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A SYMMETRIC VARIATIONAL FINITE ELEMENT–BOUNDARY INTEGRAL EQUATION COUPLING METHOD

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Abstract—This paper is concerned with the development of a mixed variational principle for coupling finite element and boundary integral methods in interface problems, using the generalized Poisson's equation as a prototype situation. One of its primary objectives is to compare the performance of fully variational procedures with methods that use collocation for the treatment of boundary integral equations. A distinctive feature of the new variational principle is that the discretized algebraic equations for the coupled problem are automatically symmetric since they are all derived from a single functional. In addition, the condition that the flux remain continuous across interfaces is satisfied naturally. In discretizing the problem within inhomogeneous or loaded regions, domain finite elements are used to approximate the field variable. On the other hand, only boundary elements are used for regions where the medium is homogeneous and free of external agents. The corresponding integral equations are discretized both by fully variational and by collocation techniques. Results of numerical experiments indicate that the accuracy of the fully variational procedure is significantly greater than that of collocation for the complete interface problem, especially for complex disturbances, at little additional computational cost. This suggests that fully variational procedures may be preferable to collocation, not only in dealing with interface problems, but even for solving integral equations by themselves.

1. INTRODUCTION

Coupled finite element (FE) and boundary element (BE) methods are especially well-suited for dealing with problems that are defined over a combination of homogeneous regions free of body forces and regions within which body loads are present or where the material is inhomogeneous, even possibly nonlinear. Such methods have been used increasingly in engineering since the early 1970s [1, 2]; their mathematical analysis was initiated with the work of Brezzi, Johnson, and Nedelec [3, 4]. Many different variants of these methods have been developed over the years; see, for example, Hsiao [5] for a recent survey.

In most applications to date, while the domain finite element part of the formulation is generally derived from a variational principle, the boundary element part is usually obtained by direct collocation of the corresponding boundary integral equation. Only in isolated cases have variational principles been used to derive all the discretized equations. Hence, the resulting algebraic systems are often non-symmetric. Since symmetry is a desirable property in numerical computations, substantial effort has been devoted to developing symmetric formulations. Representative examples of each kind of formulation are cited in Table 1.

Despite the recent progress on coupled FE/BE methods there has been little effort to assess the relative advantages and disadvantages of various procedures. The main objective of this paper is to present a fully variational symmetric coupling method, using Poisson's equation as a prototype, and to compare results of numerical experiments with those from a related collocation procedure. The fully variational method presented here is based on a coupling method [10] developed originally for solving the Helmholtz equation in combined exterior and interior regions, and applied subsequently [13] to the numerical study of propagation of incoming SH-waves through inhomogeneous valleys surrounded by a halfspace. In determining the coefficients of the system of discretized algebraic equations with this methodology, it is necessary to evaluate certain double integrals that involve Green's function or its normal derivative as a kernel. By approximating one of the variables by Dirac-delta functions one can arrive instead at a collocation procedure that involves only single integrations. It is of interest to examine how the fully variational and the collocation procedures perform in problems that involve several interior subregions. In particular, inasmuch as collocation is currently the prevalent method for discretizing integral equations, it is important to assess both the relative computational complexity of the two procedures and their relative accuracy.

In Sec. 2 we formulate the prototype problem under study and present the corresponding variational formulation. Two distinct discretization procedures are described in Sec. 3, and in Sec. 4 we give

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Table 1. Representative BE/FE methods

	Variational approach combined with collocation	Fully variational
Non-symmetric	Belytschko *et al.* [6] Cruse [7] Mathews [8]	—
Symmetric	Zienkiewicz *et al.* [2] McDonald and Wexler [1] Mathews [8]	Hamdi and Jean [9] Bielak and MacCamy [10] Costabel [11] Jeans and Mathews [12]

a numerical example and compare results from the two schemes. The main observation from this comparison is that the fully variational technique yields significantly more accurate results than the collocation procedure, especially at interfaces and other boundaries, and for increasingly complex loading. This result is dramatically illustrated in Table 2.

