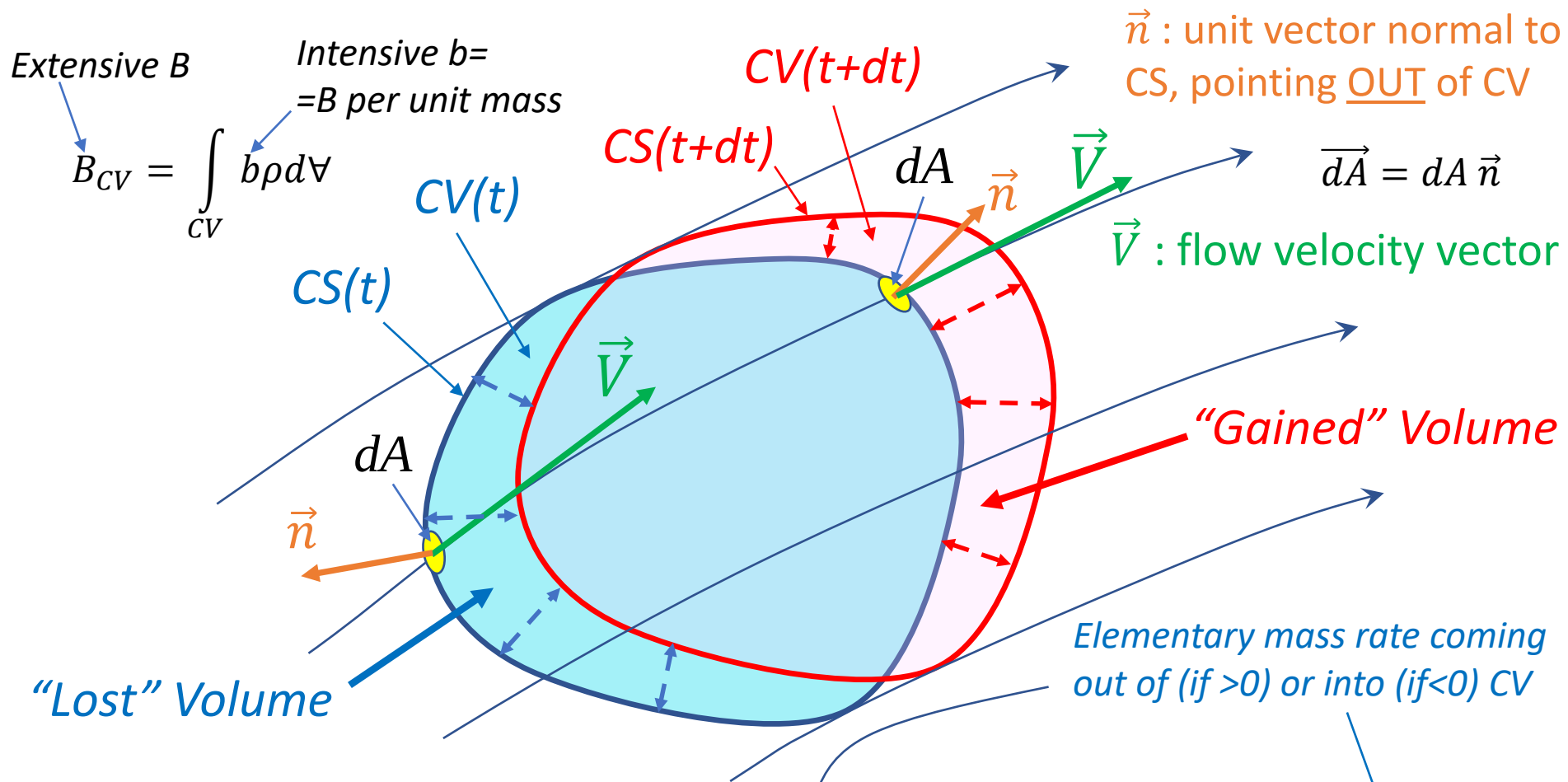


Reynolds Transport Theorem

The diagram illustrates a Control Volume (CV) at two different times, t and $t + \Delta t$. The CV is represented by a blue shaded region bounded by a Control Surface (CS). At time t , the CV is bounded by $CV(t)$ and $CS(t)$. At time $t + \Delta t$, the CV has expanded to $CV(t + \Delta t)$ and the CS has moved to $CS(t + \Delta t)$. The region between $CS(t)$ and $CS(t + \Delta t)$ is divided into two parts: a pink region labeled "Lost" Volume and a red region labeled "Gained" Volume. The flow velocity vector $\vec{V}$ is shown as a green arrow pointing outwards from the CV. The unit vector normal to the CS, pointing outwards, is denoted by $\vec{n}$. The differential area element dA is shown on the CS. The relationship $d\vec{A} = dA \vec{n}$ is indicated. The diagram also shows the relationship between the Extensive property B and the Intensive property b (per unit mass): $B_{CV} = \int_{CV} b \rho dV$.

