COMPARING HYDROSTATIC AND NONHYDROSTATIC NAVIER-STOKES MODELS OF INTERNAL WAVES

Bridget M. Wadzuk[1]
Ben R. Hodges[2], Associate Member ASCE

Internal waves occur at a variety of temporal and spatial scales and are mechanisms by which momentum, energy, nutrients and biota are transported in lakes, estuaries and coastal oceans. In stratified systems, accurate prediction of water quality effects requires modeling internal wave evolution and propagation. Lake, estuary, and coastal ocean models typically apply the hydrostatic approximation to the Navier-Stokes equations, which limits the accuracy of internal wave predictions.

The hydrostatic approximation exploits the small aspect ratio (vertical:horizontal length scale) of natural systems, which makes the vertical acceleration and dynamic pressure negligible relative to the horizontal acceleration and hydrostatic pressure (Marshall *et al*, 1997). The hydrostatic approximation also eliminates the need to solve a three-dimensional Poisson problem for dynamic pressure, thereby dramatically decreasing computational requirements. The hydrostatic approximation is adequate for large-scale ocean processes, but breaks down for scales less than ten kilometers (Kantha and Clayson, 2000). At the mesoscale (10-100 km in horizontal extent, depths of order 1000 m, and horizontal velocities of 0.1 – 1 m s^{-1}), the aspect ratio may no longer be considered small, therefore dynamic pressure and vertical acceleration are not negligible and the hydrostatic approximation will not effectively model system dynamics. The exclusion of dynamic pressure in a model may distort the balance between internal wave steepening and dispersion such that the small error of neglecting dynamic pressure accumulates into a large error in the long-term prediction of wave propagation. Thus, inclusion of vertical acceleration and nonhydrostatic pressure is necessary to properly model the evolution of internal waves (Long, 1972).

As a simple, monochromatic wave in a nonlinear system evolves, the system's nonlinearities cause the internal wave to slowly steepen. If there is no force balancing the steepening, the wave will propagate unabated, shorten, and eventually overtop itself causing a mixing event to occur. The full Reynolds-averaged Navier-Stokes (RANS) momentum equation contains both vertical acceleration (the advective term) and dynamic pressure, Eq. (1):

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\delta_{i\alpha} g \left(\frac{\partial \eta}{\partial x_\alpha} + \frac{1}{\rho_o} \frac{\partial}{\partial x_\alpha} \int_{z'}^{\eta} \rho' dz \right) - \frac{1}{\rho_o} \frac{\partial P_d}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\left\{ \nu + \nu_e \delta_{jk} \right\} \frac{\partial U_i}{\partial x_j} \right) \tag{1}$$

where i, j, k = three component space, α = horizontal two component space, U_i = slowly varying mean velocity, P_d = dynamic pressure, ρ_o = reference density, ρ' = density fluctuation, η = slowly varying mean free surface height, z' = elevation, ν = molecular viscosity, ν_e = eddy viscosity, g = gravity, and $\delta_{i\alpha}$ = Kronecker's delta. The hydrostatic approximation reduces Eq. (1) to the horizontal momentum equations and no vertical momentum equation, Eq. (2).

[1] PhD candidate, Dept. of Civil Engineering, Center for Research in Water Resources, Univ. of Texas at Austin, 10100 Burnet Rd, Bld. 119, Austin, TX 78712. E-mail: bwadzuk@mail.utexas.edu
[2] Assistant Professor, Dept. of Civil Engineering, Center for Research in Water Resources, Univ. of Texas at Austin, 10100 Burnet Rd, Bld. 119, Austin, TX 78712. E-mail: hodges@mail.utexas.edu

$$\frac{\partial U_\alpha}{\partial t} + U_j \frac{\partial U_\alpha}{\partial x_j} = -g\left(\frac{\partial \eta}{\partial x_\alpha} + \frac{1}{\rho_o}\frac{\partial}{\partial x_\alpha}\int_{z'}^{\eta}\rho' dz\right) + \frac{\partial}{\partial x_j}\left(\left\{\nu + \nu_e \delta_{jk}\right\}\frac{\partial U_\alpha}{\partial x_j}\right) \qquad (2)$$

In the full RANS vertical momentum equation, the dominant terms are advection and dynamic pressure and the local rate of change and the diffusion terms are considered negligible (Cushman-Roisin, 1994); this is demonstrated by the disappearance of the vertical momentum equation under the hydrostatic approximation. The balancing relationship is seen between dynamic pressure and vertical acceleration; advection is responsible for nonlinear wave steepening and dynamic pressure moderates the steepening rate and disperses the propagating wave into a series of solitary waves. The solitary waves formed are the fastest traveling wave for a given height and unchanging form (Turner, 1973); the larger amplitude solitary waves pass energy to the smaller amplitude solitary waves as they evolve and propagate. The rate of nonlinear steepening and the formation of solitary waves will increase as the surface mixed layer becomes thin or the amplitude of the initial basin-scale wave increases (Horn *et al*, 2001).

