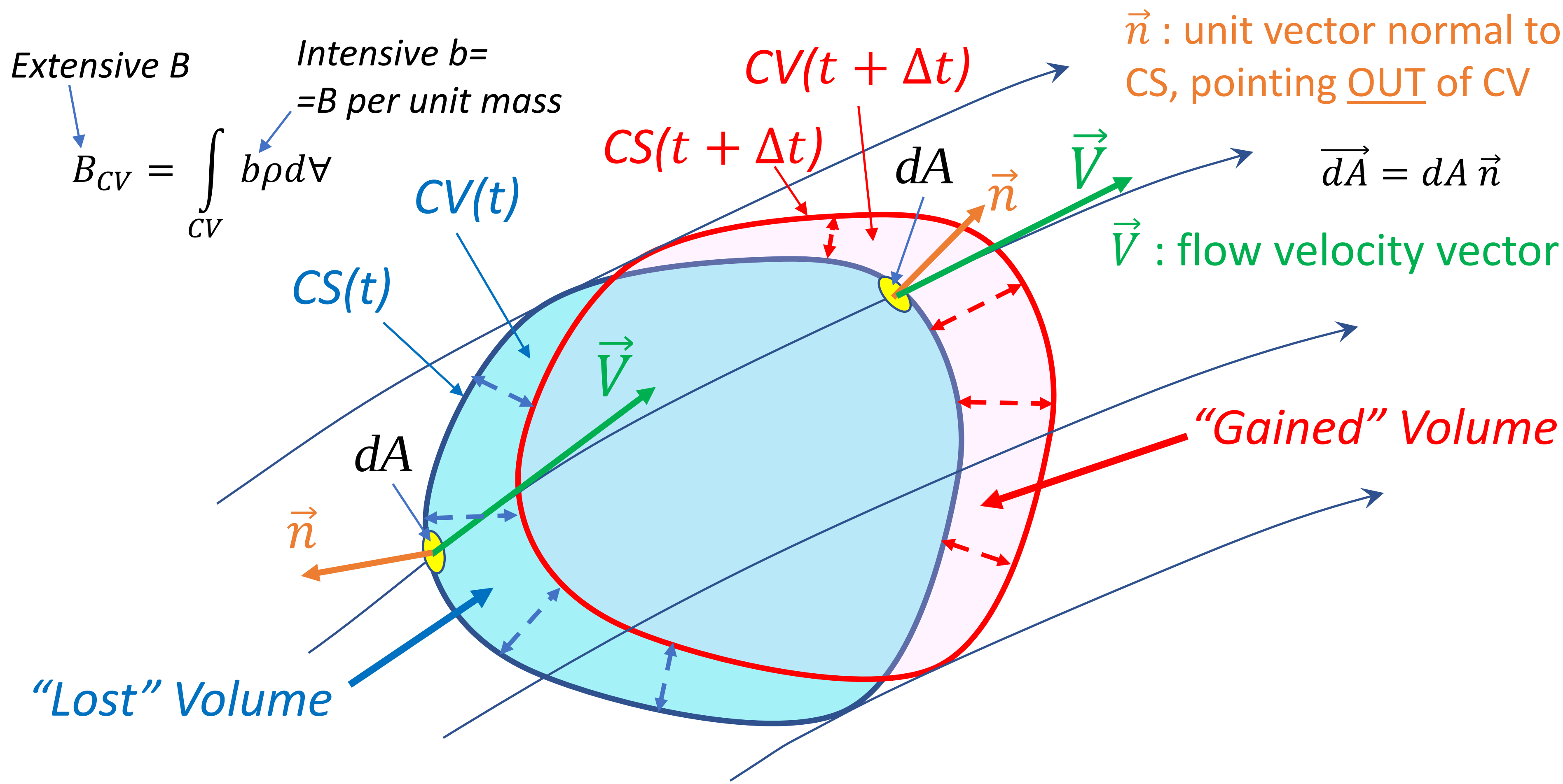
Reynolds Transport Theorem



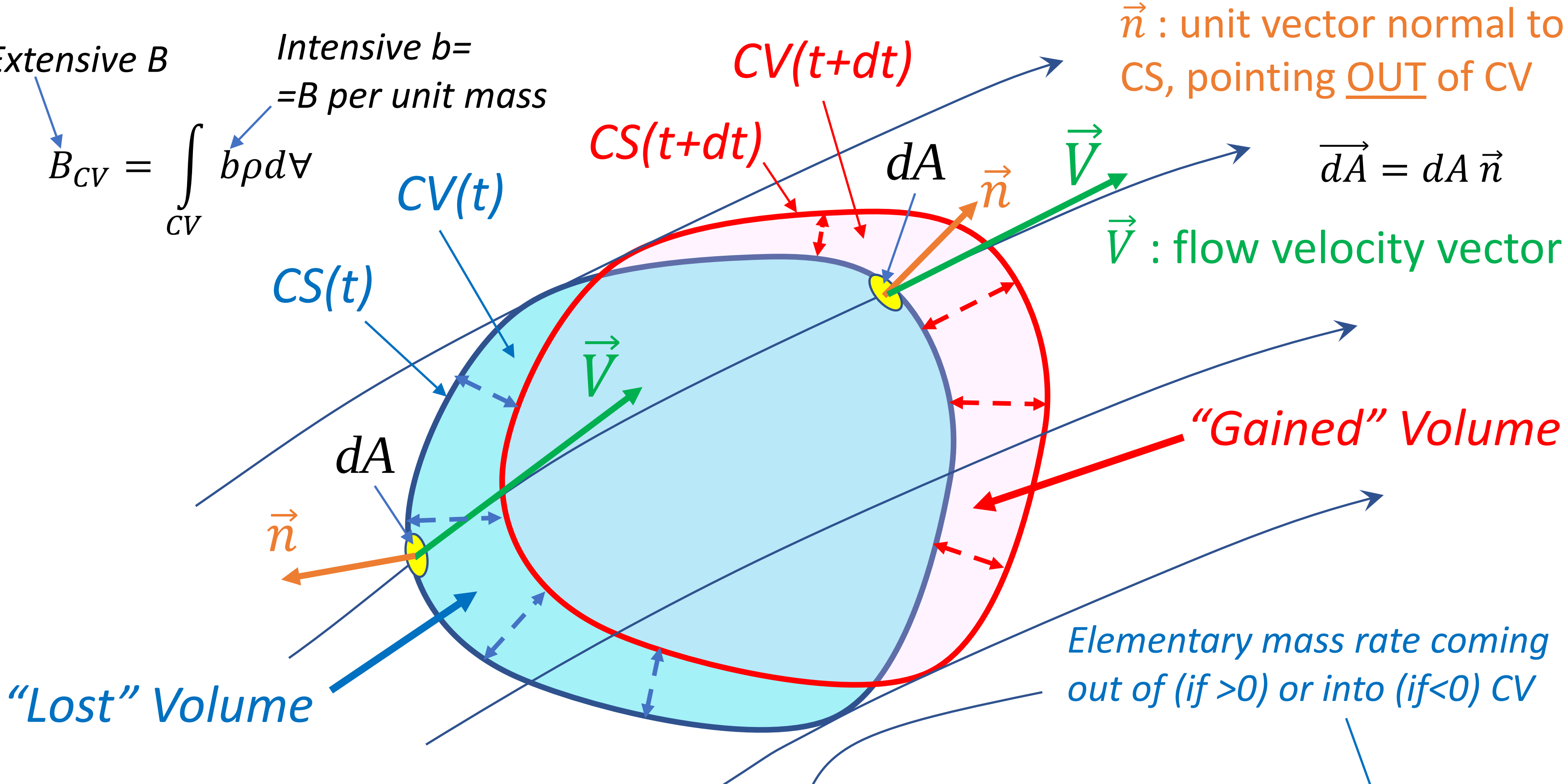
Definition:
$$\frac{dB_{CV}}{dt} = \lim_{\Delta t \to 0} \frac{B_{CV(t+\Delta t)} - B_{CV(t)}}{\Delta t}$$

