

# **A mixed symmetric BEM for multi-domain, multi-material, and crack interface problems in elastostatics**

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## **Abstract**

We discuss a variational approach that leads to a symmetric boundary element formulation suitable for multi-material and crack interface problems in heterogeneous domains arranged as assemblies of homogeneous subdomains. The variational principle is based on a Lagrangian functional comprising the system's potential energy augmented by the side imposition of the classical integral representation of the interior solution within each homogeneous subdomain. Any applied boundary tractions and all interface traction continuity conditions are automatically satisfied by the variational principle. Following a single condensation of the subdomain boundary tractions and the Lagrange multipliers, the boundary and interface displacements are left as the only unknowns. Upon discretization, there results a block-sparse system, with each block representing a single homogeneous subdomain (or part thereof). We validate the variational approach via numerical experiments entailing cracks at single and bi-material interfaces.

*Keywords: Multi-material interfaces; cracks; domain decomposition; boundary integral equations; symmetric boundary element method*

## **1 Introduction**

The Mixed Boundary Element Method (MBEM) discussed herein is a methodology that combines the direct and indirect boundary element methods together with domain decomposition ideas to arrive at a fairly flexible approach for handling a

variety of interface problems in engineering, while bypassing the need to resolve hypersingular operators. The method discussed herein draws from earlier work by Bielak and his collaborators [1, 2, 3, 4, 5].

## 2 Variational Principle

### 2.1 Statement of problem

Consider a multiply-connected region  $\Omega$  bounded by  $\Gamma$  (Fig. 1a). Let  $\Omega$  consist of  $N$  homogeneous subdomains  $\Omega_i$ , i.e.,  $\Omega = \bigcup \Omega_i$ ,  $i = 1 \dots N$ . Each of the homogeneous subdomains is occupied by a linear isotropic elastic solid characterized by the Lamé parameters  $\lambda_i$  and  $\mu_i$ . The entire assembly is constrained along the  $\Gamma_u$  part of the boundary, and is subjected to tractions on the  $\Gamma_t$  part of its boundary (Fig. 1b), with  $\Gamma_u \cap \Gamma_t = \emptyset$  and  $\Gamma_u \cup \Gamma_t = \Gamma$ . Let  $\mathbf{u}^i$  denote the displacement vector within the  $i$ -th subdomain, and  $\hat{\mathbf{U}}$  and  $\hat{\mathbf{T}}$  the prescribed boundary displacements and tractions on  $\Gamma_u$  and  $\Gamma_t$ , respectively. Then, the boundary-value problem consists of finding  $\mathbf{u}^i$  ( $\forall i$ ) such that

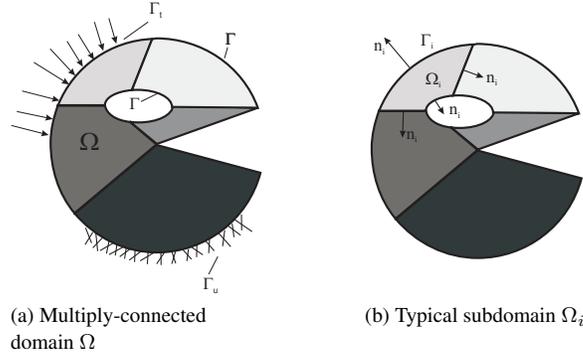


Figure 1: Domain and subdomain notation

ments and tractions on  $\Gamma_u$  and  $\Gamma_t$ , respectively. Then, the boundary-value problem consists of finding  $\mathbf{u}^i$  ( $\forall i$ ) such that

$$\mu_i \nabla \cdot \nabla \mathbf{u}^i + (\lambda_i + \mu_i) \nabla (\nabla \cdot \mathbf{u}^i) = \mathbf{0} \quad \text{in } \Omega_i, \quad (1)$$

$$\mathbf{t}^i = \hat{\mathbf{T}} \quad \text{on } \Gamma_i \cap \Gamma_t, \quad (2)$$

$$\mathbf{u}^i = \hat{\mathbf{U}} \quad \text{on } \Gamma_i \cap \Gamma_u, \quad (3)$$

$$\mathbf{u}^i = \mathbf{u}^j \quad \text{on } \Gamma_i \cap \Gamma_j, \quad (4)$$

$$\mathbf{t}^i = -\mathbf{t}^j \quad \text{on } \Gamma_i \cap \Gamma_j, \quad (5)$$

where subscripts (and superscripts)  $i$  and  $j$  denote the  $i$ -th and  $j$ -th subdomain, respectively, and  $\mathbf{t}^i$  is the traction vector, defined as:

$$\mathbf{t}^i = \boldsymbol{\sigma}^i \mathbf{n}_i, \quad (6)$$

with  $\boldsymbol{\sigma}^i$  denoting the stress tensor within each subdomain. The following constitutive law and kinematic condition also hold:

$$\boldsymbol{\sigma}^i = \lambda_i \mathbf{I} \operatorname{tr}(\boldsymbol{\mathcal{E}}^i) + 2\mu_i \boldsymbol{\mathcal{E}}^i, \quad (7)$$

$$\boldsymbol{\mathcal{E}}^i = \frac{1}{2} [\nabla \mathbf{u}^i + (\nabla \mathbf{u}^i)^T], \quad (8)$$

where  $\boldsymbol{\mathcal{E}}^i$  is the strain tensor within each subdomain, and  $\mathbf{n}_i$  denotes the normal outward vector on the subdomain's boundary. In other words, (1) is the Navier equation within each subdomain, (2) and (3) are the  $i$ -th subdomain's boundary conditions, and (4) and (5) are interface continuity conditions.