Hydrostatic models have three different errors that tend to suppress nonlinear steepening, numerical dissipation, numerical diffusion and numerical dispersion. Numerical dissipation of energy removes kinetic energy from the system, damping a wave and not allowing it to steepen. Numerical diffusion of mass alters the potential energy of a system by exchanging mass across the wave's interface, reducing stratification, which impedes steepening. Numerical dispersion causes the wave to change shape, thus acting like dynamic pressure. The mass transport scheme typically used in hydrostatic models allows numerical diffusion of mass to dominate, which erroneously introduces mixing and incorrectly models internal wave energy transport and the spatial occurrence of the mixing; numerical dispersion is present, but it is not a dominant term, therefore often rendering hydrostatic models incapable of expressing the evolution and propagation of solitary waves. Hydrostatic models show waves that propagate and steepen, but do not show the proper degeneration into a series of solitary waves; rather the wave continues to travel like a bore as numerical diffusion occurs at the wave front. This is contrary to what theory and experiment has shown to occur in an actual system (Long, 1972; Horn *et al*, 2001). The following two figures show an internal wave that evolves and propagates through time with a hydrostatic model. Both systems begin with a 10m pycnocline that varies 4 kg/m^3. Fig. 1 shows a system with a 4.4m amplitude internal wave that is adequately modeled, while Fig. 2 shows a 13m amplitude wave, which is not sufficiently modeled and the system is considered non-hydrostatic. The wave in Fig. 1 is a slow-moving basin-scale seiche that nearly maintains its' initial shape and thickness. The formation of a bore is seen within a few time steps in Fig. 2; diffusion of the wave front density gradient causes the pycnocline to increase in thickness, indicating that the steepened wave is overtopping itself and mixing is occurring. The wave in this scenario shows dramatic change in shape and the thickness increases from 10m to 15m.

The effects of wave front diffusion for the two scenarios can be seen through an analysis of the background potential energy (BPE). As mixing occurs in a closed, adiabatic system, numerical diffusion will increase the BPE (Laval *et al*, 2003). Fig. 3 shows the change in BPE at each time step compared with the initial BPE. Almost immediately, the large amplitude (non-hydrostatic) system shows an increase in BPE, which is coincident with the quick steepening of the wave front as seen in Fig. 2. The BPE change for the large amplitude system is about an order of magnitude more than the smaller amplitude system.

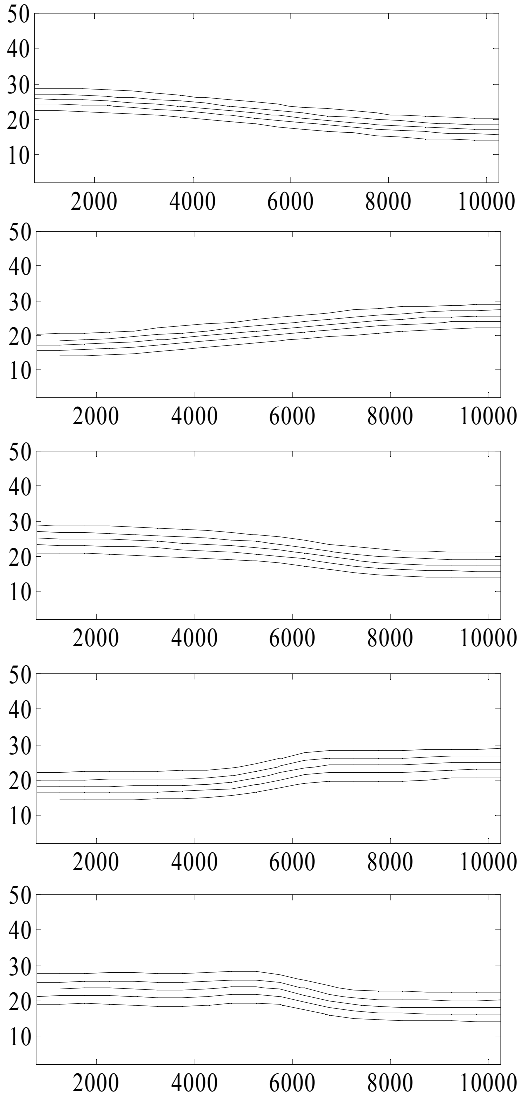

Figure 1. Hydrostatic system, with a hydrostatic model: 10km long, 50m deep, 4.4m initial wave amplitude.