Reynolds Transport Theorem

Extensive B

Intensive b= =B per unit mass

$$B_{CV} = \int_{CV} b\rho d\forall$$



$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} b\rho d\forall + \int_{CS} b\,\boxed{\rho\vec{V}\,.\,\overrightarrow{dA}} = \int_{CV} \frac{\partial(b\rho)}{\partial t} d\forall + \int_{CS} b\,\boxed{\rho\vec{V}\,.\,\overrightarrow{dA}}$$

<u>NOTE</u>: In general $\vec{V}$ is the velocity with which CS moves. Here we consider that the CS moves with the flow.

Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} b\rho d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA}$$

B= mass m ⇒ *b=1*

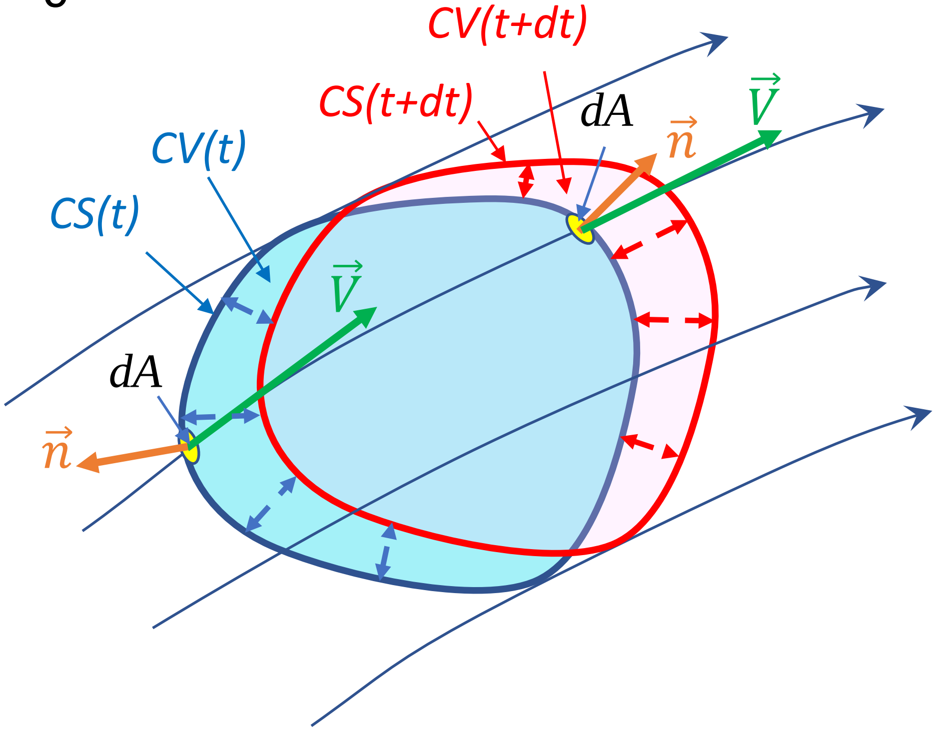
$$\frac{dm_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho\vec{V}\,.\,\overrightarrow{dA} = 0$$

Conservation of mass

$$\dot{m}_{CV} = \dot{m}_{in} - \dot{m}_{out}$$

Steady flow:

