A Second-Order Correction for Semi-Implicit Shallow Water Methods

… or, is second-order really?

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Conclusions

The C-N 2nd-order discretization of the semi-implicit shallow water continuity equation integrated over depth at *time 'n'* is not formally 2nd order accurate

However, a correction term can be derived and applied.

The effect of the correction term is insignificant unless a/D >> 1/100 and the barotropic CFL < 1

At practical grid and time scales (for inland waters), the C-N method is typically 1st order convergent

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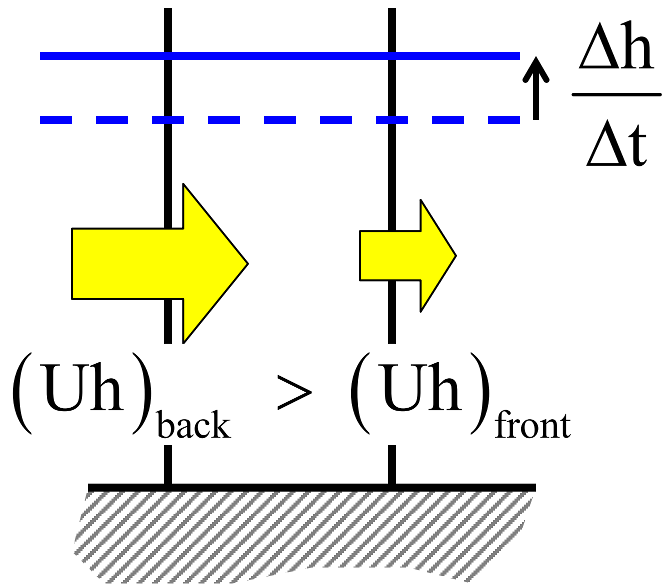
Continuity for the shallow-water equations

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_b^h u\,dz - \frac{\partial}{\partial y}\int_b^h v\,dz$$



which is simply conservation of surface height with water column fluxes:

$$\frac{\Delta h}{\Delta t} = (Uh)_{back} - (Uh)_{front} + (Vh)_{right} - (Vh)_{left}$$

…nothing new here

next

The simplest approach is backwards Euler

$$h^{n+1} = h^n - \Delta t \frac{\partial}{\partial x}\left\{ \int_0^{h^n} u^{n+1}\, dz \right\} + O\left(\Delta t^2\right)$$

Note the h^n is used in the limit – not h^{n+1}, which means this isn't *exactly* backwards Euler. However, it can be shown that between h^n and h^{n+1}

$$\Delta t \frac{\partial}{\partial x} \int_{h^n}^{h^{n+1}} u^{n+1}\, dz \sim \Delta t \frac{\partial}{\partial x}\left(h^{n+1} - h^n\right) u^{n+1}_{\left(h^{n+1}\right)} \sim O\left(\Delta t^2\right)$$

…so "fudging" the limit of integration doesn't affect the discretization accuracy

next

Fudging the integration limit is critical in efficient semi-implicit methods

Continuity

$$h^{n+1} = h^n - \Delta t \frac{\partial}{\partial x}\left\{\int_0^{h^n} u^{n+1} dz\right\} + O\left(\Delta t^2\right)$$

Momentum

$$u^{n+1} = F\left(u^n\right) - g\Delta t\left\{\frac{\partial h^{n+1}}{\partial x}\right\}$$

if limit had h^{n+1}, term would be implicit nonlinear!

substituting gives

$$h^{n+1} = h^n - \Delta t \frac{\partial}{\partial x}\left\{\int_0^{h^n} F\left(u^n\right) dz - \Delta t\, g\, h^n \frac{\partial h^{n+1}}{\partial x}\right\} + O\left(\Delta t^2\right)$$

This is a banded matrix problem that is linear in time n+1

$$h^{n+1} + \alpha \frac{\partial^2 h^{n+1}}{\partial x^2} = S^n$$

Key point

For 1st order semi-implicit, we can take a depth integration over time 'n' and obtain first order accuracy in a linear implicit free surface solution

$$h^{n+1} + \alpha \frac{\partial^2 h^{n+1}}{\partial x^2} = S^n + O(\Delta t^2)$$

There's lots of efficient ways to solve this banded matrix (nothing new here either!)

