3)

When the row and column dimensions of a matrix are equal ($m = n$) then the matrix is called <u>square</u>

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

(A.1.2.4)

The transpose of the (m x n) matrix A is the (n x m)

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

(A.1.2.5)

A *symmetric* matrix is one where $A^T = A$. An example of a symmetric matrix is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(A.1.2.6)

A *diagonal* matrix is a square matrix where elements off the main diagonal are all zero

$$A = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

(A.1.2.7)

An *identity* matrix is a diagonal matrix where all the elements are one's

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

(A.1.2.8)

An *upper triangular* matrix is one where all the elements below the main diagonal are zero

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & a_{2n} \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

(A.1.2.9)

A *lower triangular* matrix is one where all the elements above the main diagonal are zero

$$A = \begin{pmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

(A.1.2.10)

B.1.3 Matrix Arithmetic

Two (m x n) matrices A and B are _equal_ if and only if each of their elements are equal. That is

$$A = B \text{ if and only if aij = bij for i = 1,...,m and j = 1,...,n} \tag{A.1.3.1}$$

The _addition_ of vectors and matrices is allowed whenever the dimensions are the same. The sum of two (m x 1) column vectors a and b is

$$a + b = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{pmatrix} \tag{A.1.3.2}$$

Example:

Let $u = (1,-3,2,4)$ and $v = (3,5,-1,-2)$. Then

$$u + v = (1+3,-3+5,2-1,4-2) = (4,2,1,2)$$

$$5u = (5*1,5*(-3),5*2,5*4) = (5,-15,10,20)$$

$$2u - 3v = (2,-6,4,8) + (-9,-15,3,6) = (-7,-21,7,14)$$

The sum of two (m x n) matrices A and B is

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{1n} + b_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix} \tag{A.1.3.3}$$

Multiplication of a matrix A by a scalar α is defined as

$$\alpha A = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & & \alpha a_{2n} \\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

(A.1.3.4)

The *product* of two matrices A and B is defined only if the number of columns of A is equal to the number of rows of B. If A is (n x p) and B is (p x m), the product is an (n x m) matrix C

$$C = AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{1n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{m1} & a_{11}b_{12} + \cdots + a_{1n}b_{m2} & \cdots & a_{11}b_{1n} + \cdots + a_{1n}b_{mn} \\ a_{21}b_{11} + \cdots + a_{2n}b_{m1} & a_{21}b_{12} + \cdots + a_{2n}b_{m2} & & a_{21}b_{1n} + \cdots + a_{2n}b_{mn} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{1n}b_{m1} & a_{m1}b_{12} + \cdots + a_{mn}b_{m2} & \cdots & a_{m1}b_{1n} + \cdots + a_{mn}b_{mn} \end{pmatrix}$$

(A.1.3.5)

The *ij* element of the matrix C is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

(A.1.3.6)

That is the c_{ij} element is obtained by adding the products of the individual elements of the *i*-th *row* of the first matrix by the *j*-th *column* of the second matrix (i.e., "row-by-column"). The following figure shows an easy way to check if two matrices are compatible for multiplication and what the dimensions of the resulting matrix will be:

$$C_{nxm} = A_{nxp} \, B_{pxm}$$

(A.1.3.7)

Example:

Let $a = (a_1, a_2, \cdots, a_n)$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1, a_2, \cdots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Example:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, then

$$\boldsymbol{Ab} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{matrix} a_{11} b_1 + a_{12} b_2 + \cdots a_{1n} b_n \\ a_{11} b_1 + a_{22} b_2 + \cdots a_{2n} b_n \\ \\ a_{n1} b_1 + a_{n2} b_2 + \cdots a_{nn} b_n \end{matrix}$$

Matrix _division_ is not a defined operation. The identity matrix has the property that $\boldsymbol{IA} = \boldsymbol{A}$ and $\boldsymbol{AI} = \boldsymbol{A}$. If A is an (n x n) square matrix and there is a matrix X with the property that

$$\boldsymbol{AX} = \boldsymbol{I} \tag{A.1.3.8}$$

where $\boldsymbol{I}$ is the identity matrix, then the matrix X is defined to be the _inverse_ of A and is denoted A^{-1}. That is

$$\boldsymbol{AA^{-1}} = \boldsymbol{I} \text{ and } \boldsymbol{A^{-1}A} = \boldsymbol{I} \tag{A.1.3.9}$$