$$\dot{m}_{in} = \dot{m}_{out}$$

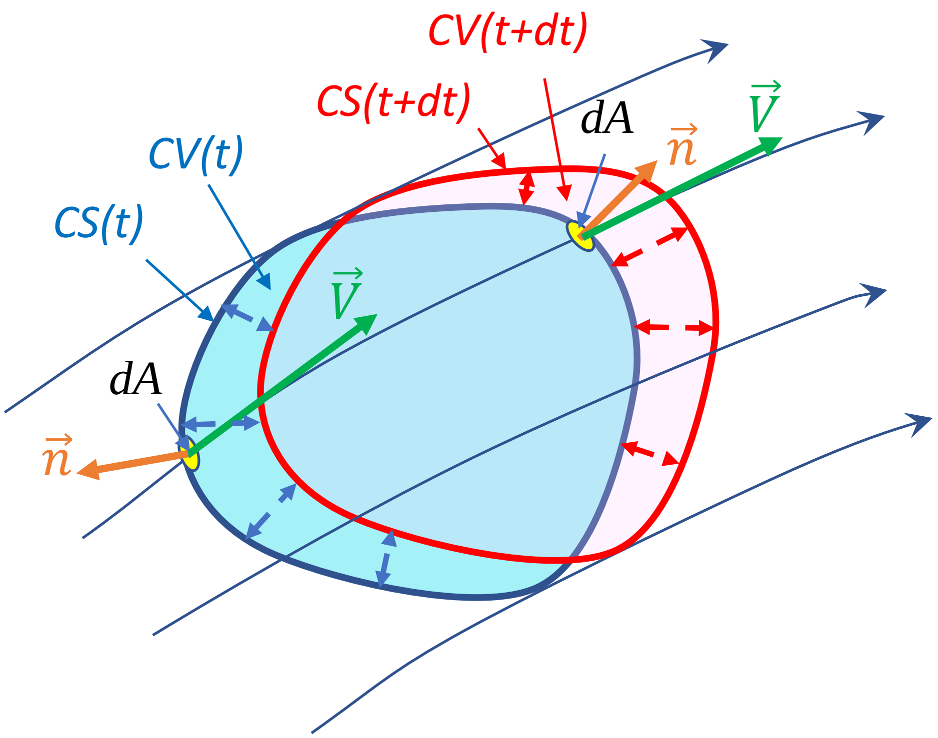


Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} b\rho d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA}$$

B= momentum $\vec{J}$ ⇒ b= $\vec{V}$

$$\frac{d\vec{J}_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho d\forall + \int_{CS} \vec{V}\rho\vec{V}\,.\,\overrightarrow{dA}$$



Steady flow:

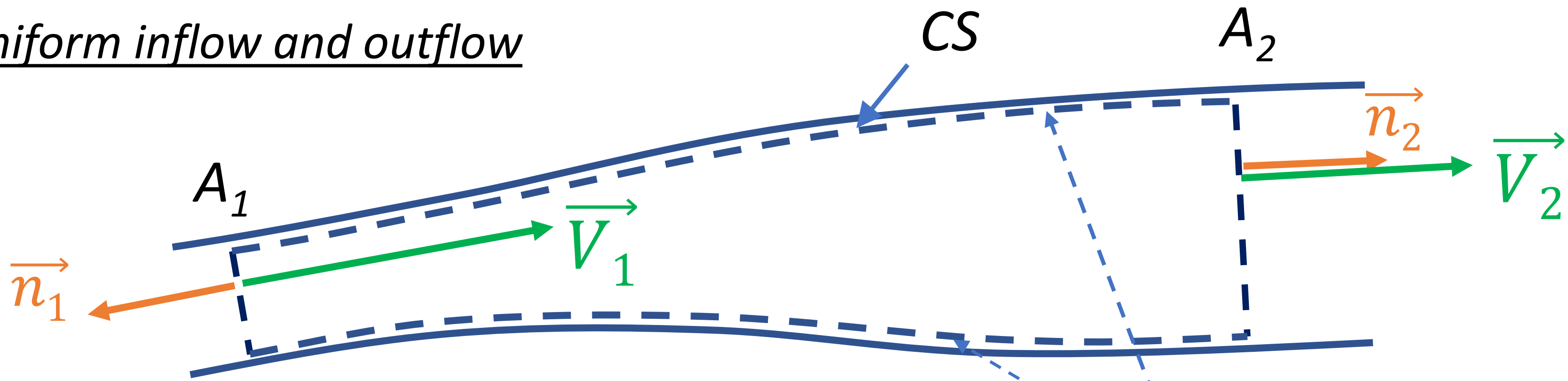
$$\frac{d\vec{J}_{CV}}{dt} = \int_{CS} \vec{V}\rho\vec{V}\,.\,\overrightarrow{dA}$$

Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} b\rho d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA}$$

B= momentum $\vec{J}$ *b= $\vec{V}$*

Uniform inflow and outflow



$$\int_{A_1} \vec{V}_1\,\rho\vec{V}_1\,.\,\vec{n}_1\,dA_1 = -\,\vec{V_1}\,\rho V_1 A_1 = -\,\vec{V_1}\,\rho Q = -\,\dot{m}\vec{V_1}$$

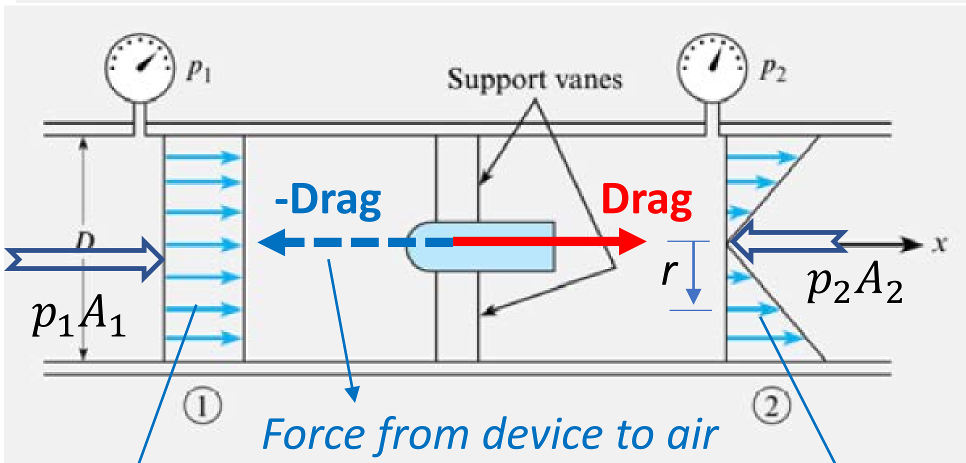
$$\int_{A_2} \overrightarrow{V_2}\,\rho\overrightarrow{V}_2\,.\,\overrightarrow{n_2}\,dA_2 = \overrightarrow{V_2}\,\rho V_2 A_2 = \overrightarrow{V_2}\,\rho Q = \dot{m}\overrightarrow{V_2}$$

Q: What about the integral over the sides of CS?

$$\sum \overrightarrow{F_{ext}} = \frac{d\vec{J}}{dt} = \dot{m}\,(\overrightarrow{V_2} - \overrightarrow{V_1})$$ *(Steady Flow)*

Example 6.6 of the Textbook

The drag force of a bullet-shaped device may be measured using a wind tunnel. The tunnel is round with a diameter of 1 m, the pressure at section 1 is 1.5 kPa gage, the pressure at section 2 is 1.0 kPa gage, and air density is 1.0 kg/m^3. At the inlet, the velocity is uniform with a magnitude of 30 m/s. At the exit, the velocity varies linearly as shown in the sketch. Determine the drag force on the device and support vanes. Neglect viscous resistance at the wall, and assume pressure is uniform across sections 1 and 2.