2^{nd} order: Crank-Nicolson discretization

$$h^{n+1} = h^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}\left\{ \int_0^{h^n} u^n \, dz + \int_0^{h^{n+1}} u^{n+1} \, dz \right\} + O\left(\Delta t^3\right)$$

$$u^{n+1} = F\left(u^n\right) - g\frac{\Delta t}{2}\left\{ \frac{\partial h^n}{\partial x} + \frac{\partial h^{n+1}}{\partial x} \right\}$$

Substituting and combining time 'n' terms into a single source

$$h^{n+1} = S^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}\left\{ \int_{h^n}^{h^{n+1}} F(u^n) \, dz \right.$$

Again, we have an implicit nonlinearity in time (n+1) depth

$$\left. - \frac{g\Delta t}{2}\left[\left(h^{n+1} - h^n\right)\frac{\partial h^n}{\partial x} + h^{n+1}\frac{\partial h^{n+1}}{\partial x} \right] \right\} + O\left(\Delta t^3\right)$$

next

Fudging the limits changes things…

$$h^{n+1} = S^n - \frac{\Delta t}{2}\frac{\partial}{\partial x}\left\{ \int_{h^n}^{h^{n+1}} F(h^n)\,dz - \frac{g\Delta t}{2}\left[\left(h^{n+1}-h^n\right)\frac{\partial h^n}{\partial x} + h^{n+1}\frac{\partial h^{n+1}}{\partial x}\right]\right\} + O\left(\Delta t^3\right)$$

…resulting in

$$h^{n+1} - \frac{g\Delta t^2}{4} h^n \frac{\partial^2 h^{n+1}}{\partial x^2} = \tilde{S}^n + O\left(\Delta t^{??}\right)$$

A nice banded matrix – but is it second order?

next

Key point

A semi-implicit shallow water approximate C-N scheme integrated over the *time 'n'* depth neglects 3 terms

$$\frac{\Delta t}{2}\frac{\partial}{\partial x}\int_{h^n}^{h^{n+1}} F(u^n)\,dz \qquad \sim O(\Delta t^2)$$

$$\frac{g\Delta t^2}{2}\frac{\partial}{\partial x}\left\{\left(h^{n+1}-h^n\right)\frac{\partial h^n}{\partial x}\right\} \qquad \sim O(\Delta t^3)$$

$$\frac{g\Delta t^2}{2}\frac{\partial}{\partial x}\left\{\left(h^{n+1}-h^n\right)\frac{\partial h^{n+1}}{\partial x}\right\} \qquad \sim O(\Delta t^3)$$

So it loses formal 2nd order accuracy! due to a term that is *linear* in time n+1

next

What do we do?

… it can be shown that

$$\frac{\Delta t}{2}\frac{\partial}{\partial x}\int_{h^n}^{h^{n+1}} F(u^n)\,dz \quad = \quad \frac{\Delta t}{2}\frac{\partial}{\partial x}\left(h^{n+1}-h^n\right)F(u^n)_{h^n} + O(\Delta t^3)$$

So the former banded matrix…

$$h^{n+1} - \alpha_1 \Delta t^2 \frac{\partial^2 h^{n+1}}{\partial x^2} = \tilde{S}^n + O\left(\Delta t^2\right)$$

…becomes:

$$h^{n+1} - \alpha_2 \Delta t \frac{\partial \beta h^{n+1}}{\partial x} - \alpha_1 \Delta t^2 \frac{\partial^2 h^{n+1}}{\partial x^2} = \hat{S}^n + O\left(\Delta t^3\right)$$