## 2.2 Variational Formulation

We now proceed to establish the variational form of the problem. We construct first an appropriate functional, and then proceed to show that the vanishing of its first variation recovers equations (1) – (5). We start with the total potential energy of the elastic system:

$$\Pi = \frac{1}{2} \sum_{i=1}^N \int_{\Omega_i} \left\{ \lambda_i (\nabla \cdot \mathbf{u}^i)^2 + 2\mu_i (\nabla \mathbf{u}^i)^2 \right\} d\Omega_i - \sum_{i=1}^N \int_{\Gamma_i} \mathbf{u}^i \cdot \hat{\mathbf{T}} d\Gamma_i \quad (9)$$

In the following, the Dirichlet condition (3) will be imposed explicitly. Integrating (9) by parts, yields:

$$\begin{aligned} \Pi &= \frac{1}{2} \sum_{i=1}^N \int_{\Omega_i} \left\{ \lambda_i \nabla \cdot [(\nabla \cdot \mathbf{u}^i) \mathbf{u}^i] - \lambda_i [\nabla (\nabla \cdot \mathbf{u}^i)] \cdot \mathbf{u}^i \right\} d\Omega_i \\ &\quad + \sum_{i=1}^N \int_{\Omega_i} \mu_i \left\{ \nabla \cdot [(\nabla \mathbf{u}^i) \cdot \mathbf{u}^i] - [\nabla \cdot (\nabla \mathbf{u}^i)] \cdot \mathbf{u}^i \right\} d\Omega_i \\ &\quad - \sum_{i=1}^N \int_{\Gamma_i} \mathbf{u}^i \cdot \hat{\mathbf{T}} d\Gamma_i. \end{aligned} \quad (10)$$

Using the divergence theorem,  $\Pi$  becomes:

$$\begin{aligned} \Pi &= \frac{1}{2} \sum_{i=1}^N \int_{\Gamma_i} [\lambda (\nabla \cdot \mathbf{u}^i) \mathbf{u}^i + 2\mu_i \boldsymbol{\mathcal{E}}^i \cdot \mathbf{u}^i] \cdot \mathbf{n}_i d\Gamma_i - \\ &\quad \frac{1}{2} \sum_{i=1}^N \int_{\Omega_i} [\lambda \nabla (\nabla \cdot \mathbf{u}^i) + 2\mu_i \nabla \cdot \boldsymbol{\mathcal{E}}^i] \cdot \mathbf{u}^i d\Omega_i - \sum_{i=1}^N \int_{\Gamma_i} \mathbf{u}^i \cdot \hat{\mathbf{T}}. \end{aligned} \quad (11)$$

For the time being, let us assume that the governing equation (1) is satisfied; then, (11) reduces to:

$$\Pi = \sum_{i=1}^N \frac{1}{2} \int_{\Gamma_i} \mathbf{t}^i \cdot \mathbf{u}^i d\Gamma_i - \sum_{i=1}^n \int_{\Gamma_i} \mathbf{u}^i \cdot \hat{\mathbf{T}} d\Gamma_i \quad (12)$$

To actually ensure that (1) holds, we make use of the following direct representation:

$$\mathbf{u}^i(\mathbf{x}) = \mathcal{S}_i[\mathbf{t}^i](\mathbf{x}) - \mathcal{D}_i[\mathbf{u}^i](\mathbf{x}), \quad \text{in } \Omega_i \quad i = 1, 2, \dots, N \quad (13)$$

where  $\mathbf{x}$  is a point interior to  $\Omega_i$ , and  $\mathcal{S}_i$  and  $\mathcal{D}_i$  represent single and double layers, respectively:

$$\mathcal{S}_i[\mathbf{t}^i](\mathbf{x}) = \sum_{j=1}^N \int_{\Gamma_j} \mathbf{u}_{ij}^T(\mathbf{x}, \mathbf{y}) \cdot \mathbf{t}^j(\mathbf{y}) d\Gamma_j(\mathbf{y}) \quad (14)$$

$$\begin{aligned} \mathcal{D}_i[\mathbf{u}^i](\mathbf{x}) &= \sum_{j=1}^N \int_{\Gamma_j} \frac{\partial \mathbf{u}_{ij}^T(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_y} \cdot \mathbf{u}^j(\mathbf{y}) d\Gamma_j(\mathbf{y}) \\ &= \sum_{j=1}^N \int_{\Gamma_j} \mathcal{T}_{ij}^T(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}^j(\mathbf{y}) d\Gamma_j(\mathbf{y}). \end{aligned} \quad (15)$$

In the above,  $\mathbf{u}_{ij}$  is the Green's function corresponding to equation (1). For two-dimensional problems it can be written as:

$$\mathbf{u}_{ij}(\mathbf{x}, \mathbf{y}) = C_1 \left( C_2 \delta_{ij} \ln r - \frac{\mathbf{r}_i \cdot \mathbf{r}_j}{r^2} \right), \quad (16)$$

with the corresponding  $\mathcal{T}_{ij}$ :

$$\mathcal{T}_{ij}(\mathbf{x}, \mathbf{y}) = \left( \frac{C_3}{r^2} \right) \left[ C_4 (n_j r_i - n_i r_j) + \left( C_4 \delta_{ij} + \frac{2r_i r_j}{r^2} \right) r_i n_i \right]. \quad (17)$$

In the above, the coefficients  $C_1$  through  $C_4$  are given as:

$$C_1 = -\frac{1}{8\pi\mu(1-\nu)}, \quad C_2 = 3 - 4\nu, \quad C_3 = -\frac{1}{4\pi(1-\nu)}, \quad C_4 = 1 - 2\nu.$$