2. STATEMENT OF PROBLEM AND VARIATIONAL FORMULATION

Let Ω be a bounded region in $\mathbf{R}^2$, with boundary $\partial\Omega$. Ω consists of two distinct parts Ω_b and Ω_f, as shown on Fig. 1; the first subdomain is homogeneous and load-free while the second may be inhomogeneous and be subjected to external loads. Let Γ be the interface between Ω_f and Ω_b, and let Γ_b, the part of $\partial\Omega$ surrounding Ω_b, be subdivided into Γ_b^u and $\Gamma_b^{u_n}$. Similarly, let $\Gamma_f = \partial\Omega \backslash \Gamma_b$ be partitioned into Γ_f^u and $\Gamma_f^{u_n}$. Let k_f be a positive function in Ω_f, p be the applied load, and k_b a positive constant. Also, let u^b, t^b, u^f and t^f be prescribed functions on Γ_b^u, $\Gamma_b^{u_n}$, Γ_f^u and $\Gamma_f^{u_n}$, respectively. We then consider the problem of finding u in Ω such that

$$\nabla\cdot(k_f\nabla u) = p \quad \text{in } \Omega_f, \tag{1a}$$

$$k_b\nabla^2 u = 0 \quad \text{in } \Omega_b, \tag{1b}$$

$$u^- = u^+ \quad \text{on } \Gamma, \tag{1c}$$

$$k_f^- u_n^- = k_b u_n^+ \quad \text{on } \Gamma, \tag{1d}$$

$$u = u^f \quad \text{on } \Gamma_f^u, \tag{1e}$$

$$u_n = t^f \quad \text{on } \Gamma_f^{u_n}, \tag{1f}$$

$$u = u^b \quad \text{on } \Gamma_b^u, \tag{1g}$$

$$u_n = t^b \quad \text{on } \Gamma_b^{u_n}, \tag{1h}$$

and denote this as Problem P. $\mathbf{n}$ is the exterior normal to Ω as well as the unit normal to Γ, pointing toward Ω_f, and the plus and minus signs indicate limits from Ω_f and Ω_b. The subscript n denotes normal derivative, that is, $u_n = \nabla u\cdot\mathbf{n}$. Physically, (1) may be given several different interpretations, but to fix ideas we will regard u as the temperature field in the closure $\bar{\Omega}$ generated by the source p and the various prescribed temperatures and fluxes on $\partial\Omega$. k_f and k_b then represent thermal conductivities.

To derive the variational form of problem P we start with the standard potential energy functional

$$\Pi[u] = \frac{1}{2}\int_{\Omega_f} k_f\nabla u\cdot\nabla u\,d\Omega + \frac{1}{2}\int_{\Omega_b} k_b\nabla u\cdot\nabla u\,d\Omega - \int_{\Omega_f} pu\,d\Omega - \int_{\Gamma_f^{u_n}} t^f u\,dS - \int_{\Gamma_b^{u_n}} t^b u\,dS. \tag{2}$$

Table 2. Relative errors, in percent, for example with $u = u_0\cos n\theta$, for different harmonics and number of elements, at three different locations on ray $\theta = 0°$. The table compares errors between fully variational and collocation approaches