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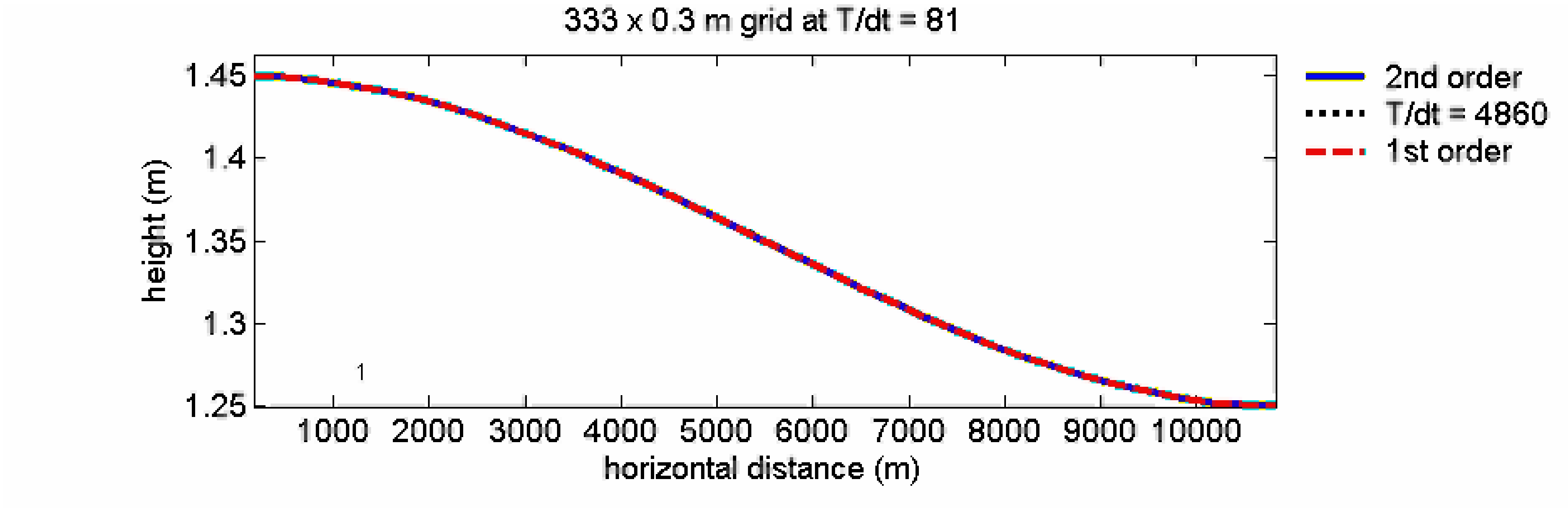
Effect of the 2nd order correction



Cosine wave

a = 0.1 m

λ = 21 km

D = 1.35 m

Navier-Stokes model using hydrostatic, Boussinesq, 2D (x-z), inviscid assumptions

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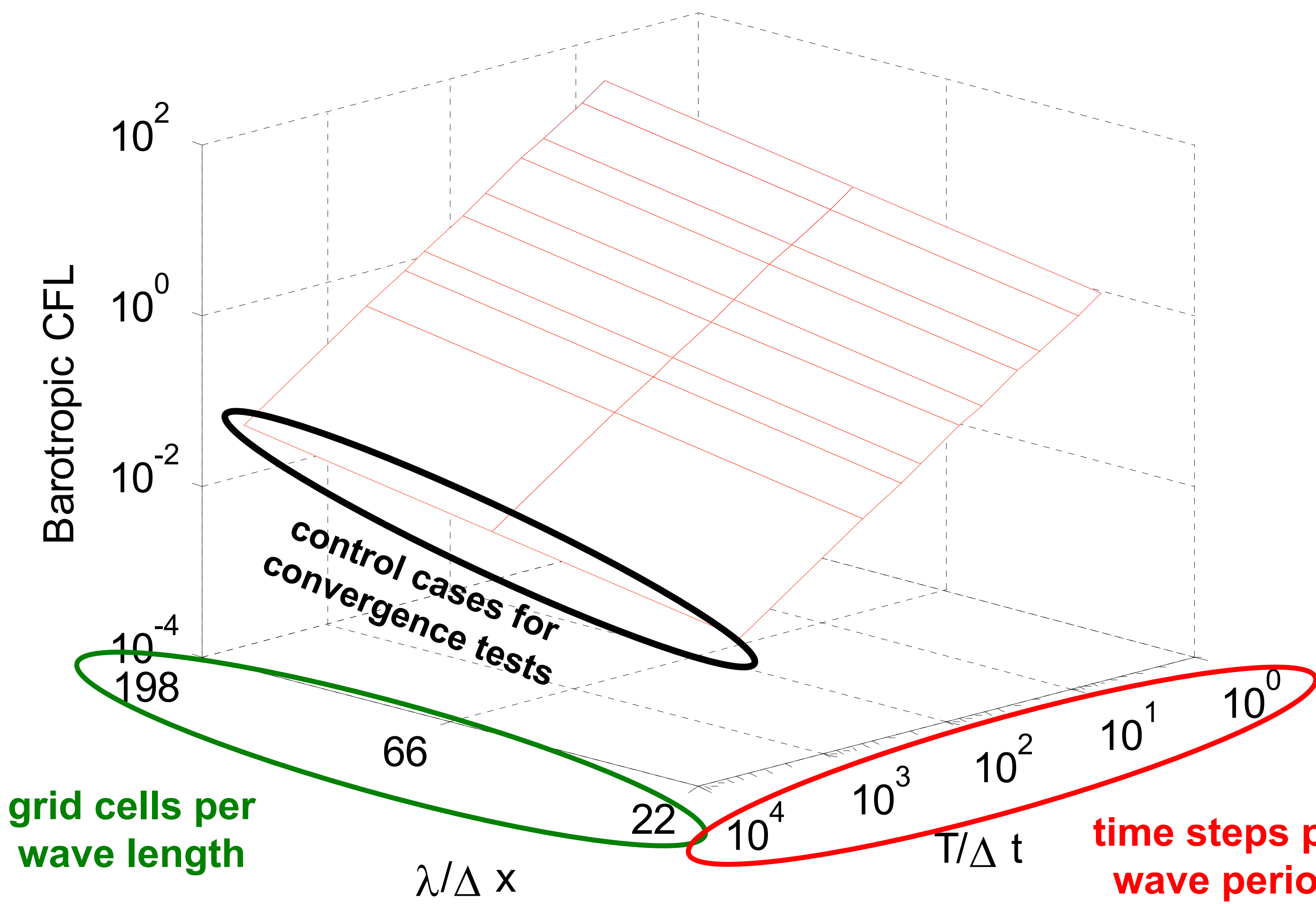
Examine a range of space and time scales