Similarly, for three-dimensional problems:

$$\mathbf{u}_{ij}(\mathbf{x}, \mathbf{y}) = C_1 \frac{1}{r} \left( C_2 \delta_{ij} + \frac{\mathbf{r}_i \cdot \mathbf{r}_j}{r^2} \right), \quad (18)$$

$$\mathcal{T}_{ij}(\mathbf{x}, \mathbf{y}) = \left( \frac{C_3}{r^2} \right) \left[ C_4 \left( \frac{n_j r_i}{r} - \frac{n_i r_j}{r} \right) + \left( C_4 \delta_{ij} + \frac{3r_i r_j}{r^2} \right) \frac{r_i n_i}{r} \right] \quad (19)$$

with

$$C_1 = -\frac{1}{16\pi\mu(1-\nu)}, \quad C_2 = 3 - 4\nu, \quad C_3 = -\frac{1}{8\pi(1-\nu)}, \quad C_4 = 1 - 2\nu.$$

In (16)-(19),  $i, j = 1, \dots, d$ , where  $d$  is the problem's dimensionality, and  $n_i$  denotes the normal vector's component along the  $i$ -th cartesian direction. In addition,

$$\mathbf{r}_i = \mathbf{x}_i - \mathbf{y}_i, \quad r^2 = \mathbf{r}_i \cdot \mathbf{r}_i, \quad i = 1, \dots, d.$$

When  $\mathbf{x}$  is on the boundary  $\Gamma_i$ , one has the following well-known jump relations on  $\Gamma_i$  for smooth  $\Gamma_i$  ( $i = 1, \dots, N$ ):

$$\mathcal{S}_i[\phi]^\pm(\mathbf{x}) = \mathcal{S}_i[\phi](\mathbf{x}), \quad (20)$$

$$\mathcal{D}_i[\phi]^\pm(\mathbf{x}) = \mp \frac{1}{2}\phi(\mathbf{x}) + \mathbf{D}_i[\phi](\mathbf{x}), \quad (21)$$

$$\frac{\partial}{\partial n_x} \mathcal{S}_i[\phi]^\pm(\mathbf{x}) = \pm \frac{1}{2}\phi(\mathbf{x}) + \mathbf{N}_i[\phi](\mathbf{x}), \quad (22)$$

where the superscript on a layer such as  $\mathcal{S}_i$  denotes the limit as  $\mathbf{x}$  approaches a point on  $\Gamma_i$  from the interior. The  $\mathcal{S}_i$ ,  $\mathcal{D}_i$ , and  $\mathcal{N}_i$  are integral operators on  $\Gamma_i$  satisfying the following symmetry relations:

$$\int_{\Gamma_i} \mathcal{S}_i[\phi](\mathbf{x}) \psi(\mathbf{x}) d\Gamma_i = \int_{\Gamma_i} \mathcal{S}_i[\psi](\mathbf{x}) \phi(\mathbf{x}) d\Gamma_i, \quad (23)$$

$$\int_{\Gamma_i} \mathcal{D}_i[\phi](\mathbf{x}) \psi(\mathbf{x}) d\Gamma_i = \int_{\Gamma_i} \mathcal{N}_i[\psi](\mathbf{x}) \phi(\mathbf{x}) d\Gamma_i. \quad (24)$$

That is,  $\mathcal{S}_i$  is self-adjoint, and  $\mathcal{D}_i$  and  $\mathcal{N}_i$  are adjoint. Moreover,  $\mathcal{S}_i$  has a weak singularity,  $\mathcal{D}_i$  is continuous in  $\mathbf{R}^2$ , and has an integrable singularity in  $\mathbf{R}^3$ . From equations (13) and (20)-(22) it follows that:

$$\frac{1}{2}\mathbf{u}^i(\mathbf{x}) = \mathcal{S}_i[\mathbf{t}^i](\mathbf{x}) - \mathcal{D}_i[\mathbf{u}^i](\mathbf{x}) \quad \text{on } \Gamma_i. \quad (25)$$

The satisfaction of (25) ensures that (1) holds automatically. Thus, next, we modify the functional  $\Pi$  of (12) by introducing (25) as a side condition with the aid of Lagrange multipliers  $\phi^i$ . There results,

$$\begin{aligned} \Pi = \sum_{i=1}^N \left\{ \frac{1}{2} \int_{\Gamma_i} \mathbf{t}^i \cdot \mathbf{u}^i d\Gamma_i - \int_{\Gamma_i} \mathbf{u}^i \cdot \hat{\mathbf{T}} d\Gamma_i + \right. \\ \left. \frac{1}{2} \int_{\Gamma_i} \left( \frac{1}{2}\mathbf{u}^i - \mathcal{S}_i[\mathbf{t}^i] + \mathcal{D}_i[\mathbf{u}^i] \right) \cdot \phi^i d\Gamma_i \right\} \quad (26) \end{aligned}$$

As it will be seen, the Lagrange multiplier  $\phi^i$  represents the density of the single-layer for the  $i$ -th subdomain. Equation (26) serves as the basis for our variational principle for the elasticity problem defined by equation (1)-(5). The first variation of the functional  $\Pi$  can be written as follows:

$$\begin{aligned} \delta\Pi = \sum_{i=1}^N \frac{1}{2} \int_{\Gamma_i} (\mathbf{u}^i - \mathcal{S}_i[\phi^i]) \cdot \delta\mathbf{t}^i d\Gamma_i + \\ \sum_{i=1}^N \frac{1}{2} \int_{\Gamma_i} \left( \frac{1}{2}\mathbf{u}^i - \mathcal{S}_i[\mathbf{t}^i] + \mathcal{D}_i[\mathbf{u}^i] \right) \cdot \delta\phi^i d\Gamma_i + \end{aligned}$$

$$\sum_{i=1}^N \int_{\Gamma_i} \frac{1}{2} \left\{ \left( \mathbf{t}^i + \frac{1}{2} \boldsymbol{\phi}^i + \mathbf{N}_i [\boldsymbol{\phi}^i] \right) - \hat{\mathbf{T}} \right\} \cdot \delta \mathbf{u}^i d\Gamma_i. \quad (27)$$

In writing (27), use was made of the self-adjointness of  $\mathcal{S}_i$ .