$$V_1 = V$$

$$V_2 = V_{max}\frac{r}{R}$$

$$\sum \overrightarrow{F_{ext}} = \frac{d\vec{J}}{dt} = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho d\forall + \int_{CS} \vec{V}\rho\vec{V} \cdot \overrightarrow{dA}$$

$$\Downarrow$$

$$\sum F_{ext,x} = \frac{dJ_x}{dt} = \int_{CS} V_x \rho\vec{V} \cdot \overrightarrow{dA}$$

$$\Downarrow$$

$$-Drag + p_1 A_1 - p_2 A_2 =$$

$$\int_{CS_1} V_1\, \rho\overrightarrow{V_1} \cdot \overrightarrow{dA_1} + \int_{CS_2} V_2\, \rho\overrightarrow{V_2} \cdot \overrightarrow{dA_2}$$

$$\Downarrow$$

$$-Drag + (p_1 - p_2)\frac{\pi D^2}{4} =$$

$$-\rho V^2 \frac{\pi D^2}{4} + \frac{9}{8}\rho V^2 \frac{\pi D^2}{4}$$

Example 6.6 of the Textbook (continued)

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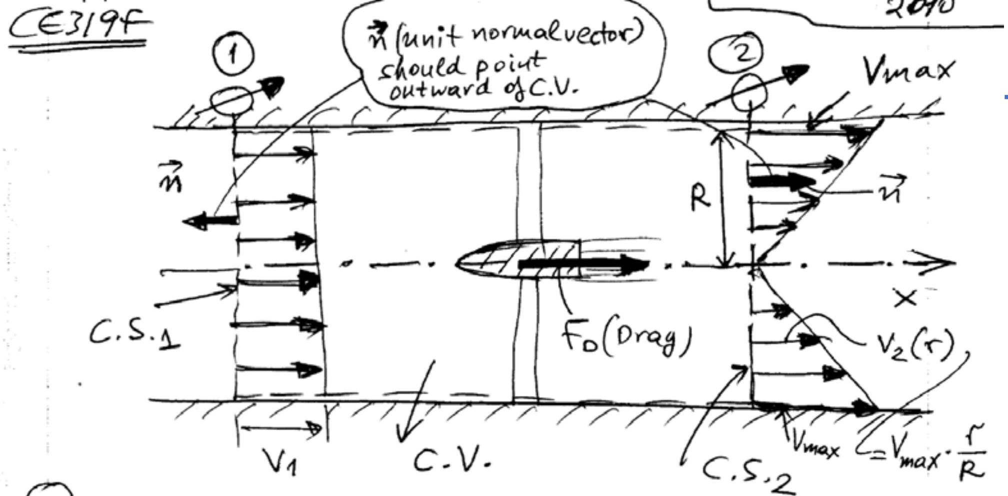


ⓐ

$$\rho \int_{C.S._1} V_x (\vec{V}\cdot\vec{n})\, dA = \rho \int_{C.S._1} V_1 (-V_1)\, dA = -\rho V_1^2 \int_{C.S._1} dA = -\rho V_1^2 A_1$$

- in station ① flow is uniform
- in station ①: $\vec{V}\cdot\vec{n} = \underbrace{|\vec{V}|}_{=V_1}\cdot\underbrace{|\vec{n}|}_{=1}\cos(180^\circ) = V_1(-1) = -V_1$

$$\int_{CS_1} V_1\, \rho \overrightarrow{V_1} \cdot \overrightarrow{dA_1} = -\rho V^2 \frac{\pi D^2}{4}$$

$$V_1 = V$$

ⓑ

$$\rho \int_{C.S._2} V_x \underbrace{(\vec{V}\cdot\vec{n})}_{= |\vec{V}|\cdot|\vec{n}|\cdot\cos(0^\circ) = V_2(r)} dA = \rho \int_{C.S._2} V_2(r)\,(V_2(r))\, dA = \rho \int_{C.S._2} V_2^2\, dA = \rho \int_0^R V_2^2(r)\, 2\pi r\, dr$$