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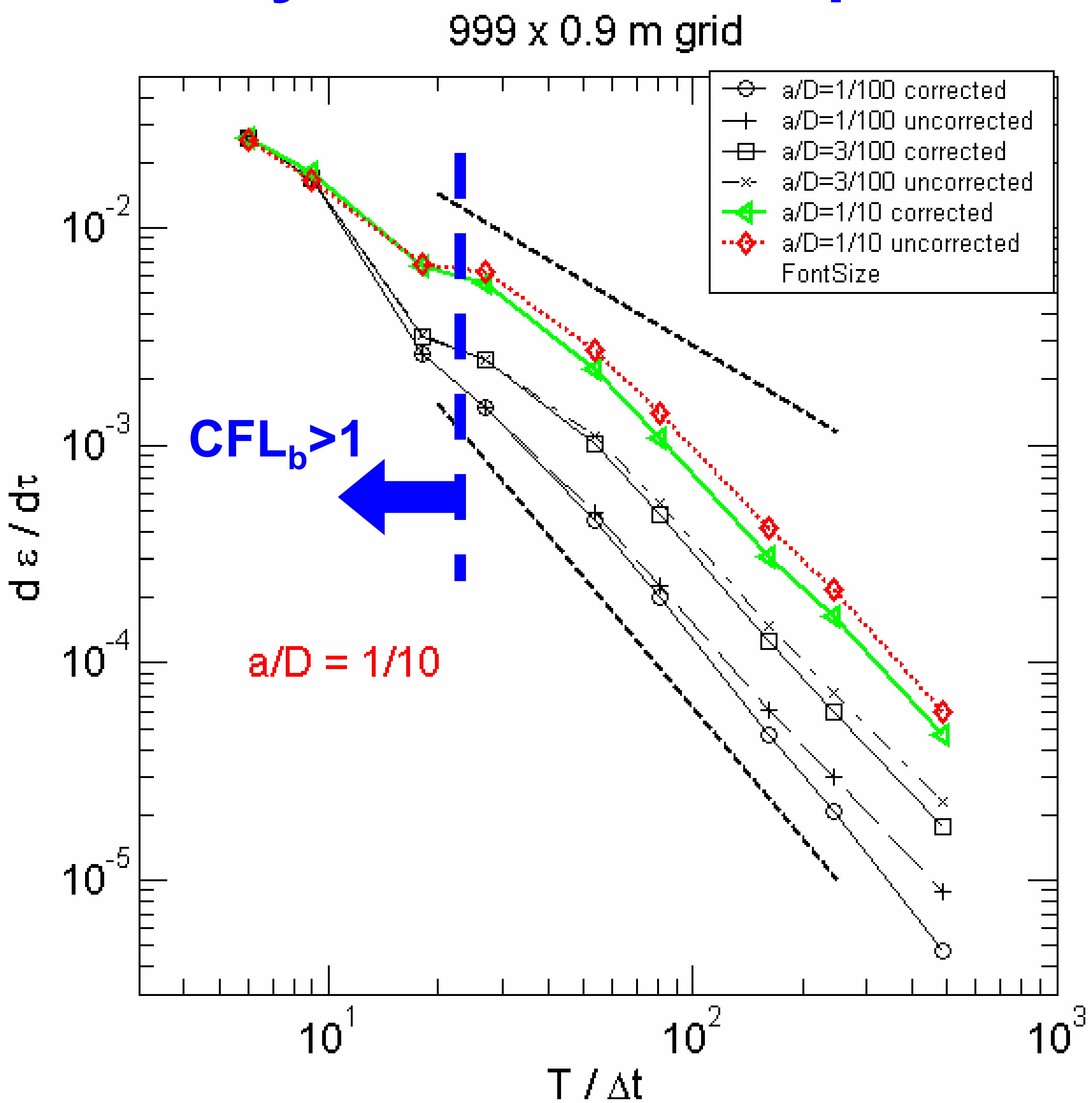
