CE 319F - Laboratory #6

Dimensional Analysis Applied to Drag Force

Objectives

The primary purpose of this laboratory experiment is to see the application of dimensional analysis to obtain dimensionless parameters that reduce the number of variables needed to represent a physical process. The physical process in this laboratory experiment is the drag (resistance) force on spheres moving through a fluid.

In the experiments, the rate at which spheres fall through liquids will be measured. Using dimensional analysis, the results apply equally well to spheres falling through gases instead of liquids or light spheres rising through gases or liquids. The results of these experiments and any others should be used only within the same range of dimensionless variables for which the tests are done.

The Theory section summarizes some concepts of objects moving through fluids. These flows are sometimes called external flows, since the fluid is outside of the object. Internal flows are flows in pipes and ducts.

Theory

Part 1: Dimensional Analysis

The drag force (F_D) on a submerged spherical object is dependent on the diameter of the sphere (D), the relative velocity between the sphere and the fluid (V), the fluid density (r), and the fluid viscosity (m). In other words, F_D = f(D, V, r, m), where f indicates a functional relationship, which just gives another way of saying that F_D depends on D, V, r, and m. The dimensions (indicated by the square brackets) of each variable are as follows:

- $[F_D] = [F] = [ML/T^2]$
- D = [L]

- $V = [L/T]$
- $r = [M/L^3]$
- $m = [FT/L^2] = [M/LT]$

Since there are 5 dimensional variables and 3 dimensions, there are two dimensionless variables for this problem. The units can be canceled in steps, beginning with the parameter which has the largest number of dimensions. In this case, the process will begin with the drag force, which has the largest number of dimensions when expressed in terms of mass. First divide the force by the density to cancel the mass dimension:

$$\frac{F_D}{\rho} = \frac{\left[\frac{ML}{T^2}\right]}{\left[\frac{M}{L^3}\right]} = \left[\frac{L^4}{T^2}\right]$$

Since T^2 is in the denominator, divide by V^2 to cancel the time dimension:

$$\frac{F_D/\rho}{V^2} = \frac{\left[\frac{L^4}{T^2}\right]}{\left[\frac{L^2}{T^2}\right]} = \left[L^2\right]$$

Finally, divide by D^2 to cancel the length dimension:

$$\frac{F_D/(\rho V^2)}{D^2} = \frac{\left[L^2\right]}{\left[L^2\right]} = [1]$$

Since the result is dimensionless, the first dimensionless variable is

$$\Pi_1 = \frac{F_D}{\rho V^2 D^2}$$

To be certain that the second dimensionless parameter is independent of the first one, the same process can be done again beginning with one of the dimensional variables which is not in the first dimensionless parameter and which has the largest number of dimensions. Thus, start with m since it is the only parameter not in the first dimensionless term (and since it contains all of the dimensions). Divide by r to cancel the mass dimension:

$$\frac{\mu}{\rho} = \frac{\left[\frac{M}{LT}\right]}{\left[\frac{M}{L^3}\right]} = \left[\frac{L^2}{T}\right]$$

Since T is in the denominator, divide by V to cancel the time dimension:

$$\frac{\mu/\rho}{V} = \frac{\left[\frac{L^2}{T}\right]}{\left[\frac{L}{T}\right]} = [L]$$

Finally divide by D to cancel the remaining length dimension:

$$\frac{\mu/(\rho V)}{D} = \frac{[L]}{[L]} = [1]$$

Since this result is dimensionless, the second dimensionless variable is

$$\Pi_2 = \frac{\mu}{\rho V D}$$

From the original relation and the two dimensionless parameters,

$$\Pi_1 = f(\Pi_2)$$

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\mu}{\rho V D}\right)$$

It can be confirmed that these two parameters are indeed dimensionless by inserting the dimensions for each term and seeing that all of the dimensions cancel.

There are some dimensionless groups that occur frequently in fluid mechanics and thus have evolved into standard forms. In this case, P_1 is similar to a parameter called a drag coefficient (C_D), where

$$C_D = \frac{F_D}{\rho V^2 A_{proj}/2}$$

A_{proj} is the projected area, i.e., the area of the object projected onto a plane perpendicular to the flow so that $A_{proj} = pD^2/4$ for a sphere. Comparison with P_1 will show that $C_D = 8P_1/p$. The general functional dependence, namely $P_1 = f(P_2)$, is not changed by constant factors.