The inverse of a (2 x 2) matrix A can be represented simply as

$$A^{-1} = \frac{1}{a_{11} a_{22} - a_{12} a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{A.1.3.10}$$

Example

If $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, then $A^{-1} = \frac{1}{2(2) - 1(1)} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$

B.1.4 Systems of Linear Equations

Consider the linear system of equations

$$Ax = b \tag{A.1.4.1}$$

where A is an $(n \times n)$ matrix, b is a column vector of constants, called the right-hand-side, and x is the unknown solution vector to be determined. This system can be written out as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{A.1.4.2}$$

Performing the matrix multiplication and writing each equation out separately, we have

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \tag{A.1.4.3a}$$

This system can also be written in the following manner

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad i = 1, \cdots n \tag{A.1.4.3b}$$

A formal way to obtain a solution using matrix algebra is to multiply each side of the equation by the inverse of A to yield

$$A^{-1}Ax = A^{-1}b \tag{A.1.4.4}$$

or, since $AA^{-1} = I$

$$x = A^{-1}b \tag{A.1.4.5}$$

Thus, we have obtained the solution to the system of equations. Unfortunately, this is not a very efficient way of solving the system of equations. We will discuss more efficient ways in the following sections.

Example: Consider the following two equations in two unknowns:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

Solve the first equation for x_2

$$x_2 = \frac{-3}{2}x_1 + 9$$

which is a straight line with an intercept of 9 and a slope of (-3/2). Now, solve the second equation for x_2

$$x_2 = \frac{1}{2}x_1 + 1$$

which is also a straight line, but with an intercept of 1 and a slope of (1/2). These lines are plotted in the following Figure. The solution is the intersection of the two lines at $x_1 = 4$ and $x_2 = 3$.
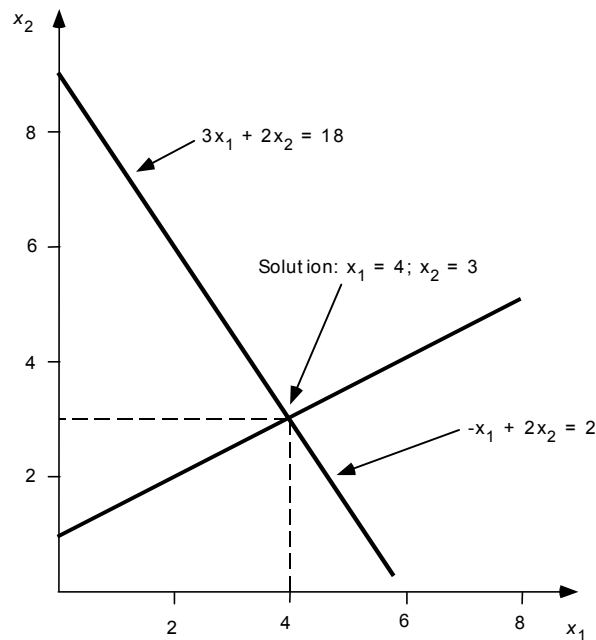

Figure B.1.4.1. Graphical solution of two simultaneous linear equations.

Each linear equation

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad \text{(A.1.4.6)}$$

represents a <u>hyperplane</u> in an n-dimensional Euclidean space (R^n), and the system of equations $Ax =$

$\boldsymbol{b}$ represents m hyperplanes. The solution of the system of equations is the intersection of all of the m hyperplanes, and can be

- the empty set (no solution)
- a point (unique solution)
- a line (non-unique solution)
- a plane (non-unique solution)

B.1.5 Systems of Linear Inequalities

A system of m linear inequalities in n unknowns can be written as

$$\boldsymbol{Ax} \leq \boldsymbol{b} \qquad\qquad\qquad (A.1.5.1)$$

or

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \leq b_n \qquad\qquad (A.1.5.2a)$$

This system of inequalities can also be written in the following manner

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad i = 1, \cdots n \qquad\qquad (A.1.5.2b)$$

Each linear inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \qquad\qquad (A.1.5.3)$$

represents a half-space in R^n, and the system of inequalities $\boldsymbol{Ax} \leq \boldsymbol{b}$ represents the intersection of m half-spaces which is a polyhedral convex set or, if bounded, a polyhedron.

B.2 Calculus

B.2.1 Functions

A <u>function</u> $f(\boldsymbol{x})$ of n variables can be written as

$$y = f(\boldsymbol{x}) \qquad\qquad\qquad (A.2.1.1)$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in R^n \ (column\,vector); \qquad y \in R^1 \ (scalar)$$

$$(A.2.1.2)$$

A <u>linear function</u> of n variables is written as

$$y = f(x) = c \cdot x = \sum_{i=1}^{n} c_i x_i = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$(A.2.1.3)$$

where c is a vector of coefficients.