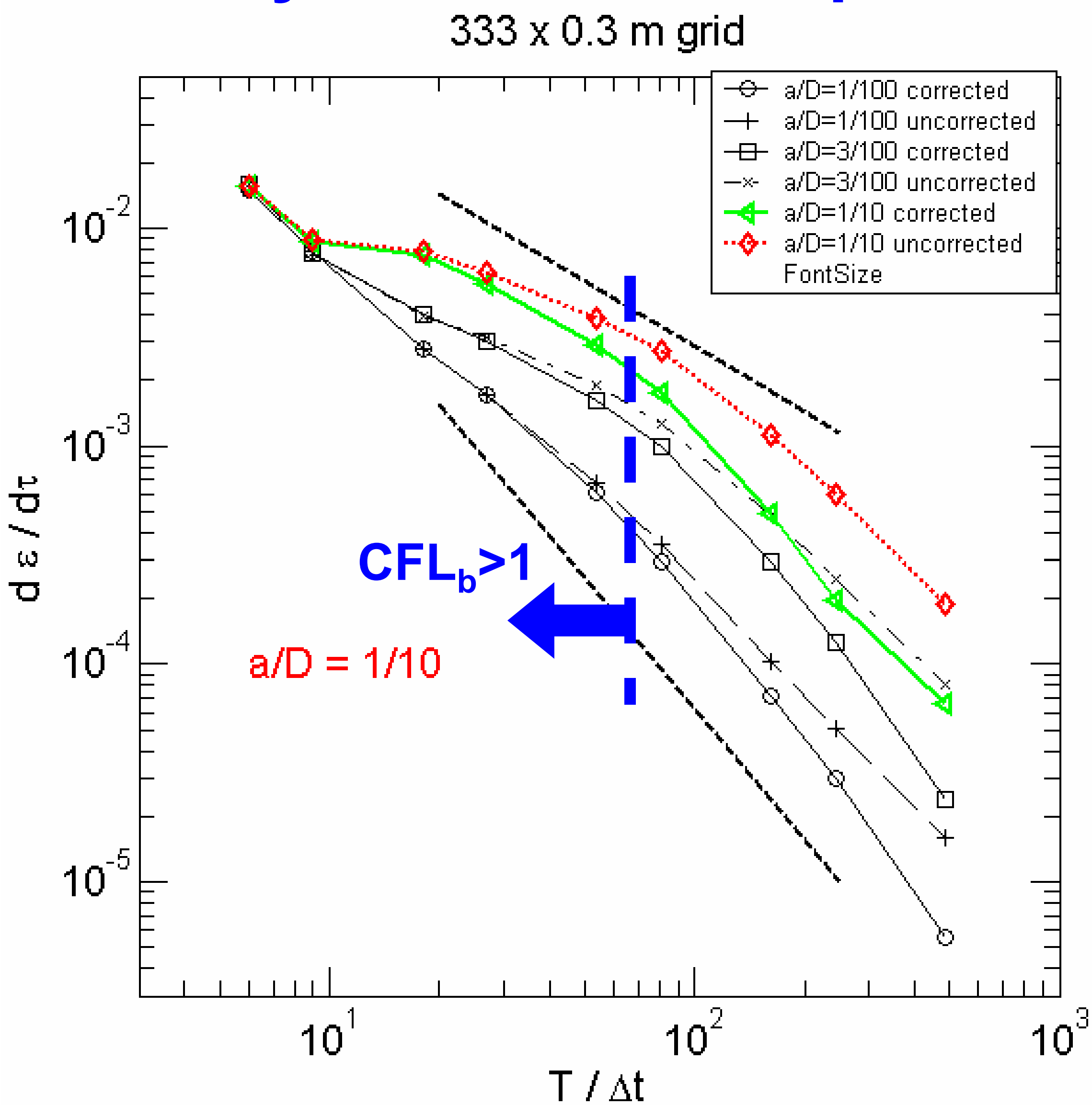
Accuracy as the time step is refined