P_2 is the inverse of a dimensionless parameter called the Reynold's number and abbreviated as Re:

$$Re = \frac{1}{\Pi_2} = \frac{\rho VD}{\mu}$$

The Reynold's number can be interpreted as the ratio of inertial forces to viscous forces.

The general functional dependence is not changed by constant factors or by combining or inverting parameters as long as the proper number of independent, dimensionless parameters is maintained. Thus, the original relation $P_1 = f(P_2)$ can be written as $C_D = f(Re)$. The functional relationship originally represented by 5 variables is now totally representated by only two variables.

Part 2: Terminal Velocity

The settling of a particle in a fluid can be analyzed by the classic laws of sedimentation formulated by Newton and Stokes. Newton's law yields the terminal particle velocity, which is the steady velocity that an object has when its submerged weight is equal to the resistance or drag force exerted by the fluid. At the terminal velocity (V_t), the sum of the forces is zero so there is no acceleration and the velocity is constant. An object's submerged weight (W_{subm}) is its weight (W_s) minus the buoyant force (F_B). The submerged will be upward if the buoyant force is greater than the weight. The terminal velocity can be found by equating the submerged weight of an object with the drag force (F_D), i.e.,

drag force = submerged weight = weight - buoyant force

$F_D = W_{subm} = W_s - F_B$

The weight (W_s) of an object with a constant density is

$W_s = r_s g"_s$

where r_s = the density of the particle, g = the acceleration due to gravity, and $"_s$ = the volume of the particle. For a spherical particle, this volume is

$$\forall_s = \frac{\pi D^3}{6}$$

If the object is displacing fluid with a constant density, the buoyant force is

$F_B = rg"_s$

where r = the density of the fluid. The submerged weight (W_{subm}) is then $W_B = (r_s - r)g"_s$.

The drag force can be written as

$$F_D = C_D A_{proj} \frac{\rho V^2}{2}$$

where C_D is a drag coefficient. Equating the submerged weight and the drag force yields

$$C_D \frac{\pi D_s^2}{4} \frac{\rho V_t^2}{2} = g(\rho_s - \rho) \frac{\pi D_s^3}{6}$$

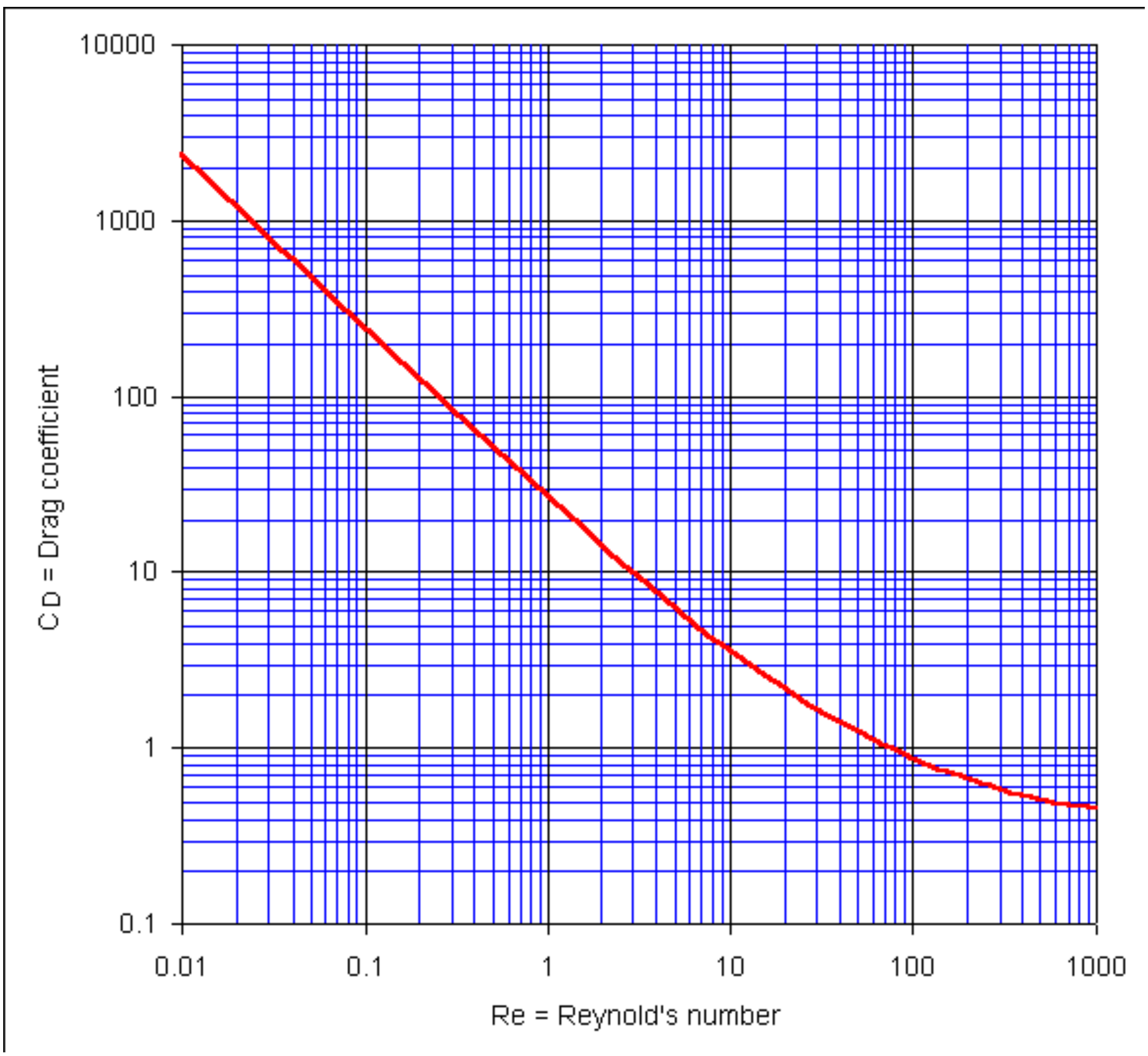
where V_t is the terminal velocity. Solving for the velocity yields Newton's law for terminal velocity, which is