			Radius (r/a)					
			2		3		8	
n	N_r	N_θ	Variational	Collocation	Variational	Collocation	Variational	Collocation
0	2	4	2.48(−3)†	2.72(−3)	1.15(−1)	1.90(−2)	2.92(−2)	9.66(−2)
	4	8	1.21(−3)	1.05(−3)	1.69(−2)	1.86(−3)	4.43(−3)	7.75(−2)
	8	16	1.57(−4)	3.82(−4)	2.21(−3)	2.60(−4)	7.57(−4)	3.94(−2)
1	2	4	5.57(−2)	2.17(+0)	2.18(−1)	1.67(+1)	7.14(−1)	5.36(+0)
	4	8	3.03(−2)	1.73(−2)	3.75(−2)	1.21(+0)	3.28(−2)	1.07(−1)
	8	16	1.67(−3)	2.99(−3)	1.83(−2)	1.14(−1)	5.09(−4)	7.26(−3)
2	2	4	1.19(+1)	1.30(+1)	1.45(+1)	2.16(+1)	1.51(+1)	1.88(+1)
	4	8	5.72(−1)	6.69(+0)	8.35(−1)	3.88(+1)	9.79(−1)	1.01(+2)
	8	16	2.63(−2)	5.19(−1)	3.22(−2)	3.42(+0)	5.29(−2)	4.66(+0)
3	2	4	6.27(+0)	9.71(+0)	6.90(+0)	1.08(+2)	7.33(+0)	1.27(+2)
	4	8	2.68(−1)	1.94(+0)	4.39(−1)	1.59(+1)	4.52(−1)	5.05(+1)
	8	16	1.67(−2)	1.34(−1)	6.14(−3)	2.69(+0)	2.76(−2)	2.15(+0)

† 2.48(−3) means 2.48×10^{-3}.

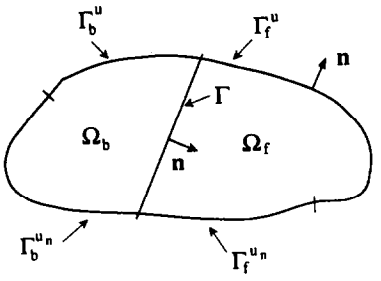

Fig. 1. Model of interface problem with nomenclature for various subdomains: (Ω_f) domain approach; (Ω_b) boundary integral approach.

We assume that the admissible functions satisfy the essential boundary conditions (1c), (1e), and (1g). Given sufficient regularity it is then straightforward to show that the variation of (2) for all admissible u vanishes if and only if (1) holds.

Now, with the aid of the divergence theorem, the second term on the right side of (2) can be rewritten as

$$\int_{\Omega_b} k_b \nabla u \cdot \nabla u \, d\Omega = -\int_{\Omega_b} k_b u \nabla^2 u \, d\Omega + \int_{\partial\Omega_b} u k_b u_n \, dS, \quad (3)$$

in which $\partial\Omega_b = \Gamma \cup \Gamma_b^u \cup \Gamma_b^{u_n}$, and if we assume that all admissible u in Ω_b are such that (1b) is satisfied, then the left-hand side of (3) can be expressed exclusively in terms of a boundary integral. To ensure that u satisfies (1b), we require that the Cauchy boundary values u and u_n on $\partial\Omega_b$ satisfy the well-known Helmholtz integral representation fomula (written for smooth $\partial\Omega_b$)

$$\tfrac{1}{2}u(\mathbf{x}) - \int_{\partial\Omega_b} u(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_y} dS_y + \int_{\partial\Omega_b} u_n(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) \, dS_y = 0, \quad \mathbf{x} \in \partial\Omega_b. \quad (4)$$

Here $G(\mathbf{x}, \mathbf{y})$ is the Green's function

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln|\mathbf{x} - \mathbf{y}|, \quad \mathbf{x}, \mathbf{y} \in \bar{\Omega}_b, \quad (5a)$$

which satisfies the symmetry relationships

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}), \quad (5b)$$

$$\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_y} = \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n_x}. \quad (5c)$$