Definition: $\frac{dB_{CV}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{B_{CV(t+\Delta t)} - B_{CV(t)}}{\Delta t}$

Reynolds Transport Theorem



$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} b \rho dV + \int_{CS} b [\rho \vec{V} \cdot d\vec{A}] = \int_{CV} \frac{\partial(b\rho)}{\partial t} dV + \int_{CS} b [\rho \vec{V} \cdot d\vec{A}]$$

NOTE: In general $\vec{V}$ is the velocity with which CS moves. Here we consider that the CS moves with the flow.

Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} b \rho dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA}$$

$B = \text{mass } m \rightarrow b=1$

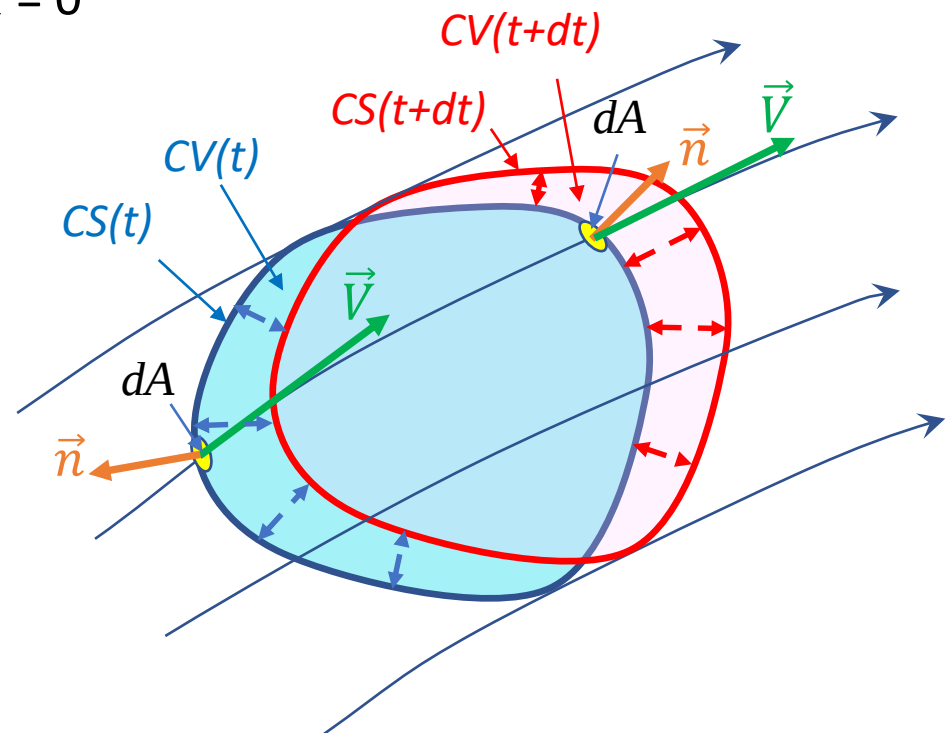
$$\frac{dm_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot \vec{dA} = 0$$