Also continuity requires: $V_1 A_1 = \int_0^R V_2 (2\pi r)\, dr \Rightarrow$

$$\Rightarrow V_1 \pi R^2 = \int_0^R V_{max}\left(\frac{r}{R}\right)(2\pi r)\, dr = \frac{2\pi V_{max}}{R}\int_0^R r^2 dr = \frac{2\pi V_{max}}{R}\frac{R^3}{3}$$

$$V_{max} = \frac{3}{2} V_1$$

$$\int_{CS_2} V_2\, \rho \overrightarrow{V_2} \cdot \overrightarrow{dA_2} = \frac{9}{8} \rho V^2 \frac{\pi D^2}{4}$$

$$V_2 = V_{max}\frac{r}{R}$$

$$V_{max} = \frac{3}{2} V$$

Applications of Reynolds Transport Theorem

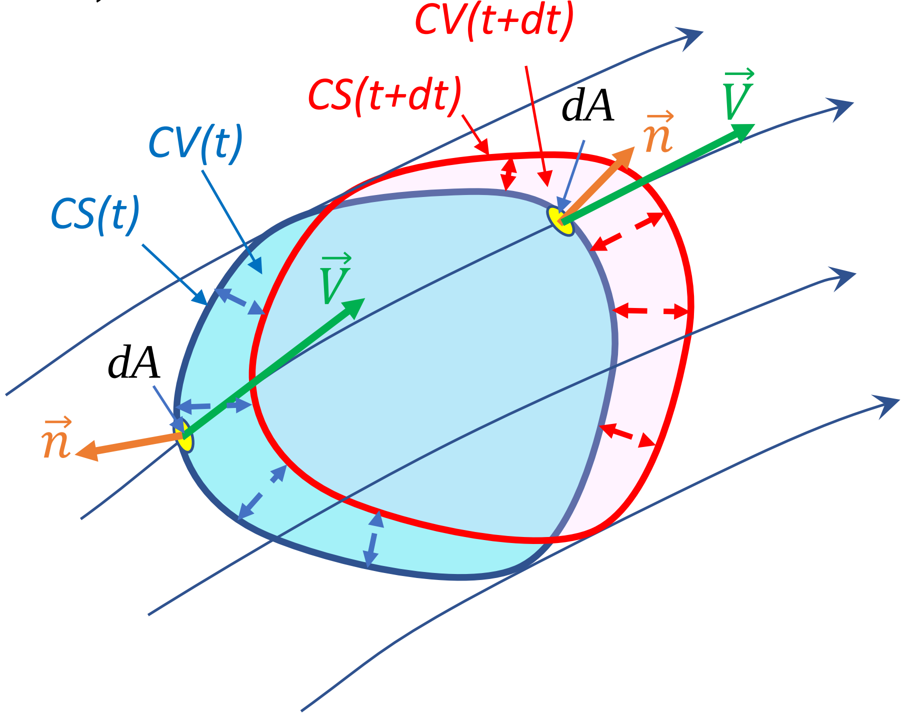
$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} b\rho d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA} = \int_{CV}\frac{\partial(b\rho)}{\partial t}d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA}$$

B= energy E (kinetic + potential) ⇒ *b=* $\left(\frac{V^2}{2}+gz\right)$

$$\frac{dE_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV}\left(\frac{V^2}{2}+gz\right)\rho d\forall + \int_{CS}\left(\frac{V^2}{2}+gz\right)\rho\vec{V}\,.\,\overrightarrow{dA}$$

Steady flow:

$$\frac{dE_{CV}}{dt} = \int_{CS}\left(\frac{V^2}{2}+gz\right)\rho\vec{V}\,.\,\overrightarrow{dA}$$



Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t}\int_{CV} b\rho d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} d\forall + \int_{CS} b\,\rho\vec{V}\,.\,\overrightarrow{dA}$$

B= volume $\forall$ ⇒ $b = 1/\rho$

$$\frac{d\forall_{CV}}{dt} = \int_{CV} \cancel{\frac{\partial(1)}{\partial t}} d\forall + \int_{CS} \vec{V}\,.\,\overrightarrow{dA} = 0$$

$$\frac{d\forall_{CV}}{dt} = \int_{CS} \vec{V}\,.\,\overrightarrow{dA}$$

$$\frac{d\forall_{CV}}{dt} = Q_{in} - Q_{out}$$