Then, by substituting (3) into (2), after setting $\nabla^2 u = 0$ within Ω_b, and introducing (4) via a Lagrange multiplier, ϕ, we obtain a modified functional $\hat{\Pi}$ that involves only boundary terms corresponding to Ω_b. In this formulation we regard u_n as an additional independent unknown, separate from u, and denote it as ψ, i.e. $u_n = \psi$ on $\Gamma_b^u \cup \Gamma$. The functional $\hat{\Pi}$ just described then becomes

$$\hat{\Pi}[u, \phi, \psi] = \frac{1}{2}\int_{\Omega_f} k_f \nabla u \cdot \nabla u \, d\Omega + \frac{1}{2}\int_{\partial\Omega_b} u k_b \psi \, dS - \frac{1}{2}\int_{\partial\Omega_b} \left[\frac{1}{2}u(\mathbf{x}) - \int_{\partial\Omega_b} u(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_y} dS + \int_{\partial\Omega_b} \psi(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) \, dS_y\right] \phi(\mathbf{x}) \, dS_x - \int_{\Omega_f} pu \, d\Omega - \int_{\Gamma_f^{u_n}} t^f u \, dS - \int_{\Gamma_f^{u_n}} t^b u \, dS. \quad (6)$$

This functional provides the basis for our variational principle. By taking the first variation of (6), and making use of (5b, c), it can be readily shown that the triad $[u, \psi, \phi]$ gives a solution to problem P if and only if the first variation $\delta\hat{\Pi}$ vanishes for admissible variations δu, $\delta\psi$ and $\delta\phi$; the admissible δus are required to satisfy homogeneous Dirichlet conditions on $\partial\Omega_b$, while $\delta\psi$ and $\delta\phi$ need not satisfy any boundary conditions, u is the solution of problem P within Ω_f, as well as on the boundaries $\Gamma_f^{u_n}$, $\Gamma_b^{u_n}$, Γ; ψ gives the normal derivative u_n on $\Gamma_f^{u_n}$, $\Gamma_b^{u_n}$, Γ; and ϕ yields the solution for u in Ω_b via

$$u(\mathbf{x}) = \int_{\partial\Omega_b} G(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \, dS_y, \quad \mathbf{x} \in \Omega_b \quad (7)$$

that is, ϕ, the Lagrange multiplier in (6), turns out to be the density of a single-layer potential that completely determines u within Ω_b. u may also be obtained in terms of the Cauchy boundary values through

$$u(\mathbf{x}) = \int_{\partial\Omega_b} u(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_y} dS_y - \int_{\partial\Omega_b} G(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \, dS_y, \quad \mathbf{x} \in \Omega_b. \quad (8)$$

Remarks. (i) The flux transition condition (1d), as well as the additional Neumann boundary conditions (1f) and (1h) are satisfied naturally by the variational principle. This means that when we approximate, as below, with finite elements within Ω, $\partial\Omega$, and Γ one only needs to impose the essential boundary conditions (1c), (1e), and (1f) on the trial functions, and corresponding homogeneous Dirichlet conditions on the test functions. (ii) If one chooses the admissible ϕ to be in the same space as ψ, i.e. at least piecewise continuous, the discretization procedure will yield a full finite element formulation for both subregions Ω_f and Ω_b, entailing the evaluation of several double integrals, as may be seen from (6). If, on the other

hand, ϕ is represented as a linear weighted combination of Dirac-delta functions, $\delta(\mathbf{y}-\mathbf{y}_i)$, this will be equivalent to solving the integral equation (4) by collocation, requiring that it be satisfied at the preselected points $\mathbf{y}_i$. In that case the double integrals reduce to single integrals. We emphasize that both approaches are based on our variational principle. To differentiate between the two procedures, however, we reserve the term fully variational for the former approach, while the latter will be referred to as collocation. (iii) Since our variational principle involves taking the variation of a trilinear functional, the corresponding discretized algebraic equations will always be symmetric, whether one chooses admissible ϕs that are piecewise continuous, discontinuous, or given by concentrated sources. Thus, our collocation procedure will also lead to a symmetric system of algebraic equations.