11 horizontal grid cells for ½ of a cosine wave

next

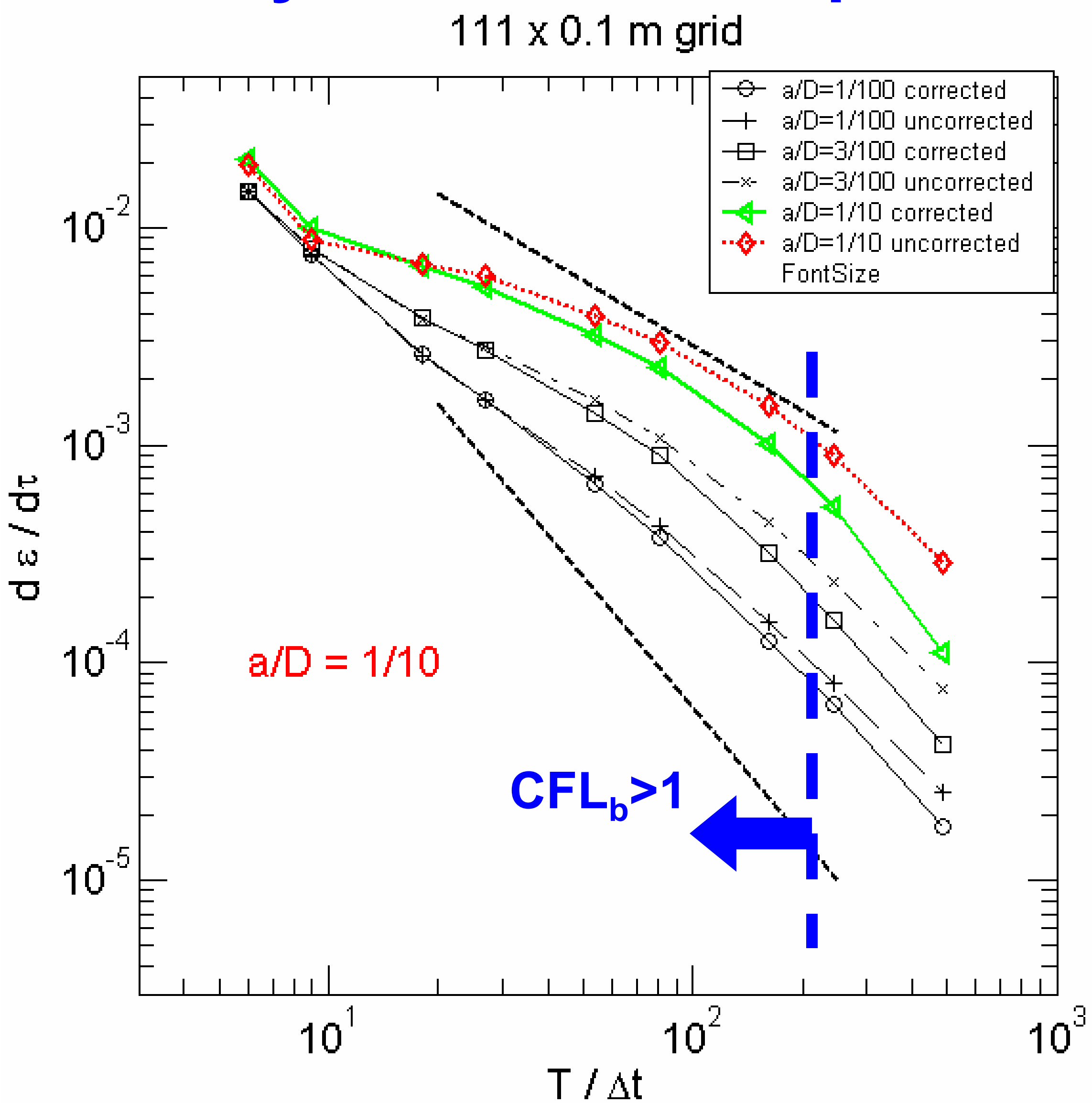
Accuracy as the time step is refined



33 horizontal grid cells for ½ of a cosine wave

next

Accuracy as the time step is refined



99 horizontal grid cells for ½ of a cosine wave

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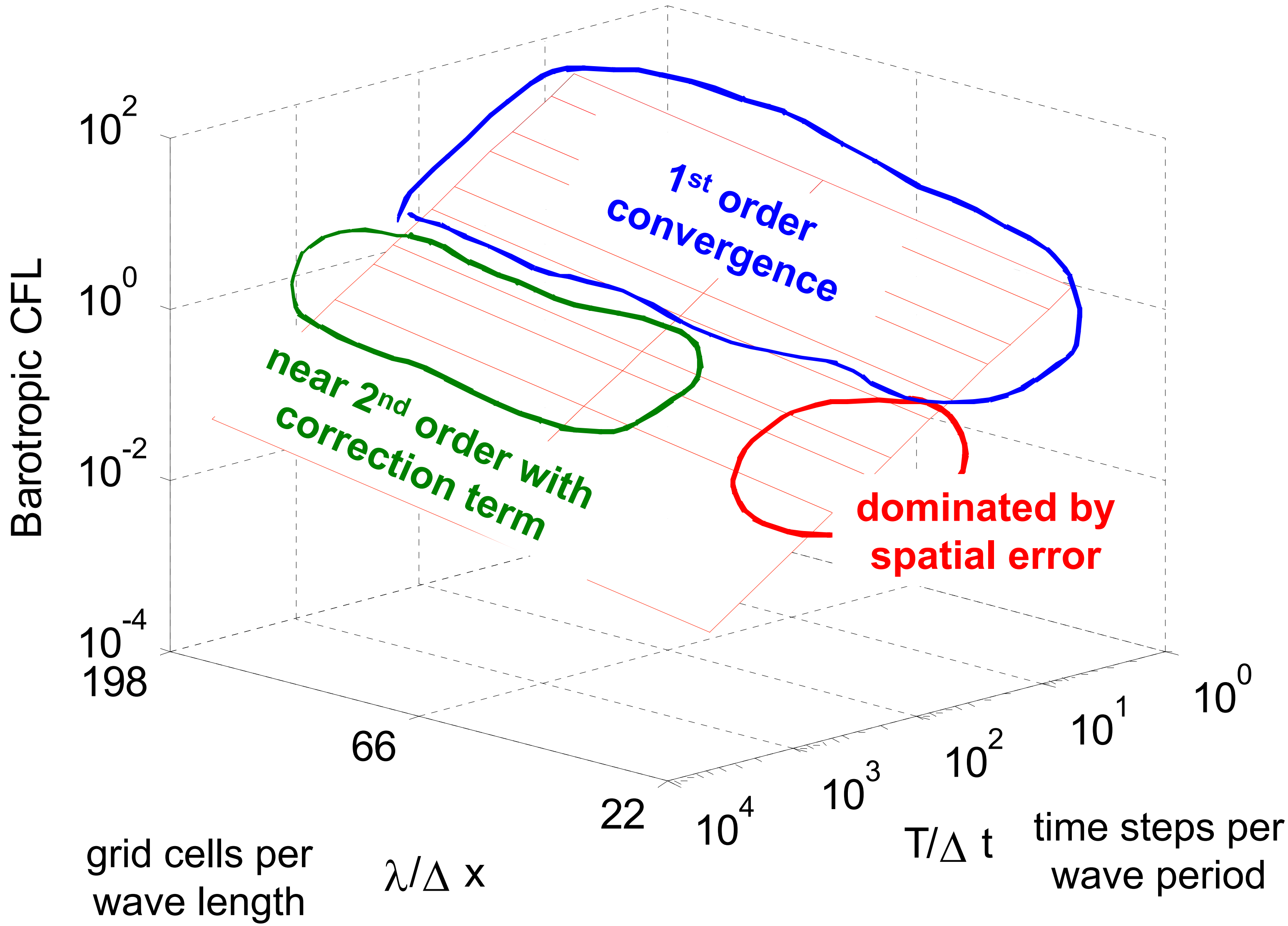
Results over space and time scales



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Analysis

At coarse grid scales, the spatial error dominates and the temporal correction term is unimportant

At finer grid scales, 2nd order requires the temporal correction term when a/D > 1/100