Up to now, no requirements have been imposed on the admissibility of  $\mathbf{u}^i$ ,  $\mathbf{t}^i$ , and  $\boldsymbol{\phi}^i$ . We will now require that the continuity of  $\mathbf{u}$  throughout  $\Omega$  be imposed as an essential condition, and that (3) be satisfied explicitly. This means that, by construction,  $\mathbf{u}^i$  will be equal to  $\mathbf{u}^j$  (and hence  $\delta \mathbf{u}^i = \delta \mathbf{u}^j$ ) on  $\Gamma_i \cap \Gamma_j$ , and it will be equal to  $\hat{\mathbf{U}}$  on  $\Gamma_u \cap \Gamma_i$ . On the other hand,  $\mathbf{t}^i$  and  $\boldsymbol{\phi}^i$  will remain unconstrained, and thus,  $\mathbf{t}^i$ ,  $\mathbf{t}^j$ ,  $\boldsymbol{\phi}^i$ , and  $\boldsymbol{\phi}^j$ , can be varied independently. Hence by setting  $\delta \Pi$  to zero for arbitrary  $\delta \mathbf{u}^i$ ,  $\delta \mathbf{t}^i$ , and  $\delta \boldsymbol{\phi}^i$  subject to the constraint  $\delta \mathbf{u}^i = \delta \mathbf{u}^j$  on  $\Gamma_i \cap \Gamma_j$ , (27) yields:

$$\mathbf{u}^i = \mathcal{S}_i [\boldsymbol{\phi}^i], \quad \text{on } \Gamma_i, \quad (28)$$

$$\frac{1}{2} \mathbf{u}^i - \mathcal{S}_i [\mathbf{t}^i] + \mathcal{D}_i [\mathbf{u}^i] = 0, \quad \text{on } \Gamma_i, \quad (29)$$

$$\mathbf{t}^i + \frac{1}{2} \boldsymbol{\phi}^i + \mathbf{N}_i [\boldsymbol{\phi}^i] + \mathbf{t}^j + \frac{1}{2} \boldsymbol{\phi}^j + \mathbf{N}_j [\boldsymbol{\phi}^j] = 0, \quad \text{on } \Gamma_i \cap \Gamma_j, \quad (30)$$

$$\frac{1}{2} \left( \mathbf{t}^i + \frac{1}{2} \boldsymbol{\phi}^i + \mathbf{N}_i [\boldsymbol{\phi}^i] \right) - \hat{\mathbf{T}} = 0, \quad \text{on } \Gamma_i \cap \Gamma, \quad (31)$$

Equation (28) suggests that the displacement  $\mathbf{u}^i$  within the subdomain  $\Omega_i$  can be expressed in terms of a single-layer  $\boldsymbol{\phi}^i$ , i.e.

$$\mathbf{u}^i = \mathcal{S}_i [\boldsymbol{\phi}^i], \quad \text{in } \Omega_i. \quad (32)$$

Then, one can prove that

$$\mathbf{t}^i(\mathbf{x}) = \frac{1}{2} \boldsymbol{\phi}^i(\mathbf{x}) + \mathbf{N}_i [\boldsymbol{\phi}^i](\mathbf{x}), \quad \text{on } \Gamma_i. \quad (33)$$

Therefore, one has

$$\mathbf{t}_i = -\mathbf{t}_j, \quad \text{on } \Gamma_i \cap \Gamma_j, \quad \mathbf{t}_i = \hat{\mathbf{T}}, \quad \text{on } \Gamma_i \cap \Gamma. \quad (34)$$

With this, we have shown that if the first variation  $\delta \Pi$  of  $\Pi$  vanishes, then equations (1)-(5) are satisfied. That the converse is also true can be shown from equation (26) using the integral representation (29). Thus, we have the following principle:

*Variational principle:*  $\mathbf{u}^i$  is a solution of the boundary value problem defined by equations (1)-(5) if and only if  $\mathbf{u}^i$ ,  $\mathbf{t}^i$ , and  $\boldsymbol{\phi}^i$  are such that the variation  $\delta \Pi$  of the functional  $\Pi$  defined by equation (26) vanishes for arbitrary variations  $\delta \mathbf{u}^i$ ,  $\delta \mathbf{t}^i$  and  $\delta \boldsymbol{\phi}^i$  subjected to  $\delta \mathbf{u}^i = \delta \mathbf{u}^j$  on  $\Gamma_i \cap \Gamma_j$ . The displacement  $\mathbf{u}^i$  in  $\Omega_i$  can be obtained either from equation (28) in terms of  $\boldsymbol{\phi}^i$ , or from equation (13) in terms of  $\mathbf{u}^i$  and  $\mathbf{t}^i$  on  $\Gamma_i$ .

*Remarks.* (i) The variational principle is valid for all subdomains. It is, therefore, clear that the idea of decomposing  $\Omega$  into subdomains is what allows this method to deal with heterogeneous, but piecewise homogeneous domains.

(ii) Upon discretization of the unknowns  $\mathbf{u}^i$ ,  $\mathbf{t}^i$ , and  $\phi^i$ , the resulting system of algebraic equations will be automatically symmetric, since the formulation is fully variational in terms of a trilinear functional.