3. FINITE ELEMENT DISCRETIZATION

We consider here the numerical solution of the variational problem, initially using standard finite element methodology. To approximate u, introduce a finite-dimensional subspace S^h in Ω_f depending on the mesh size h, and another basis on $\partial\Omega_b$ to represent ψ and ϕ. Following the usual steps, the variational principle characterized by (6) leads to a system of algebraic equations of the form

$$\begin{bmatrix} \mathbf{K}_{\Omega_f\Omega_f} & \mathbf{K}_{\Omega_f\Gamma} & 0 & 0 & 0 \\ \mathbf{K}_{\Gamma\Omega_f} & \mathbf{K}_{\Gamma\Gamma} & 0 & \frac{1}{2}\mathbf{K}_{\Gamma\psi} & -\frac{1}{2}\mathbf{K}_{\Gamma\phi} \\ 0 & 0 & 0 & \frac{1}{2}\mathbf{K}_{\Gamma_b^{u_n}\psi} & -\frac{1}{2}\mathbf{K}_{\Gamma_b^{u_n}\phi} \\ 0 & \frac{1}{2}\mathbf{K}_{\psi\Gamma} & \frac{1}{2}\mathbf{K}_{\psi\Gamma_b^{u_n}} & 0 & -\frac{1}{2}\mathbf{K}_{\psi\phi} \\ 0 & -\frac{1}{2}\mathbf{K}_{\phi\Gamma} & -\frac{1}{2}\mathbf{K}_{\phi\Gamma_b^{u_n}} & -\mathbf{K}_{\phi\psi} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\Omega_f} \\ \mathbf{u}_\Gamma \\ \mathbf{u}_{\Gamma_b^{u_n}} \\ \boldsymbol{\psi} \\ \boldsymbol{\phi} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}_{\Omega_f} \\ \mathbf{p}_\Gamma \\ \mathbf{p}_{\Gamma_b^{u_n}} \\ \mathbf{p}_\psi \\ \mathbf{p}_\phi \end{Bmatrix}, \tag{9}$$

in which $\mathbf{u}_{\Omega_f}$ are the nodal temperatures within Ω_f and on $\Gamma_f^{u_n}$; $\mathbf{u}_\Gamma$ and $\mathbf{u}_{\Gamma_b^{u_n}}$ are the nodal temperatures at the interface Γ between Ω_f and Ω_b, and on $\Gamma_b^{u_n}$, respectively; $\boldsymbol{\psi}$ represents nodal values of the flux† on $\Gamma\cup\Gamma_b^u$, and $\boldsymbol{\phi}$ is the vector of nodal values of the density of the single layer in (7). In practice, rather than solving (9) directly, it is sometimes convenient to condense $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$. Thus, one arrives at a reduced system of equations with only temperatures as unknowns

$$\begin{bmatrix} \mathbf{K}_{\Omega_f\Omega_f} & \mathbf{K}_{\Omega_f\Gamma} & 0 \\ \mathbf{K}_{\Gamma\Omega_f} & \mathbf{K}_{\Gamma\Gamma}+\mathbf{S}_{\Gamma\Gamma} & \mathbf{S}_{\Gamma\Gamma_b^{u_n}} \\ 0 & \mathbf{S}_{\Gamma_b^{u_n}\Gamma} & \mathbf{S}_{\Gamma_b^{u_n}\Gamma_b^{u_n}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\Omega_f} \\ \mathbf{u}_\Gamma \\ \mathbf{u}_{\Gamma_b^{u_n}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}_{\Omega_f} \\ \mathbf{p}_{\Gamma\phi\psi} \\ \mathbf{p}_{\Gamma_b^{u_n}\phi\psi} \end{Bmatrix}, \tag{10}$$

in which

$$\mathbf{S}_{\Gamma\Gamma} = -\tfrac{1}{2}(\mathbf{K}_{\Gamma\psi}\mathbf{K}_{\phi\psi}^{-1}\mathbf{K}_{\phi\Gamma} + \mathbf{K}_{\Gamma\phi}\mathbf{K}_{\psi\phi}^{-1}\mathbf{K}_{\psi\Gamma}) = \mathbf{S}_{\Gamma\Gamma}^T, \tag{11a}$$