B.2.2 Sets, Neighborhoods and Distance

The <u>distance</u> between two points x and y in R^n is defined as

$$d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

$$(A.2.2.1)$$

A <u>neighborhood</u> around a point x in R^n is defined as the set of all points y less than some distance ε from the point x or

$$N_\varepsilon(x) = \left\{ y \in R^n : d(x, y) < \varepsilon \right\}$$

$$(A.2.2.2)$$

A <u>closed set</u> is a set which contains all of the points on its boundary, for example a closed interval on the real line (R^1). In a <u>bounded set</u>, the distance between two points contained in the set is finite. A <u>compact set</u> is closed and bounded, examples are any finite interval $[a,b]$ on the real line or any bounded sphere in R^3.

A set S is a <u>convex set</u> if for any two points x and y in the set, the point

$$z = ax + (1-a)y$$

$$(A.2.2.3)$$

is also in the set for all a, where $0 \le a \le 1$. That is, all weighted averages of two points in the set are also points in the set. For example, all points on a line segment joining two points in a convex set are also in the set. Straight lines, hyperplanes, closed halfspaces are all convex sets. Figure 2 below illustrates a convex and a non-convex set. A real valued function $f(x)$ defined on a convex set S is a <u>convex function</u> if given any two points x and y in S,

$$f[af(x) + (1-a)f(y)] \le af(x) + (1-a)f(y)$$

$$(A.2.2.4)$$

for all *a*, where $0 \leq a \leq 1$. Figure 3 illustrates the fact that the line segment joining two points in a convex function does not lie below the function. Figure 4 shows general examples of convex and non-convex (or concave) functions. An example of a convex function is a parabola which opens upward. Linear functions (lines, planes, hyperplanes, half-spaces) are both convex and non-convex functions.
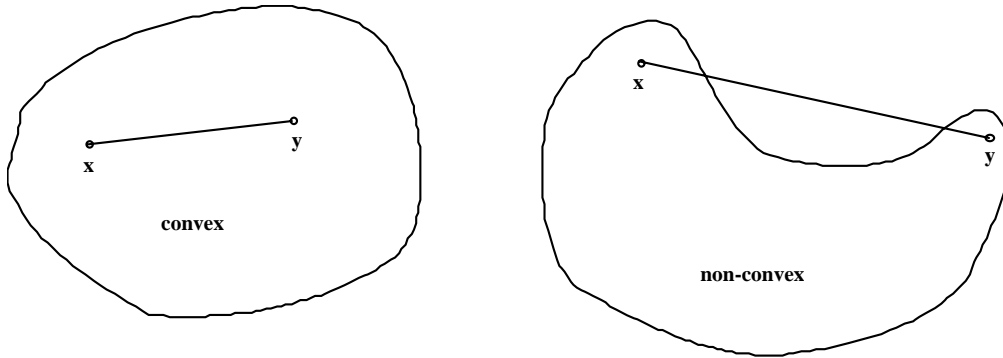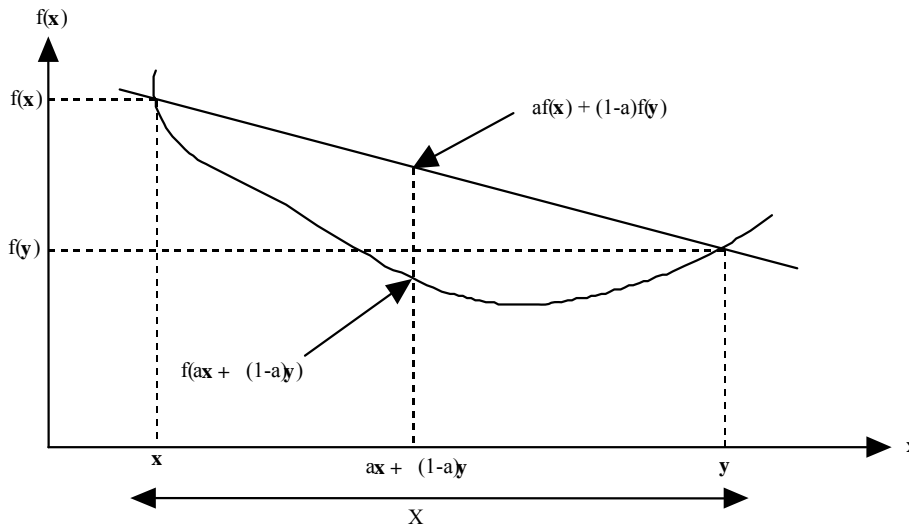

Figure A.2.2.1. General diagram of convex and non-convex sets.