(iii) The present formulation uses simultaneously the quantities  $\mathbf{u}^i$ ,  $\mathbf{t}^i$  and  $\phi^i$  as independent unknowns. The boundary integral equation method based on single- (or double-) layer potentials such as equation (25) is called *indirect* in the literature since the auxiliary variable  $\phi^i$  does not arise naturally in the formulation, and physical quantities of interest, such as displacements and tractions, are obtained in terms of the auxiliary variable. In the present situation, however,  $\phi^i$  was introduced merely as a Lagrange multiplier to enforce the representation of equation (1) as a side condition. Upon taking the first variation of the resulting functional, we found that the Lagrange multiplier can be given an interesting physical interpretation, i.e., it represents the density of a single layer in the indirect formulation of the elasticity problem. Thus, there is a duality between  $\mathbf{u}^i$ ,  $\mathbf{t}^i$ , and  $\phi^i$ . Because our formulation embodies both the physical quantities  $\mathbf{u}^i$ ,  $\mathbf{t}^i$  and the auxiliary function  $\phi^i$  as unknowns, one may view it as a *direct-indirect* boundary integral equation method.

(iv) It is important to point out that the interface condition in (5) and boundary condition (2) are satisfied naturally by the variational principle. This means that when we use approximations there are no restrictions that need to be imposed on the approximants of the interfacial tractions. This implies that each  $\mathbf{t}^i$  is coupled to  $\mathbf{u}^i$  only within each subdomain. Since each  $\phi^i$  is also coupled to  $\mathbf{u}^i$  only within each subdomain,  $\mathbf{t}^i$  and  $\phi^i$  may be condensed, leaving  $\mathbf{u}^i$  as the only unknown. It is of practical interest that the condensation be done separately for  $\mathbf{t}^i$  and  $\phi^i$  since both are coupled to  $\mathbf{u}^i$  but not to each other. In fact, it is straightforward to verify that *only one* condensation need actually be performed explicitly, since the second one is given by the transpose of the first.

(v) Upon discretization, there results a block-sparse system, with each block representing a single subdomain. We note that each subdomain can itself be divided into smaller subdomains, thereby introducing additional sparsity to the final algebraic system. Thus, the formulation can also be seen as a domain decomposition method.

### 3 Discrete Forms

We discuss next the discretization of the variational principle (27). To this end we use standard isoparametric elements to approximate the displacements, the tractions, and the Lagrange multipliers or layer densities. Figures 2a and 2b depict typical discretizations of a 2D and a 3D boundary element, respectively. On an element  $e$  we approximate the displacements  $\mathbf{u}_e^i$ , the tractions  $\mathbf{t}_e^i$ , and the layer densities  $\phi_e^i$  by:

$$\mathbf{u}_e^i = \Psi_e^T \mathbf{U}_e, \quad \mathbf{t}_e^i = \Psi_e^T \mathbf{T}_e, \quad \phi_e^i = \Psi_e^T \Phi_e, \quad (35)$$

where  $\Psi_e$  is a vector of element shape functions, and  $\mathbf{U}_e$ ,  $\mathbf{T}_e$  and  $\Phi_e$  are vectors of unknown nodal displacements, tractions, and layer densities, respectively.

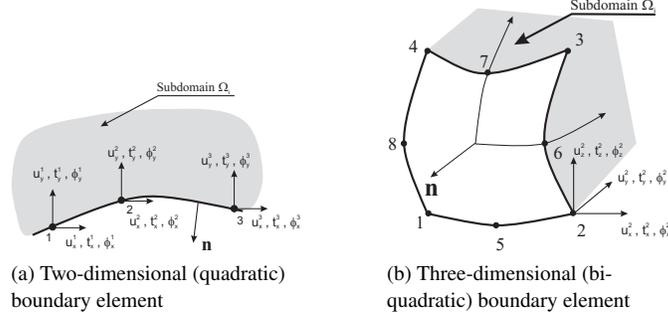


Figure 2: Typical isoparametric elements

Substitution of (35) into (26) leads to:

$$\begin{aligned}
 \Pi = \sum_{i=1}^N \sum_{e=1}^{N_i} & \left[ \frac{1}{2} \mathbf{T}_e^T \int_e \Psi \Psi^T d\Gamma_e \mathbf{U}_e - \hat{\mathbf{T}}^T \int_e \Psi \Psi^T d\Gamma_e \mathbf{U}_e + \right. \\
 & \frac{1}{4} \Phi_e^T \int_e \Psi \Psi^T d\Gamma_e \mathbf{U}_e - \frac{1}{2} \mathbf{T}_e^T \int_e \mathbf{H}_s \Psi^T d\Gamma_e \Phi_e + \\
 & \left. \frac{1}{2} \Phi_e^T \int_e \mathbf{H}_d \Psi^T d\Gamma_e \mathbf{U}_e \right], \quad (36)
 \end{aligned}$$

where  $\mathbf{H}_s$  and  $\mathbf{H}_d$  represent discrete forms of the single and double layers, respectively. Using the definitions:

$$\mathbf{G}_i = \sum_{e=1}^{N_i} \int_e \Psi \Psi^T d\Gamma_e, \quad \mathbf{G}_i^S = \sum_{e=1}^{N_i} \int_e \mathbf{H}_s \Psi^T d\Gamma_e, \quad \mathbf{G}_i^D = \sum_{e=1}^{N_i} \int_e \mathbf{H}_d \Psi^T d\Gamma_e, \quad (37)$$

equation (36) can be rewritten as:

$$\Pi = \frac{1}{2} \sum_{i=1}^N \left[ \mathbf{T}_i^T \mathbf{G}_i \mathbf{U}_i - 2\hat{\mathbf{T}}^T \mathbf{G}_i \mathbf{U}_i + \frac{1}{2} \Phi_i^T \mathbf{G}_i \mathbf{U}_i - \mathbf{T}_i^T \mathbf{G}_i^S \Phi_i + \Phi_i^T \mathbf{G}_i^D \mathbf{U}_i \right]. \quad (38)$$