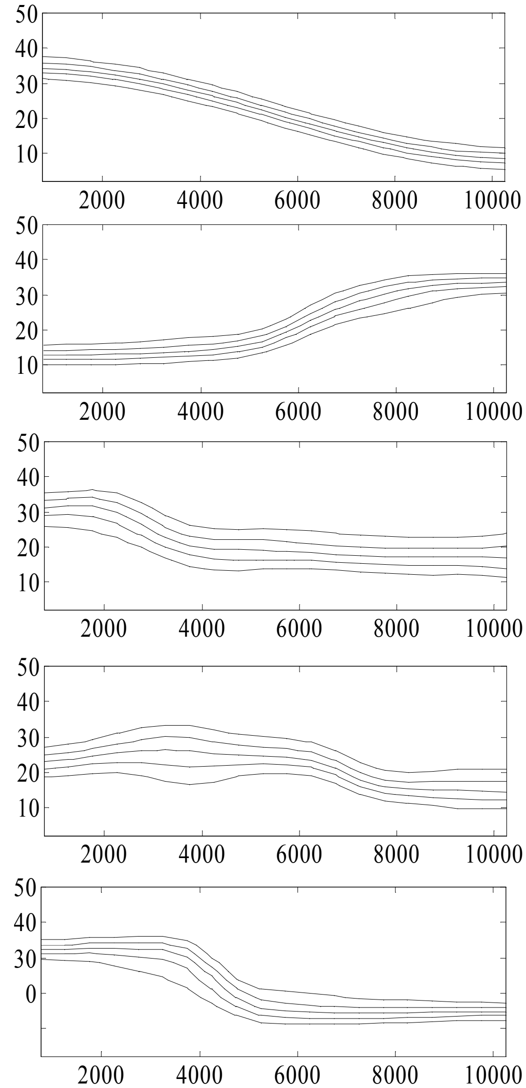

Figure 2. Non-hydrostatic system, with a hydrostatic model: 10km long, 50m deep, 13m initial wave amplitude.

In our research, hydrostatic and nonhydrostatic models are applied to several test cases to examine the propagation of internal waves within a system. Application of both models allows comparison of model results and quantification of model skill in the simulation of internal wave dynamics. The nonhydrostatic model resolves both wave steepening and dispersion effects so the correct shape of the propagating internal wave is retained. In contrast, the hydrostatic model does not account for the dispersive forces, thus nonlinear wave steepening is allowed to progress unabated and the shape of the propagating internal wave is incorrect. Comparisons are used to analyze where dynamic pressure has a contributing role in the physics of the system. Quantification of dynamic pressure effects allows assessment of the circumstances appropriate for the hydrostatic assumption. As dynamic pressure is proportional to vertical acceleration, the ratio between the modeled vertical and horizontal accelerations is investigated as an indicator of the dynamic state of a region. Where this ratio is large, a nonhydrostatic state is presumed to exist and dynamic pressure needs to be resolved; where the ratio is small, the hydrostatic state should predominate. This ratio is used to analyze model performance and our ability to *a priori* select the correct model type for a particular internal wave regime.

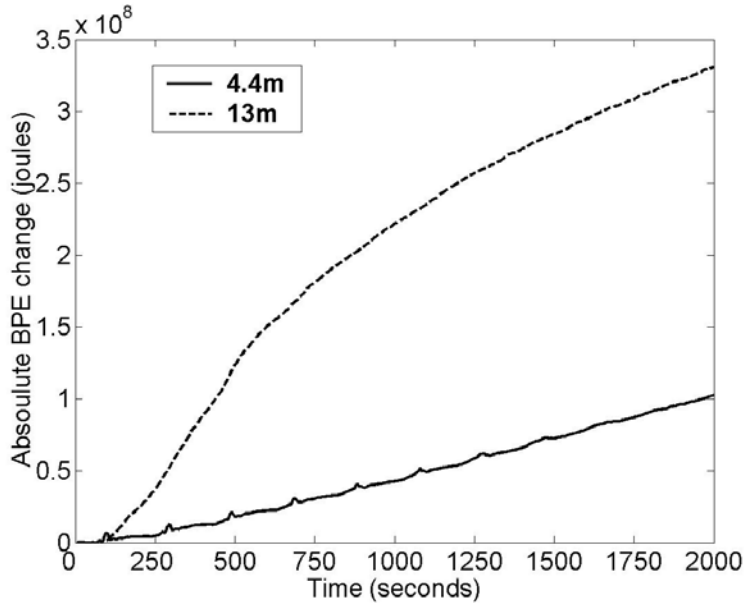

Figure 3. Comparison of background potential energy.

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