Conservation of mass

$$\dot{m}_{CV} = \dot{m}_{in} - \dot{m}_{out}$$

Steady flow:

$$\dot{m}_{in} = \dot{m}_{out}$$



Applications of Reynolds Transport Theorem

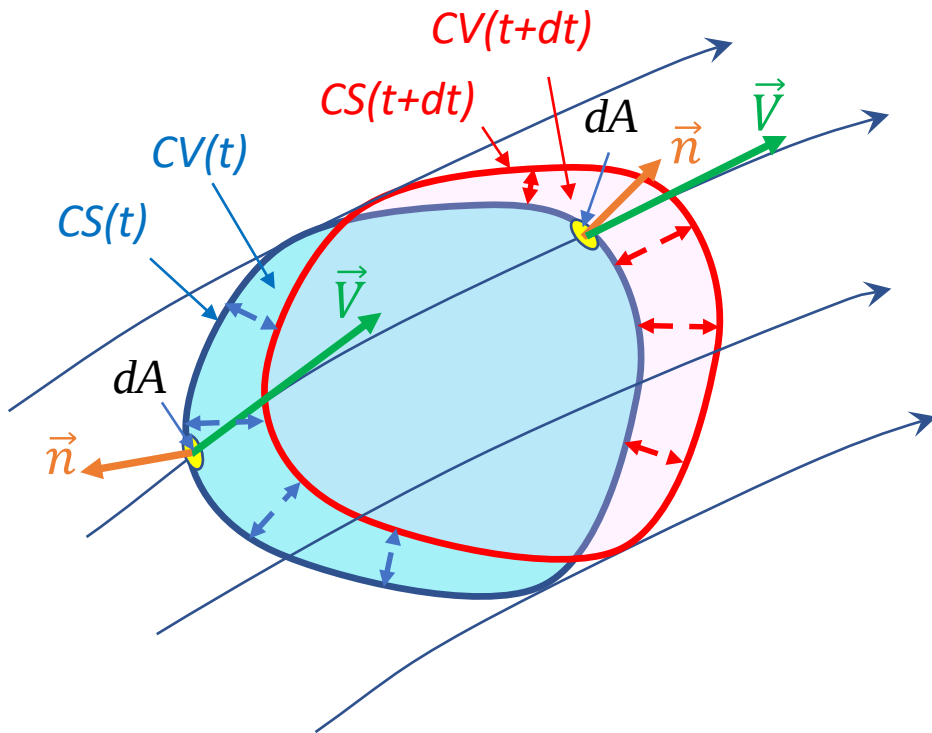
$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} b \rho dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA}$$

$B = \text{momentum } \vec{J} \Rightarrow b = \vec{V}$

$$\frac{d\vec{J}_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} \vec{V} \rho dV + \int_{CS} \vec{V} \rho \vec{V} \cdot \vec{dA}$$

Steady flow:

$$\frac{d\vec{J}_{CV}}{dt} = \int_{CS} \vec{V} \rho \vec{V} \cdot \vec{dA}$$

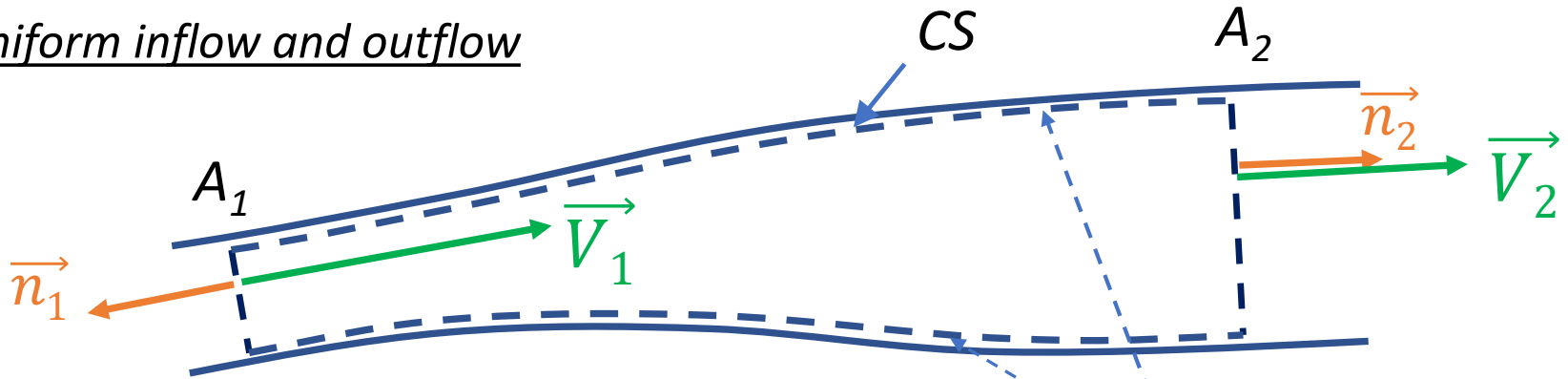


Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} b \rho dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA}$$