$$\mathbf{S}_{\Gamma\Gamma_b^{u_n}} = -\tfrac{1}{2}(\mathbf{K}_{\Gamma\psi}\mathbf{K}_{\phi\psi}^{-1}\mathbf{K}_{\phi\Gamma_b^{u_n}} + \mathbf{K}_{\Gamma\phi}\mathbf{K}_{\psi\phi}^{-1}\mathbf{K}_{\psi\Gamma_b^{u_n}}) = \mathbf{S}_{\Gamma_b^{u_n}\Gamma}^T, \tag{11b}$$

$$\mathbf{S}_{\Gamma_b^{u_n}\Gamma_b^{u_n}} = -\tfrac{1}{2}(\mathbf{K}_{\Gamma_b^{u_n}\psi}\mathbf{K}_{\phi\psi}^{-1}\mathbf{K}_{\phi\Gamma_b^{u_n}} + \mathbf{K}_{\Gamma_b^{u_n}\phi}\mathbf{K}_{\psi\phi}^{-1}\mathbf{K}_{\psi\Gamma_b^{u_n}}) = \mathbf{S}_{\Gamma_b^{u_n}\Gamma_b^{u_n}}^T, \tag{11c}$$

and

$$\mathbf{p}_{\Gamma\phi\psi} = \mathbf{p}_\Gamma + \mathbf{K}_{\Gamma\psi}\mathbf{K}_{\phi\psi}^{-1}\mathbf{p}_\phi - \mathbf{K}_{\Gamma\phi}\mathbf{K}_{\psi\phi}^{-1}\mathbf{p}_\psi, \tag{11d}$$

$$\mathbf{p}_{\Gamma_b^{u_n}\phi\psi} = \mathbf{p}_{\Gamma_b^{u_n}} + \mathbf{K}_{\Gamma_b^{u_n}\psi}\mathbf{K}_{\phi\psi}^{-1}\mathbf{p}_\phi - \mathbf{K}_{\Gamma_b^{u_n}\phi}\mathbf{K}_{\psi\phi}^{-1}\mathbf{p}_\psi. \tag{11e}$$

Observe that the shape functions for u must be piecewise continuous, while those for ψ and ϕ may be discontinuous. In our implementation of the standard finite element formulation we use identical shape functions for both ψ and ϕ, and denote this approach as fully variational. The collocation method results when ϕ is approximated by a set of Dirac-delta functions, i.e. $\phi(\mathbf{x}) = \Sigma_i \phi_i \delta(\mathbf{x}-\mathbf{x}_i)$, while ψ is still approximated by piecewise polynomials ($\mathbf{x}_i$ is taken as the midpoint at the ith element). In both formulations, the matrices $\mathbf{K}_{\Omega_f\Omega_f}$, $\mathbf{K}_{\Omega_f\Gamma}$, $\mathbf{K}_{\Gamma\Omega_f}$, and $\mathbf{K}_{\Gamma\Gamma}$ are sparse as they represent partitions of the standard symmetric stiffness matrix $\mathbf{K}$ within Ω. $\mathbf{K}_{\Gamma\psi}$ is small, and also sparse as it involves only products of the approximations for u and ψ, as its transpose $\mathbf{K}_{\psi\Gamma}$; $\mathbf{K}_{\Gamma\phi}$, $\mathbf{K}_{\phi\Gamma}$, $\mathbf{K}_{\phi\Gamma_b^u}$, $\mathbf{K}_{\Gamma_b^u\phi}$, $\mathbf{K}_{\psi\phi}$, and $\mathbf{K}_{\phi\psi}$, on the other hand, are full but small since ψ and ϕ are defined only on Γ_b. It is straightforward to verify that the systems (9) and (10) are symmetric. For instance, to show that $\mathbf{K}_{\Gamma\phi} = \mathbf{K}_{\phi\Gamma}$ one need only make use of the relationship

$$\int_{\partial\Omega_b}\int_{\partial\Omega_b} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y} u(\mathbf{y})\phi(\mathbf{x})\,\mathrm{d}S_y\,\mathrm{d}S_x = \int_{\partial\Omega_b}\int_{\partial\Omega_b} \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y} \phi(\mathbf{y})u(\mathbf{x})\,\mathrm{d}S_y\,\mathrm{d}S_x. \tag{12}$$

† Scaled by $1/k_b$.