$$V_t = \sqrt{\frac{4}{3}\left(\frac{\rho_s}{\rho} - 1\right)\frac{gD_s}{C_D}}$$

The drag coefficient has different values depending on whether the flow surrounding the particle is laminar or turbulent (i.e., depending on the value of Re). For smooth, spherical particles with Re < 1000, the drag coefficient can be approximated by

$$C_D = \frac{24}{Re} + \frac{3.0}{\sqrt{Re}} + 0.34$$

This relationship between the drag coefficient and Reynold's number for smooth spheres is shown below.



Drag Coefficients for Smooth Spheres

For Reynold's numbers less than about 0.5, the flow is called a creeping flow. The accelerations (inertial forces) of the fluid particles around the object are so small that they are insignificant. The first term dominates the equation, so $C_D = 24/Re$. Substituting this value into Newton's law yields Stoke's law for terminal velocity for $Re < 0.5$, namely,

$$V_t = \frac{(\rho_s - \rho) g D_s^2}{18\mu}$$

Laboratory Apparatus

The apparatus consists of three cylindrical tubes, each containing a different liquid. The density and viscosity of each liquid are known. There are also spheres of known diameter and specific weight.

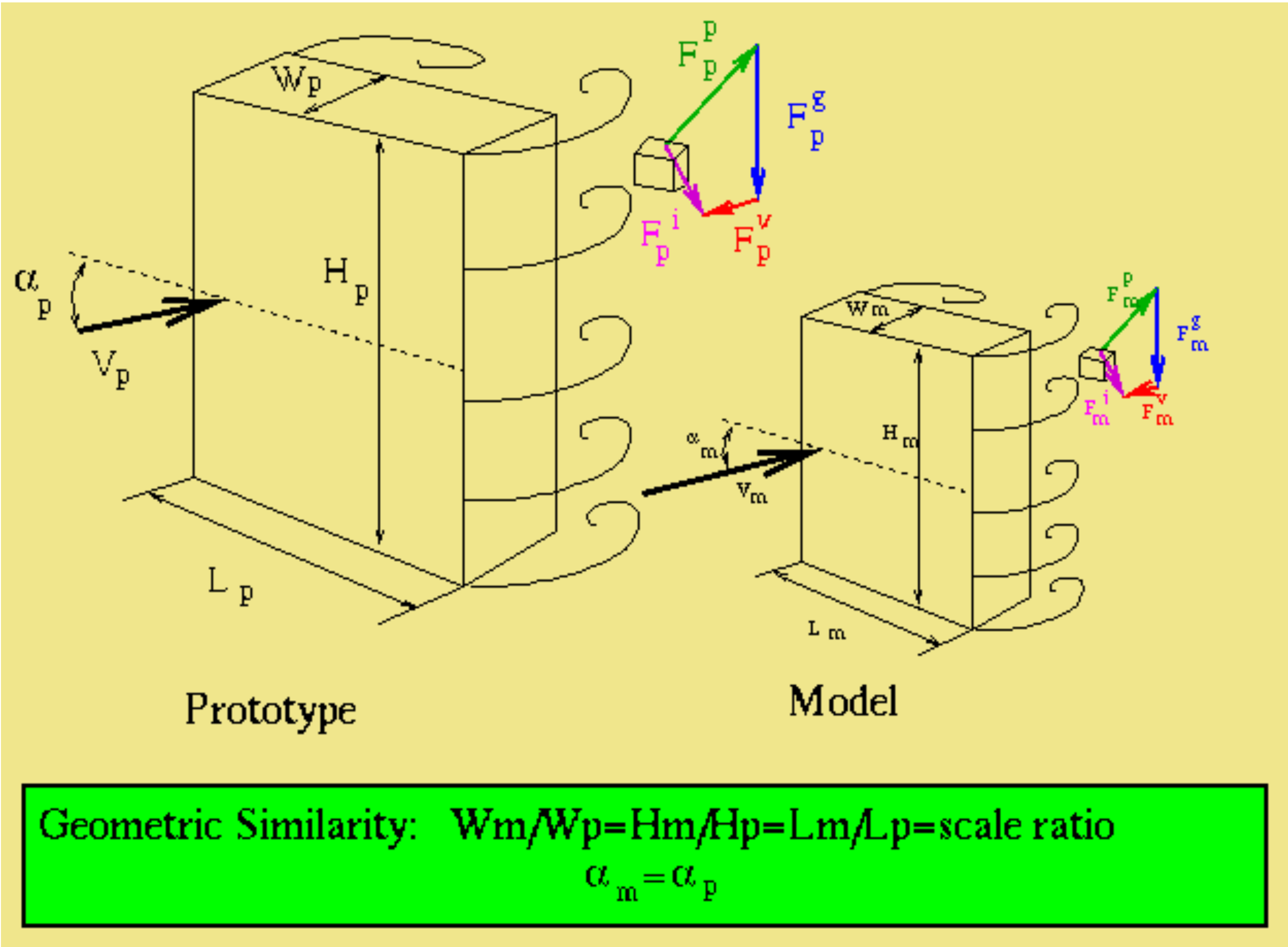
Laboratory Apparatus

Procedures

1. Release a spherical object of known size and density near the top of one of the tubes.
2. Determine the terminal velocity by letting the object come to its steady velocity and then measuring the time required for it to fall a known distance.
