Procedure to Generate Uniform Random Variates from Each Copula

The Gaussian Copula

The Gaussian copula may be generated by first obtaining a set of correlated normally distributed variates v_1 and v_2 using Choleski's decomposition, and then transforming these to uniform variables $u_1 = \Phi(v_1)$ and $u_2 = \Phi(v_2)$, where Φ is the cumulative standard normal. Then, the pair (u_1, u_2) represents draws from the Gaussian copula.

The FGM Copula

Generating draws from the FGM copula is easiest done using the conditional distribution approach (see Nelsen, 2006; pg 41). Note that the conditional distribution function of U_2, given $U_1 = u_1$, may be written as:

$$C_{2|1}(u_1, u_2) = \Pr[U_2 \le u_2 \mid U_1 = u_1] = \underset{\Delta u_1 \to 0}{Lim} \frac{C_\theta(u_1 + \Delta u_1, u_2) - C_\theta(u_1, u_2)}{\Delta u_1} = \frac{\partial C_\theta(u_1, u_2)}{\partial u_1} \qquad (1)$$

Thus, a general algorithm to draw from a copula $C_\theta(u_1, u_2)$ would be (see Johnson, 1987, Ch. 3):

(1) Draw two independent uniform random variates (u_1, v_2).

(2) Set $u_2 = C_{2|1}^{-1}(u_1, v_2)$., where $C_{2|1}^{-1}$ denotes the pseudo-inverse of $C_{2|1}$.

The vector (u_1, u_2) is generated from the copula *C*. For the FGM copula, the above algorithm is:

(1) Draw two independent uniform random variates (u_1, v_2).

(2) Set $u_2 = 2v_2 / (B + A), A = 1 + \theta(1 - 2u_1), B = \sqrt{A^2 - 4(A-1)v_2}$.[1] (2)

[1] Note that $C_{2|1}(u_1, u_2) = u_2 A + [-(A-1)]u_2^{\,2}$, where A is as just defined. Setting this equal to $v_2 (0 < v_2 < 1)$ and solving the quadratic for u_2 in the unit interval provides one root solution: $u_2 = \dfrac{A - \sqrt{A^2 - 4(A-1)v_2}}{2(A-1)}$. Multiplying and dividing by $A + \sqrt{A^2 - 4(A-1)v_2}$ and simplifying, we get the desired result.

The Clayton Copula

For the Clayton copula, we draw variates using the conditional distribution approach discussed earlier.

(1) Draw two independent uniform random variates (u_1, v_2).

(2) Set $u_2 = [u_1^{-\theta}(v_2^{-\theta/(1+\theta)} - 1) + 1]^{-1/\theta}$ (3)

The Gumbel Copula

For the Gumbel copula, the conditional distribution is not directly invertible (see Venter, 2001), and so we use another way to generate variates using the following general algorithm (see Nelsen, 2006, Genest and Rivest, 1993):

(1) Generate two independent uniform variates (v_1, v_2).

(2) Set $w = K_C^{-1}(v_2), K_C(t) = t - \dfrac{\varphi(t)}{\varphi^{'}(t)}$ (4)

(3) Set $u_1 = \varphi^{-1}[v_1\varphi(w)]$ and $u_2 = \varphi^{-1}[(1-v_1)\varphi(w)]$.

The desired pair is then (u_1, u_2). In the above algorithm, the function $K_C(t)$ is the distribution function of the random variable $C_\theta(U_1, U_2)$, where U_1 and U_2 are uniform random variables with an Archimedean copula *C* generated by φ. For the Gumbel copula, the above algorithm is:

(1) Generate two independent uniform variates (v_1, v_2).

(2) Set $K_C(w) = w\left(1 - \dfrac{\ln(w)}{\theta}\right) = v_2$, and solve numerically for $0 < w < 1$. (5)

(3) Set $u_1 = \exp[v_1^{1/\theta}\ln(w)]$ and $u_2 = \exp[(1-v_1)^{1/\theta}\ln(w)]$.

The Frank Copula

Using the conditional distribution approach discussed earlier, one can draw variates from the Frank copula as follows:

(1) Draw two independent uniform random variates (u_1, v_2).

(2) Set $u_2 = -\dfrac{1}{\theta}\ln\left(1 + \dfrac{v_2(1-e^{-\theta})}{v_2(e^{-\theta u_1} - 1) - e^{-\theta u_1}}\right)$ (6)

The Joe Copula

Random variates from the Joe copula can be obtained using the general algorithm for Archimedean copulas, discussed under Gumbel's copula:

(1) Generate two independent uniform variates (v_1, v_2).

(2) Set $K_C(w) = w - \frac{1}{\theta} \frac{\left[\ln(1-(1-w)^{\theta})\right]\left[(1-(1-w)^{\theta}\right]}{(1-w)^{\theta-1}} = v_2$, and solve numerically for $0 < w < 1$. (7)

(3) Set $u_1 = 1 - \left[1 - \left[1-(1-w)^{\theta}\right]^{v_1}\right]^{1/\theta}$ and $u_2 = 1 - \left[1 - \left[1-(1-w)^{\theta}\right]^{1-v_1}\right]^{1/\theta}$.

References:

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