At *practical* grid and time scales (CFL_b > 1), the correction term is irrelevant as the convergence is 1st order in all cases

…what about 1st order methods?

next

Theta method

The arbitrary weighting of ‘n’ and ‘n+1’ terms is often used to reduce C-N numerical dispersion

Theoretically 1st order accurate

1st order accuracy is only obtained for barotropic CFL < 1

… as one might expect, the 2nd order correction has no significant effect on accuracy

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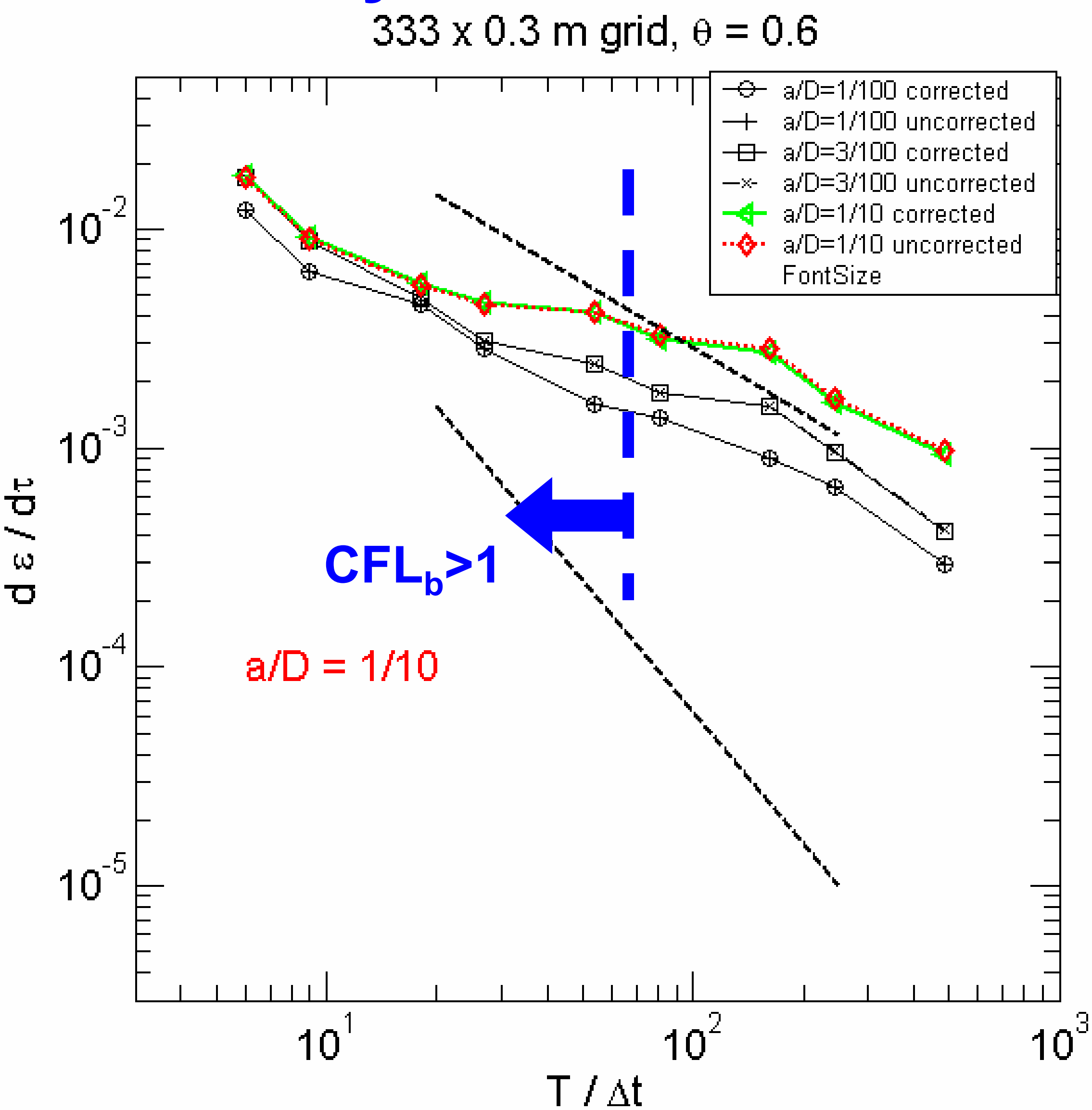
Accuracy for the theta method



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