

GRID-SCALE DEPENDENCY OF SUBGRID-SCALE STRUCTURE EFFECTS IN HYDRAULIC MODELS OF RIVERS AND STREAMS

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ABSTRACT

A key difficulty in applying numerical models for engineering hydraulics is coping with subgrid-scale heterogeneity in both benthic structure and fluid flow. This difficulty is exemplified by the effects of Large Woody Debris (LWD) on stream and river hydrodynamics. In the past, LWD effects have typically been modeled by calibrating eddy viscosities or Manning's 'n' to approximate the increased drag. However, the spatial structure of flow near LWD contributes to viability of aquatic habitat, so the prior methods are unsuitable for hydraulic models used for instream-flow habitat analysis. Furthermore, resolving the flow field around LWD with Large Eddy Simulation (LES) techniques would require an impractical amount of computer power, and coarse-grid Reynolds-averaged Navier Stokes (RANS) models lead to grid-dependency of the drag effects for subgrid-scale structure. To address these problems, a new conceptual model is developed. The new approach applies a spatial filter to the Reynolds-averaged Navier-Stokes equation, effectively creating a combination of RANS and LES. The advantage of this approach is the heterogeneity in subgrid-scale turbulence structure can be directly modeled without a grid-scale dependency.

INTRODUCTION

The complexities boundary structure in natural environments provides a wide set of unsolved problems for numerical modeling. At the largest scales of environmental flow modeling (e.g. atmospheric boundary layer, global or regional ocean circulation [1]), the subgrid-scale boundary structure can be parameterized as a simple roughness and considered homogenous over a model grid cell. For such cases,

the spatial variation in the surface roughness is lost in the uncertainties and inaccuracies of the turbulence models. On the other hand, at smaller modeling scales (e.g. flow around one or a few buildings or around a boulder in a river), attempts are made to model the flow patterns around an individual structural element, so the grid is designed to follow the structure boundary [2]. While the grid for such a model is several orders of magnitude finer than those in the large-scale circulation modeling, it shares an important characteristic: the roughness of the subgrid-scale boundary is reasonably considered homogeneous. However, between these two extremes are a large class of problems having both engineering and scientific interest, where it is impractical to model at a fine-enough scale to capture the flow patterns around individual boundary structures, and yet the structures are significantly heterogeneous in scale and/or distribution so as to invalidate the presumption of subgrid-scale boundary homogeneity. While it may be possible to carefully calibrate a model to reflect the heterogeneity in a turbulence model, such calibration will be inherently sensitive to the model grid.

As an example, consider a river laden with large woody debris (LWD) as shown at low flow conditions in Fig. 1. It is impractical model at the centimeter scale that would be necessary to capture the flow around each piece of debris. At more practical scales of $O(1\text{ m}) - O(10\text{ m})$, grid cells may have zero, one, or several pieces of LWD. As the model grid is coarsened or refined, the number of LWD pieces in an individual grid cell will be altered. Thus, any subgrid-scale model must be able to *a priori* adjust for the relationship between the size of the flow field around the structure and the size of the grid cell.



Figure 1: Sulphur River (Texas) large woody debris at low-flow conditions. (Photo courtesy of Texas Water Development Board)

In this paper, we use the steady two-dimensional flow around a circular cylinder as a simple basis for developing and demonstrating a new model concept for subgrid-scale heterogeneity. In future work, we plan to alter parameters such as permeability, aspect ratio and orientation to provide more general results. In the following we describe: 1) the existence of subgrid scale heterogeneity; 2) the grid dependence of subgrid scale heterogeneity using current numerical modeling techniques and 3) the conceptual model we proposed to address the problem.

NOMENCLATURE

In this paper Einstein's summation convention is used. For convenience, subscript c, b, a and o respectively indicate "cell", "background", "accelerated" and "obstructed" regions around a circular cylinder. The following are the variables and notations in this paper:

C_{ij}	Cross term
L_{ij}	Leonard term
R_{ij}	Reynolds term
U	spatially-filtered time averaged velocity
U	grid-scale mean velocity
g	gravitational acceleration
p	pressure
t	time
u	actual velocity
\tilde{u}	sub-filter-scale velocity
u	local velocity
u'	subgrid-scale velocity
Δt	a certain time period
x_i	local coordinates
α, β, γ	empirical coefficients
ρ	density
τ_{ij}	viscous stress
\forall	volume
$\bar{\quad}$	time average operator
$\langle \quad \rangle$	spatial filter

DISCUSSION

A commercial CFD code, Fluent (standard k-epsilon turbulence model) is used to model the two-dimensional fine-scale steady flow field around a circular cylinder of diameter 'D'. The computational domain (Fig. 2) extends from $-15D$ at the inflow to $25D$ at the outflow, and from $-9D$ to $9D$ in the cross-flow direction. The Reynolds number is 3900 (based on cylinder diameter and free-stream velocity). This flow condition was selected as a starting point due to readily available experimental data [3].