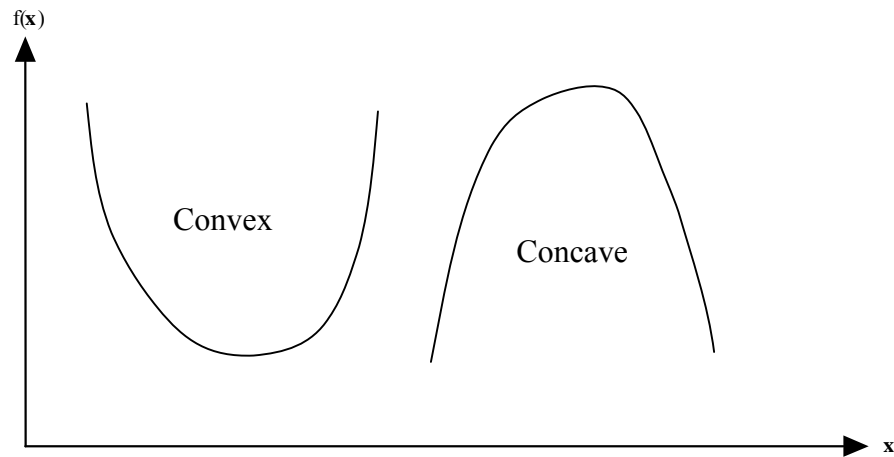**Figure A.2.2.2.** General diagram of a convex function.

Figure A.2.2.3. General diagram of a convex function and a concave function.

B.2.3 Derivatives

The derivative of a function of a single scalar variable $f(x)$ is defined as

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(A.2.3.1)

The partial derivatives of a function f of the variables x and y are defined as

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

(A.2.3.2)

That is, to find the partial derivative of a multivariable function with respect to one independent variable x_i, regard all other independent variables as fixed and find the usual derivative with respect to x_i. The partial derivative of a function of several variables $f(x)$ with respect to a particular component of x, x_i, evaluated at a point x^o is

$$\frac{\partial f}{\partial x_i} = \frac{\partial f(x)}{\partial x_i}\bigg|_{x^o}$$

(A.2.3.3)

The partial derivative of $f(x)$ with respect to the vector x is a row vector of partial derivatives of the function or the gradient vector

$$\nabla f(\boldsymbol{x}) = \frac{\partial f}{\partial \boldsymbol{x}} = \left[\frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right]$$

(A.2.3.4)

B.3 Vectors Calculus

B.3.1 Coordinate Systems

Typical coordinate systems used in groundwater problems include: Rectangular: x, y, z; and Cylindrical: r, θ, z where $x = r\cos\theta$ and $y = r\sin\theta$. Let A be a vector function of (x, y, z) or (r, θ, z), respectively, then

$$\boldsymbol{A} = A_x \boldsymbol{i} + A_y \boldsymbol{j} + A_z \boldsymbol{k} = A_r \boldsymbol{r} + A_\theta \boldsymbol{\theta} + A_z \boldsymbol{k}$$

(A.3.1.1)

where $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ and $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{k})$ are unit vectors in the (x, y, z) or (r, θ, z) directions, respectively.

B.3.2 Basic Operators

The gradient operator, del (from the Greek nabla) or ∇, is defined in rectangular coordinates as the vector

$$\nabla(\cdot) = \begin{bmatrix} \dfrac{\partial(\cdot)}{\partial x} \\ \dfrac{\partial(\cdot)}{\partial y} \\ \dfrac{\partial(\cdot)}{\partial z} \end{bmatrix} = \frac{\partial(\cdot)}{\partial x}\boldsymbol{i} + \frac{\partial(\cdot)}{\partial y}\boldsymbol{j} + \frac{\partial(\cdot)}{\partial z}\boldsymbol{k}$$

(A.3.2.1)

The major operators: Gradient, "del" or , $\nabla(\cdot)$; Divergence, "div" or $\nabla \cdot (\cdot)$; and Laplacian, "del dot del" or $\nabla \cdot \nabla(\cdot) = \nabla^2(\cdot)$ can be defined in the rectangular and cylindrical coordinate systems as:

Gradient

$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial x}\boldsymbol{i} + \frac{\partial(\cdot)}{\partial y}\boldsymbol{j} + \frac{\partial(\cdot)}{\partial z}\boldsymbol{k}$$

(Rectangular)

(A.3.2.2)

$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial r}\boldsymbol{r} + \frac{1}{r}\frac{\partial(\cdot)}{\partial \theta}\boldsymbol{\theta} + \frac{\partial(\cdot)}{\partial z}\boldsymbol{k}$$

(Cylindrical)

(A.3.2.3)

Divergence

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(Rectangular) $\qquad$ (A.3.2.4)

Proof:

$$\nabla \cdot \boldsymbol{A} = \left[\frac{\partial (\cdot)}{\partial x} \boldsymbol{i} + \frac{\partial (\cdot)}{\partial y} \boldsymbol{j} + \frac{\partial (\cdot)}{\partial z} \boldsymbol{k} \right] \cdot \left[A_x \boldsymbol{i} + A_y \boldsymbol{j} + A_z \boldsymbol{k} \right]$$

$$= \frac{\partial A_x}{\partial x} \boldsymbol{i} \cdot \boldsymbol{i} + \frac{\partial A_y}{\partial x} \boldsymbol{i} \cdot \boldsymbol{j} + \frac{\partial A_z}{\partial x} \boldsymbol{i} \cdot \boldsymbol{k}$$

$$+ \frac{\partial A_x}{\partial y} \boldsymbol{j} \cdot \boldsymbol{i} + \frac{\partial A_y}{\partial y} \boldsymbol{j} \cdot \boldsymbol{j} + \frac{\partial A_z}{\partial y} \boldsymbol{j} \cdot \boldsymbol{k}$$

$$+ \frac{\partial A_x}{\partial z} \boldsymbol{k} \cdot \boldsymbol{i} + \frac{\partial A_y}{\partial z} \boldsymbol{k} \cdot \boldsymbol{j} + \frac{\partial A_z}{\partial z} \boldsymbol{k} \cdot \boldsymbol{k}$$

$$= \frac{\partial A_x}{\partial x}(1) + \frac{\partial A_y}{\partial x}(0) + \frac{\partial A_z}{\partial x}(0)$$

$$+ \frac{\partial A_x}{\partial y}(0) + \frac{\partial A_y}{\partial y}(1) + \frac{\partial A_z}{\partial y}(0)$$

$$+ \frac{\partial A_x}{\partial z}(0) + \frac{\partial A_y}{\partial z}(0) + \frac{\partial A_z}{\partial z}(1)$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r} \frac{\partial (rA_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

(Cylindrical) $\qquad$ (A.3.2.5)

Laplacian

$$\nabla \cdot \nabla (\cdot) = \nabla^2 (\cdot) = \frac{\partial^2 (\cdot)}{\partial x^2} + \frac{\partial^2 (\cdot)}{\partial y^2} + \frac{\partial^2 (\cdot)}{\partial z^2}$$

(Rectangular) $\qquad$ (A.3.2.6)

e.g.,

$$\nabla \cdot \nabla \boldsymbol{A} = \nabla^2 \boldsymbol{A} = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}$$

$$\nabla \cdot \nabla (\cdot) = \nabla^2 (\cdot) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial (\cdot)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 (\cdot)}{\partial \theta^2} + \frac{\partial^2 (\cdot)}{\partial z^2}$$

(Cylindrical) $\qquad$ (A.3.2.7)

e.g.,

$$\nabla \cdot \nabla \boldsymbol{A} = \nabla^2 \boldsymbol{A} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial z^2}$$

B.3.3 Various Groundwater Relations

The Piezometric head, $h = p/\gamma + z$, is a scalar quantity and the gradient of this quantity is a column vector

$$\nabla h = \begin{bmatrix} \dfrac{\partial h}{\partial x} \\[2mm] \dfrac{\partial h}{\partial y} \\[2mm] \dfrac{\partial h}{\partial z} \end{bmatrix} = \frac{\partial h}{\partial x}\boldsymbol{i} + \frac{\partial h}{\partial y}\boldsymbol{j} + \frac{\partial h}{\partial z}\boldsymbol{k}$$

(A.3.3.1)

The hydraulic conductivity, $\boldsymbol{K}$, is a tensor whose common form in three dimensions is

$$\boldsymbol{K} = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix}$$

(A.3.3.2)