3. Calculate the submerged weight from $W_{subm} = (r_s - r)g''_s$.
4. Calculate the drag coefficient from

$$C_D = \frac{F_D}{\rho V_t^2 A_{proj}/2} = \frac{W_{subm}}{\rho V_t^2 A_{proj}/2}$$

5. Calculate the Reynolds number from Re = rV_tD/m.
6. Plot C_D vs. Re on the drag coefficient graph.
7. Repeat this procedure for a few different spheres in different liquids.
8. Note that the results are consistent in terms of C_D and Re and are independent of the specific values of D, V, r, and m.



The shown (elementary) force vectors are those acting on the elementary fluid cubes (fluid particles), also shown on the graphs. The pressure force vector (superscript p) is due to the pressures applying on all sides of the fluid cube, the viscous force vector (superscript v) is due to the shear stresses acting on all sides of the fluid cube, the gravity force vector (superscript g) is the weight of the cube, and the inertial force vector (superscript i) is the product of the mass of the fluid cube times the fluid acceleration vector.

Note 1: The shown fluid cubes are at the same relative location with respect to the body (e.g. building) and are also scaled by the same scale ratio.

Note 2: As shown in the graph, due to Newton's law applying on each fluid cube, the vector sum of the pressure, the viscous, and the gravity forces, equals the inertial force vector. In other words, the pressure, the viscous, the gravity, and the inertial force vectors form a closed polygon!