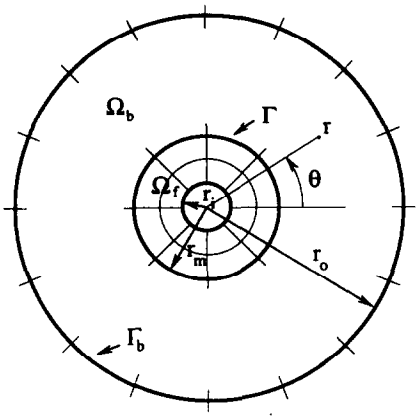

Fig. 2. Domain and mesh for numerical example.

4. NUMERICAL RESULTS

In order to illustrate the applicability of our coupled FE/BE method and to assess the relative accuracy of the fully variational and the collocation discretization procedures described in the preceding section we consider here a particular problem defined over an annular region, as shown on Fig. 2. The inner annulus ($r_i \leqslant r \leqslant r_m$) is inhomogeneous with conductivity $k_f = k_0/r$ (k_0 is constant), while the outer one ($r_m \leqslant r \leqslant r_o$) has constant conductivity $k_b = k_0$. The temperature is prescribed on $r = r_i$ as $u = u_0 \cos n\theta$, and the outer boundary is insulated, i.e. $u_r = 0$ on $r = r_o$. This problem has a closed-form solution (see the Appendix), which is depicted on Fig. 3 for the first few harmonics. In the numerical experiments r_i, r_m, and r_o were set to a, $3a$ and $8a$, respectively.

In all the calculations, three-noded quadratic isoparametric elements are used to represent the interface Γ ($r = 3a$), the exterior boundary Γ_b ($r = 8a$), and the temperature, u, and flux, ψ, on these boundaries. The annulus ($3a \leqslant r \leqslant 8a$) is divided into N_r elements in the radial direction, and N_θ in the angular direction. Twice that many ($2N_\theta$) elements are used on the outer boundary Γ_b. For the collocation approach, the Lagrange multiplier ϕ (density of the single layer) is approximated by a set of sources of unknown amplitude. This amounts to satisfying the Helmholtz representation formula (4) only at the nodes. Thence the designation of collocation method. For the fully variational procedure, ϕ is approximated by the same quadratic approximating functions as u and ψ. Within the inhomogenous region Ω_f ($a \leqslant r \leqslant 3a$), a total of $N_r \times N_\theta$ eight-noded isoparametric elements are used to approximate u. With the shape functions defined, all entries of the individual submatrices in (9) are evaluated by Gauss–Legendre numerical integration. 3×3 points per element within Ω_f, and three Gaussian points per boundary element on Γ_b are used with fully variational procedure. Even though double integration on the boundary is required for the entries that contain the Green's function and its normal derivative, the computational complexity is not severe due to the small number of integration points. Integration over singular terms involving the Green's function is carried out by first subtracting off the singularities. The collocation procedure entails only a single integration of the operators in (6) containing Green's function or its normal derivative, a seeming advantage over the fully variational approach. We found, however, that eight integration points were required for sufficient accuracy. This makes the two methods practically equivalent in their complexity. Moreover, as will be seen from the numerical results, the fully variational procedure provides significantly higher accuracy.

Table 2 shows the relative error, in percent, of the temperature u at three different locations on the horizon axis $\theta = 0$, for different harmonics n, and for various numbers of radial (N_r) and angular (N_θ) elements. One point is in the middle of the heterogeneous region, the second at the interface between the two regions, and the third one at the outer boundary. Results along other rays are omitted since they are practically indistinguishable from those shown.