By taking the variation of  $\Pi$  and requiring that it vanish for arbitrary variations  $\delta \mathbf{U}$ ,  $\delta \mathbf{T}$ , and  $\delta \Phi$ , there result the following equations valid within the  $i$ -th subdomain:

$$\mathbf{G}_i \mathbf{U}_i - \mathbf{G}_i^S \Phi_i = \mathbf{0}, \quad (39)$$

$$\frac{1}{2} \mathbf{G}_i \mathbf{U}_i - \left( \mathbf{G}_i^S \right)^T \mathbf{T}_i + \mathbf{G}_i^D \mathbf{U}_i = \mathbf{0}, \quad (40)$$

$$\frac{1}{2} \mathbf{G}_i \mathbf{T}_i - \mathbf{G}_i^T \hat{\mathbf{T}} + \frac{1}{4} \mathbf{G}_i^T \Phi_i + \frac{1}{2} \left( \mathbf{G}_i^D \right)^T \Phi_i = \mathbf{0}, \quad (41)$$

or, equivalently, for the entire domain  $\Omega$ :

$$\sum_{i=1}^N \left\{ \begin{array}{c} \left[ \begin{array}{ccc} \mathbf{0} & \mathbf{G} & -\mathbf{G}^S \\ \mathbf{G}^T & \mathbf{0} & \left[ (\mathbf{G}^D)^T + \frac{1}{2} \mathbf{G}^T \right] \\ -(\mathbf{G}^S)^T & (\mathbf{G}^D + \frac{1}{2} \mathbf{G}) & \mathbf{0} \end{array} \right]_i \left[ \begin{array}{c} \mathbf{T} \\ \mathbf{U} \\ \mathbf{\Phi} \end{array} \right]_i = \\ \left[ \begin{array}{c} \mathbf{0} \\ 2\mathbf{G}^T \hat{\mathbf{T}} \\ \mathbf{0} \end{array} \right]_i \end{array} \right\} \quad (42)$$

The system of equations (42) may be solved by taking advantage of sparsity and the presence of the various zero block submatrices. In three dimensions, the system has  $24n$  degrees of freedom (DOF), where  $n$  is the number of boundary elements in a subdomain. Since  $\mathbf{T}_i$  and  $\mathbf{\Phi}_i$  are uncoupled between subdomains, and  $\mathbf{G}_i^S$  is a nonsingular matrix, we can reduce the number of unknowns to  $9n + 6$  using condensation. Specifically,  $\mathbf{\Phi}_i$  and  $\mathbf{T}_i$  can be expressed at the subdomain level in terms of  $\mathbf{u}_i$  using (39) and (40). There results:

$$\phi_i = (\mathbf{G}_i^S)^{-1} \mathbf{G}_i \mathbf{U}_i, \quad \mathbf{T}_i = (\mathbf{G}_i^S)^{-T} \left[ \mathbf{G}_i^D + \frac{1}{2} \mathbf{G}_i \right] \mathbf{U}_i. \quad (43)$$

Substitution of (43) into (42) yields the discrete problem characterized by the following system of algebraic equations:

$$\mathcal{K} \mathbf{u} = \mathcal{P}, \quad \text{with } \mathcal{K} = \sum_{i=1}^N \mathbf{K}_i, \quad \mathcal{P} = \sum_{i=1}^N \mathbf{P}_i \quad (44)$$

where

$$\mathbf{K}_i = \mathbf{G}_i^T (\mathbf{G}_i^S)^{-T} \left[ \mathbf{G}_i^D + \frac{1}{2} \mathbf{G}_i \right] + \left[ (\mathbf{G}_i^D)^T + \frac{1}{2} \mathbf{G}_i^T \right] (\mathbf{G}_i^S)^{-1} \mathbf{G}_i, \quad (45)$$

$$\mathbf{P}_i = 2\mathbf{G}_i^T \hat{\mathbf{T}} \quad (46)$$

The summations in the above equations denote assembly of the corresponding individual terms at the subdomain level. Equation (44) is essentially the discretized version of condition (31) expressing the continuity of tractions across all interfaces. Moreover, note that  $\mathbf{K}_i$  in (45) is symmetric, since  $\mathbf{K}_i = \mathbf{K}_i^T$ . After solving for  $\mathbf{U}$ , the tractions  $\mathbf{T}$  and the layer densities  $\mathbf{\Phi}$  can be readily obtained from (43).

## 4 Numerical Results

Next, we discuss a subset of our numerical experiments using the described approach. Specifically, we focus on 4 crack problems, 3 involving two-dimensional

domains, and 1 in three-dimensions. In these problems, the conventional application of BEM has involved the use of various crack tip and singularity elements [7]. We will show next that by using the described MBEM, no special treatment is needed in order to resolve fracture problems. Figure 3(a) shows the geometry of an elastic plate with a central crack under uniaxial tension (top), together with the nodal arrangement and the deformed configuration (bottom). A semi-analytical solution of the stress intensity factor  $K_I$  accurate to 1% has been provided by Bowie [6]; specifically, Bowie computed  $K_I$  to be 2.830. It is desirable that the computed values of the mode I stress intensity factor  $K_I$  not be very sensitive to the mesh size near the crack tips. We carried out a series of numerical experiments to examine the sensitivity of  $K_I$  to  $a/l$  ranging from 0.05 to 2.85, where  $a$  is the half length of the crack, and  $l$  is the nodal separation near the crack tips. Our computed values are presented in relative error form, where Relative Error  $E_{K_I} = [K_I(\text{MBEM}) - K_I(\text{Bowie})] / K_I(\text{Bowie})$ , in percent. In Fig. 3b, open circles are the results of the MBEM, the straight line represents the semi-analytical solution of [6], and stars denote the results from [7], where quarter point singularity elements were used. The results from MBEM stay within an error margin of 1% as long as the mesh size  $l$  is less than the crack size  $a$ . Also, it is noteworthy that  $K_I$  converges monotonically as the mesh is refined. On the other hand, mesh refinement does not improve the results calculated using singularity elements, as can be seen from the dashed-starred line in Fig. 3b. Similarly to Fig. 3, Fig. 4 shows the