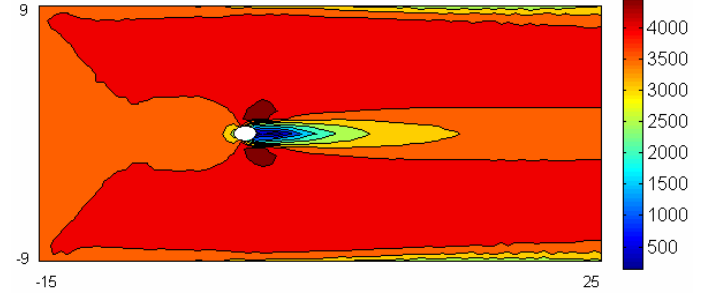


Figure 2: Local Reynolds number contours around a circular cylinder (white circle) in the computational domain

Normalized velocity is represented by the local Reynolds number as shown in Fig. 2. The flow field illustrated in Fig. 2 can be considered the subgrid-scale velocity field for a 1-cell coarse grid model. Typical LWD diameters are of order 10 cm, so the computational domain corresponds to a model grid scale of 1.8×4 m, and the selected Reynolds number corresponds to a velocity of approximately 4 cm/s (i.e. a relatively low velocity with respect to typical river conditions). Thus, this can be considered a single model cell with with subgrid-scale inhomogeneity in the bottom boundary structure that affects both subgrid and resolved scales of motion.

Our present models for tracking turbulent flow over complex boundary include large eddy simulation (LES) and Reynolds averaged Navier-Stokes equations (RANS). However, both LES and RANS cannot satisfactory account for the subgrid-scale inhomogeneity in physical features. For LES method, the subgrid-scale velocity is presumed to arise from, and be satisfactorily modeled by, the energy-containing eddies that resolved on the grid. Nevertheless, for a coarse grid model like the example we showed in Fig. 1, eddies generated from the cylinder cannot be resolved at the 1-cell coarse grid because the cylinder is a subgrid-scale obstruction. Furthermore, the effects of those eddies cannot be predicted from the resolved flow. Thus, LES cannot resolve the effects of subgrid-scale obstruction physical feature.

RANS methods assume that the subgrid scale is a homogeneous turbulence field characterized by the Reynolds stresses, which are defined using the unsteady fluctuations from the grid-scale (local) mean velocity. At coarser grid resolutions (i.e. treating Fig. 2 as a single grid cell), there is clearly subgrid scale inhomogeneity in the local mean velocity that would contribute to the nonlinear terms in the Navier-Stokes equations. Extending the RANS approach [4], we can consider the subgrid scale as a local difference between the grid-scale mean velocity and the local velocity such that:

$$u = U + u' \quad (1)$$

where u is the local velocity, U is the grid-scale mean velocity and u' is the subgrid-scale contribution. By applying different grids to our model results of Fig. 2, we can compute the grid-scale mean velocity and subgrid-scale velocity that would be associated with a “perfect” subgrid scale model (i.e. a model the reproduced U exactly at the grid scale). Figure 3 shows the best possible representation of the velocity field around the circular cylinder for two different grid scales. The grid-scale mean velocities with finer and coarser grids differ in both magnitude and direction. Thus, a perfectly designed subgrid model must account for the relationship between the subgrid scales of heterogeneity and the size of the grid.

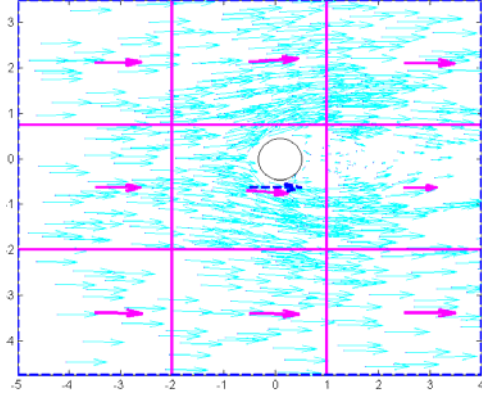


Figure 3: Local and grid-scale mean velocity fields near a circular cylinder with finer and coarser grid cell, Local velocity (---), Grid-scale mean velocity for finer grid cells (---) and Grid-scale mean velocity for coarser grid cells (---).