The tabulated results clearly show convergence of both the fully variational and the collocation methods. Naturally, the number of elements required to attain a desired accuracy increases with the order of the harmonic loading. Notice, however, that while both methods give comparable results for the axisymmetric case†, the fully variational method leads to vastly more accurate results for increasing harmonics. The largest differences, of up to two orders of magnitude, are observed at the boundaries Γ and Γ_b, where the actual temperatures are small, but large errors therein also affect the solution within the inhomogeneous region.

To sum up, from the numerical results it appears that the proposed coupled FE/BE method provides

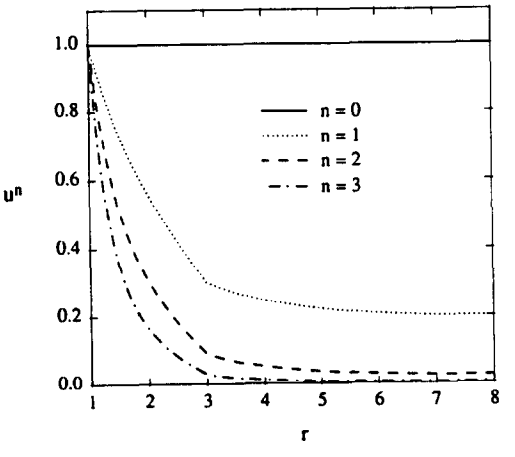

Fig. 3. Exact solution $u_n(r)[u(r, \theta)/u_0 = u_n(r) \cos n\theta]$ for numerical example, corresponding to axisymmetric case and next three harmonics.

† Theoretically, there should be no error since the exact solution is unity throughout.

an accurate procedure for solving interface problems. These results also indicate that it is advantageous to use the fully variational rather than the more common collocation methods. Not only is the first method more accurate, including a more uniform distribution of error throughout the computational domain, but the increased accuracy is obtained practically without computational penalty. While the procedure has been illustrated here only for the case of one homogeneous and one inhomogeneous region, the extension to multiple regions (including domain decomposition) is quite straightforward. The resulting system of algebraic equations is symmetric, block banded and can be solved efficiently using direct or iterative procedures. Finally, since the difference between the fully variational method and the collocation method lies only in the way the boundary integrals are discretized, it is natural that the relative advantages of the fully variational procedure should apply also to the solution of boundary integral equations by themselves.

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APPENDIX—EXACT SOLUTION

The exact solution to the problem presented in the numerical results section is given by

$$u = \begin{cases} (c_1 r^{\lambda_1} + c_2 r^{\lambda_2}) \cos n\theta, & \text{for } r_i \leqslant r \leqslant r_m \\ (c_3 r^n + c_4 r^{-n}) \cos n\theta, & \text{for } r_m \leqslant r \leqslant r_o, \end{cases} \tag{13}$$

where

$$\lambda_{1,2} = \tfrac{1}{2}[1 \pm \sqrt{1 + 4n^2}] \tag{14a}$$

$$c_{1,2} = \pm \frac{u_0 B_{2,1}}{A_1 B_2 - A_2 B_1} \tag{14b}$$

$$c_3 = r_m^n \frac{c_1 r_m^{\lambda_1} + c_2 r_m^{\lambda_2}}{r_m^{2n} + r_o^{2n}} \tag{14c}$$

$$c_4 = c_3 r_o^{2n} \tag{14d}$$

$$A_{1,2} = r_i^{\lambda_{1,2}} \tag{14e}$$

$$B_{1,2} = \left[\frac{k_0}{k_h}\frac{\lambda_{1,2}}{r_m} - \frac{r_m^{2n} - r_o^{2n}}{r_m^{2n} + r_o^{2n}}\right] r_m^{\lambda_{1,2}}. \tag{14f}$$