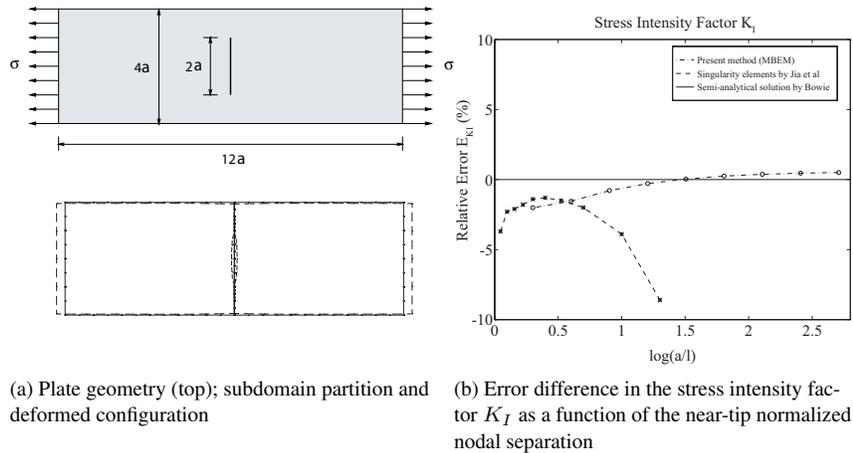


Figure 3: Two-dimensional plate with a central crack

geometry and the results pertaining to a plate with two edge cracks under uniaxial tension. In this case the semi-analytical solution [6] for  $K_I$  is 2.737.

The third problem examined the application of MBEM to a problem in which the bond between two regions occupied by dissimilar elastic materials is weakened by cracks. The problem represents idealization of two dissimilar metallic materials welded together with flaws or cracks developed along the original weld line owing, for example, to faulty joining techniques. The problem has been solved analytically

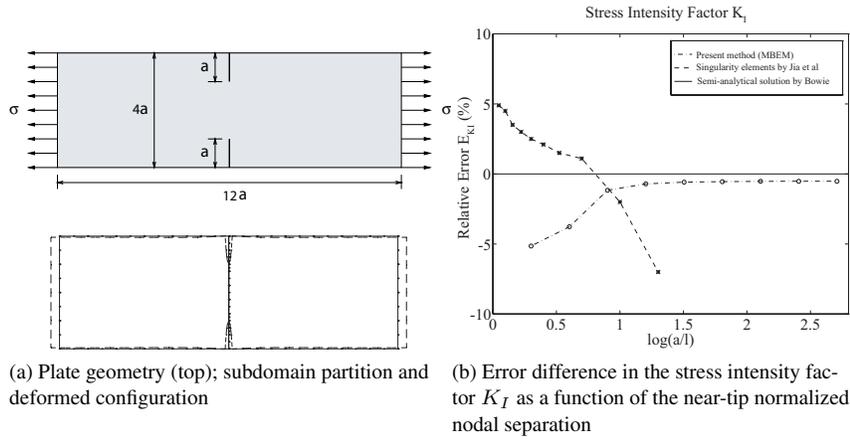


Figure 4: Two-dimensional plate with two edge cracks

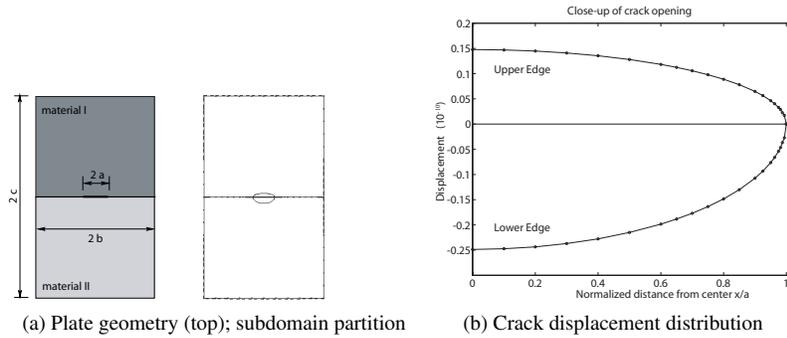


Figure 5: A bimaterial plate with a central crack

for two semi-infinite elastic bodies joined along a straight-line segment [8]. Here, we used two finite plates ( $2b \times 2c$ ), instead of two half-planes (Fig. 5(a)). The interfacial crack of fixed length  $2a$  was subjected to uniform pressure  $p$ , and is assumed to be free of shear stresses. The origin of the coordinate system is at the center of the crack and the  $x$  axis coincides with the interface. The upper plate is copper with shear modulus 45.6GPa and Poisson's ratio 0.35; it is bonded to a lower aluminum plate with shear modulus 27.0GPa and Poisson's ratio 0.34. Figure 5(b) depicts the comparison of the displacements along the upper and lower edges between the present method and the analytical solution of [8], showing excellent agreement. To verify that the MBEM result will converge to the theoretical solution as the plate dimensions become unbounded, the plate dimensions were varied and the results are shown in Table 1. Finally, the three-dimensional problem draws from a problem described by Sinclair [9] for a cracked cylinder along its centerline. Here, the stress field is prescribed ahead, behind, and on the crack face using the expressions derived in [9]. Next, the resulting displacement field is computed everywhere