The grid dependency of grid-scale mean velocity leads to grid dependency for the subgrid scale velocity. In Fig. 4, u_1' is the subgrid scale velocity calculated from grid-scale mean velocity of the coarser grid cell in Fig. 3; u_2' is the subgrid-scale velocity calculated from grid scale mean velocity of the finer grid cell in Fig. 3. As shown in Fig. 4, these two different subgrid-scale velocities are very different in both magnitude and direction. It suggests the subgrid scale flow field is not homogeneous, which invalidates the RANS assumption.

Thus, if an eddy viscosity is used as a subgrid-scale closure, it follows that the eddy viscosity must *a priori* be a function of the grid scale. In effect, the eddy viscosity must be a calibration parameter that includes the relationship between the grid scale and subgrid-scale physical inhomogeneity. Therefore, in the presence of subgrid-scale inhomogeneity it appears that the closure term cannot be invariant with the grid scale. We believe the inability of present RANS models to explicitly account for the relationship between the grid scale and subgrid-scale inhomogeneity is a principle contributor to the calibration problems for models of the natural environment.

Based on the insights above, we propose to employ a spatial filter to RANS method. This new model could be thought as an effective combination between LES and RANS. An advantage of this model is the explicit inclusion of the grid-scale spatial filter of the Reynolds stress term, which provides the framework for considering the heterogeneity in the subgrid-scale turbulence field implied by the existence of subgrid-scale physical boundary structure.

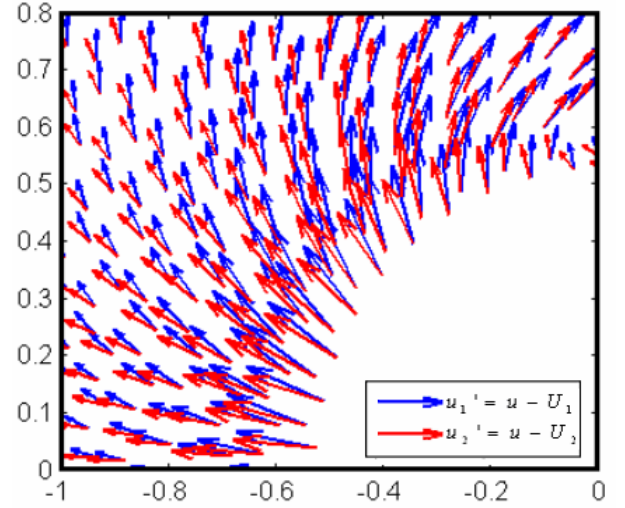


Figure 4: Grid dependency of subgrid-scale velocity

SPATIALLY-FILTERED REYNOLDS-AVERAGED NAVIER-STOKES EQUATIONS

Let us consider the unsteady RANS equations without any turbulence modeling in a Cartesian coordinate system suitable for simple open channel flow (i.e. the ‘x’ direction is aligned with a constant sloping bottom):

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} - g \frac{\partial x_i}{\partial z} + \frac{1}{\rho} \frac{\partial \bar{\tau}_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j} (\overline{u'_i u'_j}) \quad (2)$$

where \bar{u} is a time-averaged velocity and u' is a velocity fluctuation defined respectively as:

$$\bar{u} \equiv \frac{1}{\Delta t} \int u \, dt \quad (3)$$

$$u' \equiv \bar{u} - u \quad (4)$$

In the LES formalism with a finite-volume or finite-difference method, the model grid is considered a “top-hat” or “box” implicit filter that *a priori* limits the resolved motions on the grid. Rather than directly discretizing the Navier-Stokes equations in LES, a spatial filter indicated by $\langle \rangle$, is applied to obtain equations for the resolvable velocity. The resulting set of equations are for the continuous velocity and pressure fields that can be represented at the spatial filter scale. If we apply the LES filtering formalism to the unsteady RANS equations, we obtain:

$$\frac{\partial}{\partial t} \langle \bar{u}_i \rangle + \frac{\partial}{\partial x_j} \langle \bar{u}_i \rangle \langle \bar{u}_j \rangle = -\frac{1}{\rho} \frac{\partial}{\partial x_i} \langle \bar{p} \rangle + g \frac{\partial x_i}{\partial z} + \frac{1}{\rho} \frac{\partial \langle \bar{\tau}_{ij} \rangle}{\partial x_j} - \frac{\partial}{\partial x_j} \langle \overline{u'_i u'_j} \rangle + \frac{\partial}{\partial x_j} \langle \bar{u}_i \rangle \langle \bar{u}_j \rangle - \frac{\partial}{\partial x_j} \langle \bar{u}_i \bar{u}_j \rangle \quad (5)$$