$B = \text{momentum } \vec{J} \quad b = \vec{V}$

Uniform inflow and outflow



$$\int_{A_1} \vec{V}_1 \rho \vec{V}_1 \cdot \vec{n}_1 dA_1 = - \vec{V}_1 \rho V_1 A_1 = - \vec{V}_1 \rho Q = - \dot{m} \vec{V}_1$$

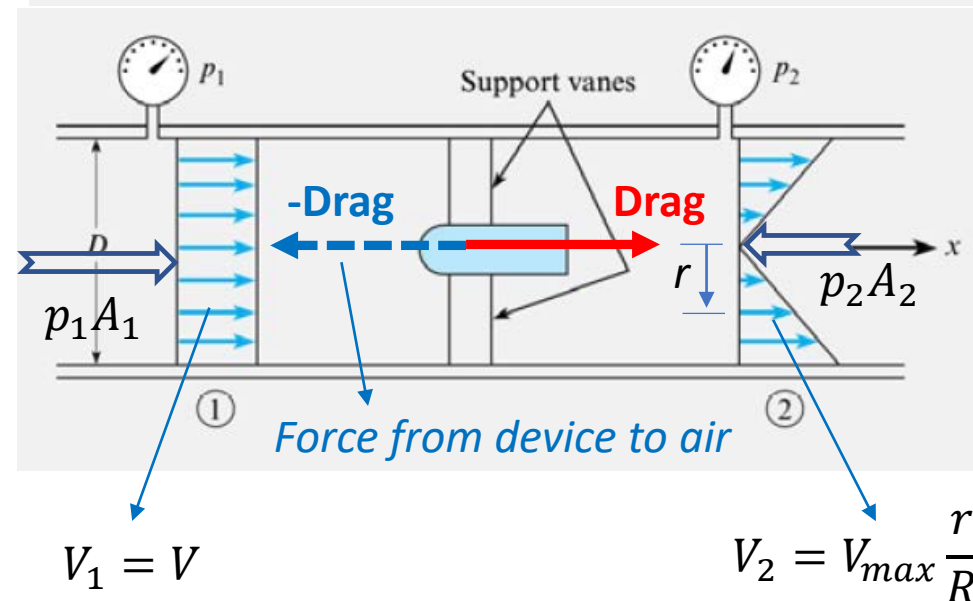
$$\int_{A_2} \vec{V}_2 \rho \vec{V}_2 \cdot \vec{n}_2 dA_2 = \vec{V}_2 \rho V_2 A_2 = \vec{V}_2 \rho Q = \dot{m} \vec{V}_2$$

Q: What about the integral over the sides of CS?

$$\sum \vec{F}_{ext} = \frac{d\vec{J}}{dt} = \dot{m} (\vec{V}_2 - \vec{V}_1) \quad (\text{Steady Flow})$$

Example 6.6 of the Textbook

The drag force of a bullet-shaped device may be measured using a wind tunnel. The tunnel is round with a diameter of 1 m, the pressure at section 1 is 1.5 kPa gage, the pressure at section 2 is 1.0 kPa gage, and air density is 1.0 kg/m^3 . At the inlet, the velocity is uniform with a magnitude of 30 m/s. At the exit, the velocity varies linearly as shown in the sketch. Determine the drag force on the device and support vanes. Neglect viscous resistance at the wall, and assume pressure is uniform across sections 1 and 2.