In order to have similarity of the flow patterns between the model and the prototype (also called Kinematic Similarity) we must have Dynamic Similarity (or similarity of the shown force polygons):

$$\frac{\vec{F}^v_p}{\vec{F}^v_m} = \frac{\vec{F}^p_p}{\vec{F}^p_m} = \frac{\vec{F}^g_p}{\vec{F}^g_m} = \frac{\vec{F}^i_p}{\vec{F}^i_m} \quad (1)$$

where the subscript ``p'' stands for prototype and ``m'' for model. The superscript ``v'' stands for viscous, ``p'' for pressure, ``g'' for gravity, and ``i'' for inertial. The inertial force is defined as:

$$\vec{F}_i = M\vec{a} \quad (2)$$

where M is the mass of the fluid particle and a the acceleration.

Due to similarity the following proportionality relations can be written:

$$F^v \sim \mu\frac{dV}{dy}(Area) \sim \mu\frac{V}{L}L^2 \sim \mu VL \quad (3)$$

$$F^p \sim \Delta p(Area) \sim \Delta pL^2 \quad (4)$$

$$F^g \sim \gamma(Volume) \sim \rho gL^3 \quad (5)$$

$$F^i \sim Ma \sim \rho L^3\frac{V}{T} \sim \rho L^3\frac{V^2}{L} \sim \rho L^2V^2 \quad (6)$$

where (Delta p) the difference of pressure from the ambient pressure

Using the above equations we can get:

- from equating the first fraction with the last fraction in equation (1)

  $$\frac{\mu_p V_p L_p}{\mu_m V_m L_m} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} \quad (7)$$

  or

  $$Re_p = \frac{\rho_p V_p L_p}{\mu_p} = \frac{\rho_m V_m L_m}{\mu_m} = Re_m \quad (8)$$

  where Re is the Reynolds number

- from equating the third fraction with the last fraction in equation (1)

  $$\frac{\rho_p g_p L_p^3}{\rho_m g_m L_m^3} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} \quad (9)$$

  or

  $$Fr_p^2 = \frac{V_p^2}{g_p L_p} = \frac{V_m^2}{g_m L_m} = Fr_m^2 \quad (10)$$

  where Fr is the Froude number

- from equating the second fraction with the last fraction in equation (1)

$$\frac{\Delta p_p L_p^2}{\Delta p_m L_m^2} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} \tag{11}$$

or

$$CP_p = \frac{\Delta p_p}{\frac{1}{2}\rho_p V_p^2} = \frac{\Delta p_m}{\frac{1}{2}\rho_m V_m^2} = CP_m \tag{12}$$

where CP is the pressure coefficient

As mentioned in class it is not feasible to enforce equality of the Reynolds and the Froude numbers at the same time. Thus, depending on the application we can only force equality of the Reynolds number (internal flow or external flow in unbounded fluid) or of the Froude number (flow of body through or under free surface, flow over a spill-way). The equality of the pressure coefficients results AUTOMATICALLY from the equality of Re or Fr numbers, and thus does not need to be enforced.

So, once the dynamic similarity has been enforced we can:

- relate the pressure in the prototype to that (measured) in the model via the equality of the pressure coefficients (see Ex. 8.6 of the textbook).
- relate the total force on the prototype (Fp) to the total force (measured) on the model (Fm) by using the dynamic similarity relations, e.g. in a case where only the Reynolds number is involved:

$$\frac{F_p}{F_m} = \frac{\mu_p V_p L_p}{\mu_m V_m L_m} = \frac{\Delta p_p L_p^2}{\Delta p_m L_m^2} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} \tag{13}$$

You can use any of the fractions in the above equation to determine Fp/Fm. In Ex. 8.7 the textbook uses (together with the fact that CPm=CPp)

$$\frac{F_p}{F_m} = \frac{\Delta p_p L_p^2}{\Delta p_m L_m^2} \tag{14}$$

In the solution of Ex. 8.7 which was presented in class I used:

$$\frac{F_p}{F_m} = \frac{\mu_p V_p L_p}{\mu_m V_m L_m} \tag{15}$$

We could have also used:

$$\frac{F_p}{F_m} = \frac{\rho_p L_p^2 V_p^2}{\rho_m L_m^2 V_m^2} \tag{16}$$

All of the above choices will give us the same answer for Fp/Fm, due to dynamic similarity (which has been enforced via equality of Reynolds numbers in the case of Ex. 8.7).

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