$x/a$	Plate dimensions $2b \times 2c$	Upper edge		Lower edge	
		$v/pa$	$u/pa$	$v/pa$	$u/pa$
0.05	$2.5a \times 4a$	-0.0242	1.661	-0.0334	-2.687
	$5a \times 8a$	-0.0197	1.475	-0.0282	-2.482
	$10a \times 16a$	-0.0188	1.447	-0.0271	-2.436
	$20a \times 32a$	-0.0185	1.432	-0.0268	-2.436
	unbounded	-0.0182	1.420	-0.0266	-2.436
0.45	$2.5a \times 4a$	-0.2197	1.442	-0.2991	-2.400
	$5a \times 8a$	-0.1784	1.320	-0.2520	-2.217
	$10a \times 16a$	-0.1670	1.294	-0.2422	-2.177
	$20a \times 32a$	-0.1662	1.287	-0.2404	-2.177
	unbounded	-0.1650	1.270	-0.2376	-2.177
0.85	$2.5a \times 4a$	-0.4276	0.766	-0.5483	-1.408
	$5a \times 8a$	-0.3472	0.781	-0.4590	-1.301
	$10a \times 16a$	-0.3311	0.766	-0.4469	-1.287
	$20a \times 32a$	-0.3284	0.757	-0.4451	-1.286
	unbounded	-0.3213	0.748	-0.4432	-1.283

Table 1: Comparison of displacements ( $10^{-11}$ ) along the crack interface for different dimensions of the rectangular plate.

(details can be found in [10]): Fig. 6 depicts the geometry, mesh, and a comparison of the displacement field between the MBEM and the solution in [9].

## 5 Conclusions

We presented a variational principle suitable for integral equation formulations in two- and three-dimensional elastostatics. The principle hinges on the side imposition of an integral equation, but, through condensation, leads to a discrete displacement-only problem. The formulation is capable of handling multi-material, and multi-domain interface problems. We discussed numerical experiments, drawing from fracture mechanics, that validated the approach.

## References

- [1] Bielak J. and MacCamy R. An exterior problem in two-dimensional elastodynamics. *Q. Appl. Math.*, **41**, pp. 143–159, 1983
- [2] Bielak J. and MacCamy R. Symmetric finite element and boundary integral coupling methods for fluid-solid interaction. *Q. Appl. Math.*, **49**, pp. 107–119, 1991
- [3] Zeng X., Kallivokas L. F., Bielak J., Stable localized symmetric integral

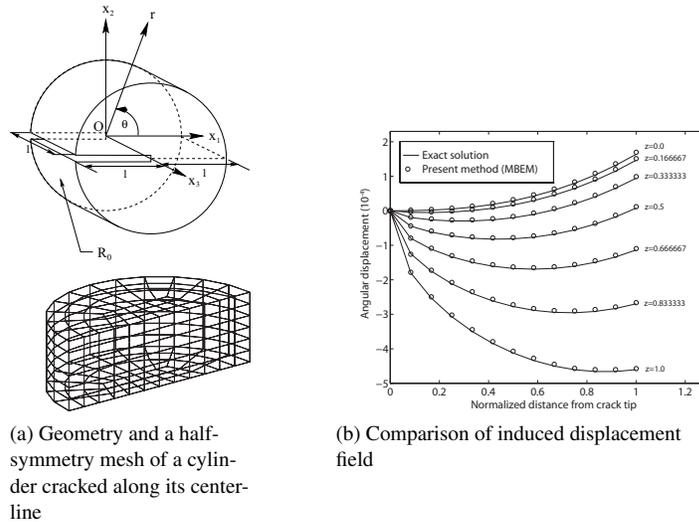


Figure 6: A three-dimensional crack problem

- equation method for acoustic scattering problems. *J. Acoust. Soc. Am.*, **91**(5), pp. 2510–2518, 1992
- [4] Zeng X. and Bielak J. Stability assessment of a unified variational boundary integral method applicable to thin scatterers with corners. *Comp. Meth. Appl. Mech. Eng.*, **111**, 305–321, 1994
- [5] Kallivokas L. F., Juneja T., Bielak J., A symmetric Galerkin BEM variational framework for multi-domain interface problems. *Comp. Meth. Appl. Mech. Eng.*, **194**(34-35), pp. 3607–3636, 2005
- [6] Bowie O. L. Rectangular tensile sheet with symmetric edge cracks. *J. Appl. Mech. ASME*, **31**, pp. 208–212, 1994
- [7] Jia Z. H., Shippy D. J., and Rizzo R. J. On the computation of two-dimensional stress intensity factors using the boundary element method. *Int. J. Num. Meth. Eng.* **26**, pp. 2739-2753, 1988
- [8] Rice J. R. and Sih G. C. Plane problems of cracks in dissimilar media. *J. Appl. Mech. ASME*, **32**, pp. 418–423, 1965
- [9] Sinclair G. Asymptotic singular eigenfunctions for the three-dimensional crack. *Proceedings of the 7th Canadian Congress of Applied Mechanics, Sherbrook, Quebec, Canada*, pp. 295–296, 1979
- [10] Yi H. Mixed Boundary Element Method – Theory and Applications in Solid Mechanics. *Dissertation*, submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy, Department of Civil and Environmental Engineering, Carnegie Mellon University, Pittsburgh, PA 15213, 1997