The term $\boldsymbol{K} \cdot \nabla h$ is the product of the matrix $\boldsymbol{K}$ with the vector ∇h or (using "row by column" multiplication)

$$\boldsymbol{K} \cdot \nabla h = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial h}{\partial x} \\[2mm] \dfrac{\partial h}{\partial y} \\[2mm] \dfrac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} K_x \dfrac{\partial h}{\partial x} \\[2mm] K_y \dfrac{\partial h}{\partial y} \\[2mm] K_z \dfrac{\partial h}{\partial z} \end{bmatrix} = K_x \frac{\partial h}{\partial x}\boldsymbol{i} + K_y \frac{\partial h}{\partial y}\boldsymbol{j} + K_z \frac{\partial h}{\partial z}\boldsymbol{k}$$

(A.3.3.3)

Now $\nabla \cdot \boldsymbol{K} \cdot \nabla h$ is the dot product of the vector $\nabla(\cdot)$ with the vector $\boldsymbol{K} \cdot \nabla h$ or

$$\nabla \cdot \boldsymbol{K} \cdot \nabla h = \begin{bmatrix} \dfrac{\partial(\cdot)}{\partial x} \\[2ex] \dfrac{\partial(\cdot)}{\partial y} \\[2ex] \dfrac{\partial(\cdot)}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} K_x \dfrac{\partial h}{\partial x} \\[2ex] K_y \dfrac{\partial h}{\partial y} \\[2ex] K_z \dfrac{\partial h}{\partial z} \end{bmatrix}$$

$$= \left[\frac{\partial(\cdot)}{\partial x}\boldsymbol{i} + \frac{\partial(\cdot)}{\partial y}\boldsymbol{j} + \frac{\partial(\cdot)}{\partial z}\boldsymbol{k} \right] \cdot \left[K_x \frac{\partial h}{\partial x}\boldsymbol{i} + K_y \frac{\partial h}{\partial y}\boldsymbol{j} + K_z \frac{\partial h}{\partial z}\boldsymbol{k} \right]$$

$$= \frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z} \right) \tag{A.3.3.4}$$

APPENDIX C. GAMS INSTALLATION

C.1 Introduction

1. Run setup.exe: In most cases, you will run the file \systems\win\setup.exe directly from the GAMS distribution CD. There are many ways to do this; for example, you can use the Windows Explorer to browse the CD; open the \systems\win directory and double click setup.exe. The setup program will first prompt you for the name of the directory in which to install GAMS. We call this directory the 'GAMS directory'. You may accept the default choice or pick another directory. Please remember that if you want to install two different versions of GAMS, they should be in separate directories (for example c:\gams and c:\gams old). The GAMS software will be installed in the directory you choose.

2. Copy the license file: If no license file is found in the GAMS directory, you will be prompted for one during the installation. If you are not sure if you have a license file, or do not have it yet, choose 'No' when asked if you wish to copy a license file. You can always do this later. If no valid license file is found, GAMS will still function in demonstration mode and will only solve small problems. For example, all demonstration and student systems are shipped without a license. If you have a license file that you wish to copy to the GAMS directory at this time, answer 'Yes' to the license file prompt. You will now be given the opportunity to browse the file system and find the license file you wish to copy. For example, if your system came with a license file on a diskette, browse to the A: drive and select gamslice.txt. When you have found the correct file, choose 'open' to perform the copy.

3. Create project files: If this is the first installation of GAMS with the GAMS IDE on your system, setup will create a default GAMS project in a subdirectory of the WINDOWS directory. If this is not the first such installation, your existing GAMS projects will be preserved.

4. Choose default solvers: Run the GAMS IDE by double clicking gamside.exe from the GAMS directory. To view or edit the default solvers, choose File ! Options ! Solvers from the IDE. You can accept the existing defaults if you wish, but most users will want to pick default solvers for each model type. It is a good idea to review the solver defaults when installing a new GAMS system or when updating a license file.

5. Run a few models to test the GAMS system: The online help for the IDE (Help → Help Topics → Guided Tour) describes how to copy a model from the GAMS model library, run it, and view the solution. To test your installation, run the following models from the GAMS model library:

> LP: trnsport (objective value: 153.675)
> NLP: chenery (objective value: 1058.9)
> MIP: bid (optimal solution: 15210109.512)
> MINLP: procsel (optimal solution: 1.9231)
> MCP: scarfmcp (no objective function)
> MPSGE: scarfmge (no objective function)

If there are any problems during these test runs, read the section on Troubleshooting.