In a model, the spatially-filtered time-averaged velocity is what we are attempting to resolve on the model grid, so for simplicity in notation let us define:

$$U \equiv \langle \bar{u} \rangle \quad (6)$$

The sub-filter-scale velocity (i.e. what cannot be resolved on the model grid) can be defined from:

$$\tilde{u} \equiv \bar{u} - U \quad (7)$$

Note that an unconventional tilde is applied to indicate the sub-filter scale velocity to remind us that this velocity is time-averaged and therefore does not include turbulent (i.e. sub-time scale) fluctuations. A similar simplified notation can be made for the resolved pressure as $P = \langle \bar{p} \rangle$. Substituting and expanding, we can write the spatial-filter applied to the RANS equations as:

$$\frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x_j} U_i U_j = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + g \frac{\partial x_i}{\partial z} + \frac{1}{\rho} \frac{\partial \langle \bar{\tau}_{ij} \rangle}{\partial x_j} - \left\langle \frac{\partial R_{ij}}{\partial x_j} \right\rangle - \left\langle \frac{\partial C_{ij}}{\partial x_j} \right\rangle - \frac{\partial L_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j} \langle \overline{u'_i u'_j} \rangle \quad (8)$$

where the RHS includes the classic terms derived by Leonard [5]:

$$\begin{aligned} L_{ij} &= \langle U_i U_j \rangle - U_i U_j \\ C_{ij} &= U_i \tilde{u}_j + \tilde{u}_i U_j \\ R_{ij} &= \tilde{u}_i \tilde{u}_j \end{aligned} \quad (9)$$

The principle difference between the above equation structure and prior LES work is the inclusion of the Reynolds-stress term that is spatially filtered. Thus, one could arguably apply this formalism to add an anisotropic subgrid-scale turbulence closure to LES. However, for the purposes of this work, we take the viewpoint that the large eddies are not resolved, so we will interpret the above as a RANS approach with additional closure terms for representing the relationships between the grid scale (resolved) flow and the subgrid-scale flow. It is worthwhile to notice that this new mathematical formation explicitly includes a grid-scale spatial filtered Reynolds stress term $\langle \overline{u'_i u'_j} \rangle$, which allows us to link the heterogeneity of subgrid-scale turbulence field and existence of subgrid-scale physical boundary. In the new formalism, the sub-filter scale velocity (\tilde{u}) is already time-averaged, so it represents longer-time-scale anisotropy in the unresolved velocity field that is caused by the subgrid-scale structure; i.e. this represents the divergence of the free-stream flow around an obstacle. Hence, one of the crucial points for this new model is to quantify how the free-stream flow diverges around the subgrid-scale structure. In the next section, we will explore the different flow regions around a circular cylinder as a possible prototype for a subgrid-scale structure model.

FLOW REGIONS AROUND CIRCULAR CYLINDER

Based on Prantl's boundary layer theory [6, 7], the circular cylinder flow has four different regions as shown in Fig. 5. Corresponding to this theory, we consider a grid cell of volume ∇_c that can be divided into 4 sub-volumes: 1) a background volume ∇_b that is unaffected by sub-grid scale inhomogeneity, 2) an obstructed volume ∇_o in which the flow is slowed, 3) an accelerated volume ∇_a in which the obstruction leads to local flow acceleration, and 4) a wake volume ∇_w where the turbulence is enhanced by the wake of the object.

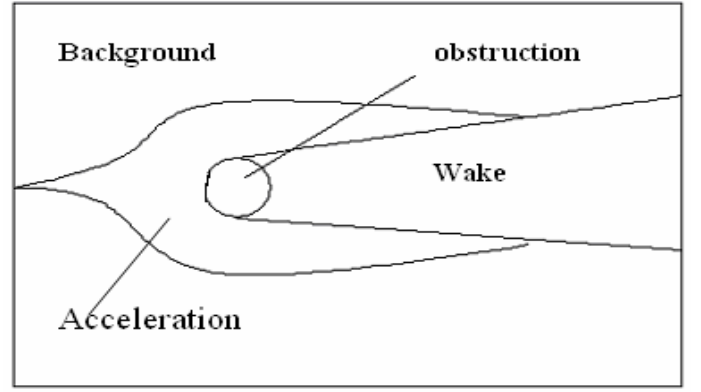


Figure 5: Flow regions around a circular cylinder

The characteristic scales of the time-averaged velocity in each subregion are sub-filter scale velocities: $\tilde{u}_b, \tilde{u}_o, \tilde{u}_a$ and \tilde{u}_w . Let us consider the simplest possible model where the resolved flow is only in the U direction ($V=0$) such that the flow is decelerated through an obstruction and accelerated around the obstruction. The characteristics velocities in the regions are modeled by an empirical parameter and the resolved velocity as:

$$\begin{aligned} \tilde{u}_b &= U & \tilde{v}_b &= 0 & \tilde{w}_b &= 0 \\ \tilde{u}_o &= \alpha U & \tilde{v}_o &= 0 & \tilde{w}_o &= 0 \\ \tilde{u}_a &= \beta U & \tilde{v}_a &= \gamma U & \tilde{w}_a &= 0 \\ \tilde{u}_w &= U & \tilde{v}_w &= 0 & \tilde{w}_w &= 0 \end{aligned} \quad (10)$$

where, $\alpha < 1$, $\beta > 1$, and $-1 < \gamma < 1$.

Based on the new spatially-filtered Reynolds-averaged Navier-Stokes equations and flow regions theory, we can analytically calculate the cross term in the equation below:

$$\left\langle \frac{\partial C_{11}}{\partial x} \right\rangle = \frac{1}{\nabla_c} \left[\nabla_b \frac{\partial C_{11(b)}}{\partial x} + \nabla_o \frac{\partial C_{11(o)}}{\partial x} + \nabla_a \frac{\partial C_{11(a)}}{\partial x} + \nabla_w \frac{\partial C_{11(w)}}{\partial x} \right] \quad (11)$$

where

$$\begin{aligned} C_{11(b)} &= U \tilde{u}_b = U^2 \\ C_{11(o)} &= U \tilde{u}_o = \alpha U^2 \\ C_{11(a)} &= U \tilde{u}_a = \beta U^2 \\ C_{11(w)} &= U \tilde{u}_w = U^2 \end{aligned} \quad (12)$$

which provides:

$$\left\langle \frac{\partial C_{11}}{\partial x} \right\rangle = \frac{U^2}{\nabla_c} [\nabla_b + \alpha \nabla_o + \beta \nabla_a + \nabla_w] \quad (13)$$

Thus, once we have the empirical coefficients for the relationships between the background (free-stream) flow and the flow in the object-affected regions, the cross terms is analytically calculable. Similar approaches can be used for the other cross, Leonard and Reynolds terms. The key insight is that the modeling relationships (i.e. α, β, γ) are determined empirically without regard to the model grid spacing, but their implementation implicitly includes the effect of the grid scale.

CONCLUSIONS

Subgrid-scale heterogeneity is studied using flow around a circular cylinder. Existing practical coarse-grid models do not account for the scale relationship subgrid-scale obstruction and the grid, which leads to grid-scale dependency of subgrid scale features. A new spatial-filtered RANS method provides a framework to account for the subgrid-scale heterogeneity. This new approach includes the subgrid-scale heterogeneity and an explicit way of quantifying grid-scale dependency.

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REFERENCES

- [1] Avissar, R., 1992, "Conceptual aspect of a statistical-dynamic approach to represent landscape subgrid-scale heterogeneities in atmospheric models," *Journal of Geophysical research- atmospheres*, **97(D3)**, pp. 2729-2742.
- [2] Crowder, D.W. and Diplas, P., 2000, "Using two-dimensional hydrodynamic models at scales of ecological importance," *Journal of Hydrology*, **230**, pp. 172-191.
- [3] Ma, X., Karamanos, G.S., Karniadakis, G.E., 2000, "Dynamics and low-dimensionality of a turbulent near wake," *J. Fluid. Mech.*, **410**, pp. 29-65.
- [4] Speziale, C.G., 1996, "Modeling of turbulent transport equations," *Simulation and modeling of turbulent flows*, pp.185-242, Oxford University Press, New York.
- [5] Leonard, A., 1974, "Energy cascade in large eddy simulations of turbulent fluid flows," *Adv. in Geophys.*, **18A**, pp.237.
- [6] Prandtl, L., 1925. "Uber die ausgebildete turbulenz," *ZAMM*, **5**, pp. 331-340.
- [7] Zdravkovich, M. M., 1997, *Flow around circular cylinders*, I, pp.3-4., Oxford University Press.