$$\sum \vec{F}_{ext} = \frac{d\vec{J}}{dt} = \frac{\partial}{\partial t} \int_{CV} \vec{V} \rho dV + \int_{CS} \vec{V} \rho \vec{V} \cdot \vec{dA}$$



$$\sum F_{ext,x} = \frac{dJ_x}{dt} = \int_{CS} V_x \rho \vec{V} \cdot \vec{dA}$$



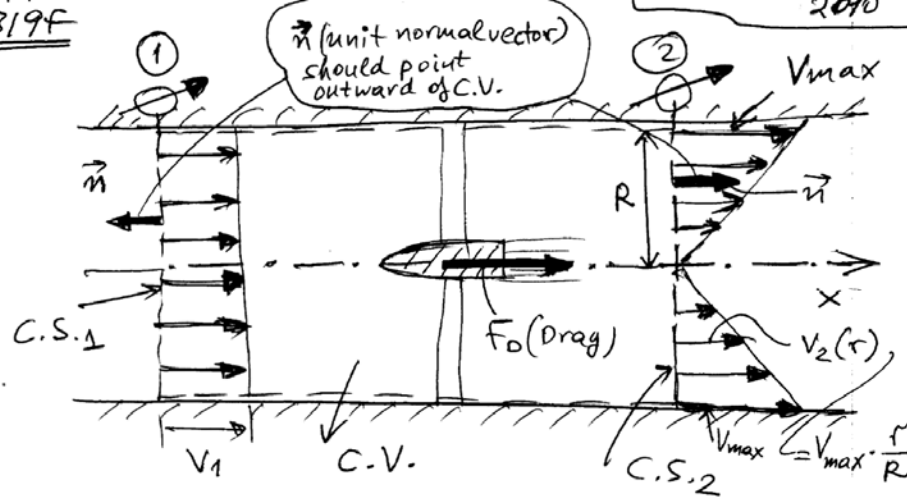
$$-Drag + p_1 A_1 - p_2 A_2 =$$

$$\int_{CS_1} V_1 \rho \vec{V}_1 \cdot \vec{dA}_1 + \int_{CS_2} V_2 \rho \vec{V}_2 \cdot \vec{dA}_2$$



$$-Drag + (p_1 - p_2) \frac{\pi D^2}{4} =$$

$$-\rho V^2 \frac{\pi D^2}{4} + \frac{9}{8} \rho V^2 \frac{\pi D^2}{4}$$



(a)

$$\oint_{C.S.1} V_x (\vec{V} \cdot \vec{n}) dA = \oint_{C.S.1} V_1 (-V_1) dA = -\oint_{C.S.1} V_1^2 dA = -\oint_{C.S.1} V_1^2 A_1$$

• in station ① flow is uniform

• in station ①: $\vec{V} \cdot \vec{n} = |\vec{V}| |\vec{n}| \cos(180^\circ) = V_1 (-1) = -V_1$
 $\underbrace{|\vec{V}|}_{=V_1} \underbrace{|\vec{n}|}_{=1}$

$$\int_{C.S.2} V_2 \rho \vec{V}_2 \cdot \vec{dA}_2 = \frac{9}{8} \rho V^2 \frac{\pi D^2}{4}$$

$$V_2 = V_{max} \frac{r}{R}$$

$$V_{max} = \frac{3}{2} V$$

$$\int_{C.S.1} V_1 \rho \vec{V}_1 \cdot \vec{dA}_1 = -\rho V^2 \frac{\pi D^2}{4}$$

$$V_1 = V$$

$$\textcircled{b} \oint_{C.S.2} V_x (\vec{V} \cdot \vec{n}) dA = \oint_{C.S.2} V_2(r) (V_2(r)) dA = \oint_{C.S.2} V_2^2 dA$$

$$= |\vec{V}| \cdot |\vec{n}| \cdot \cos(0^\circ) = V_2(r)$$

$$= \oint_0^R V_2^2(r) 2\pi r dr$$

Also continuity requires: $V_1 A_1 = \int_0^R V_2 (2\pi r) dr \Rightarrow$
 $\Rightarrow V_1 \pi R^2 = \int_0^R V_{max} \left(\frac{r}{R}\right) (2\pi r) dr = \frac{2\pi V_{max}}{R} \int_0^R r^2 dr = \frac{2\pi V_{max}}{R} \frac{R^3}{3}$

$$V_{max} = \frac{3}{2} V_1$$

Applications of Reynolds Transport Theorem

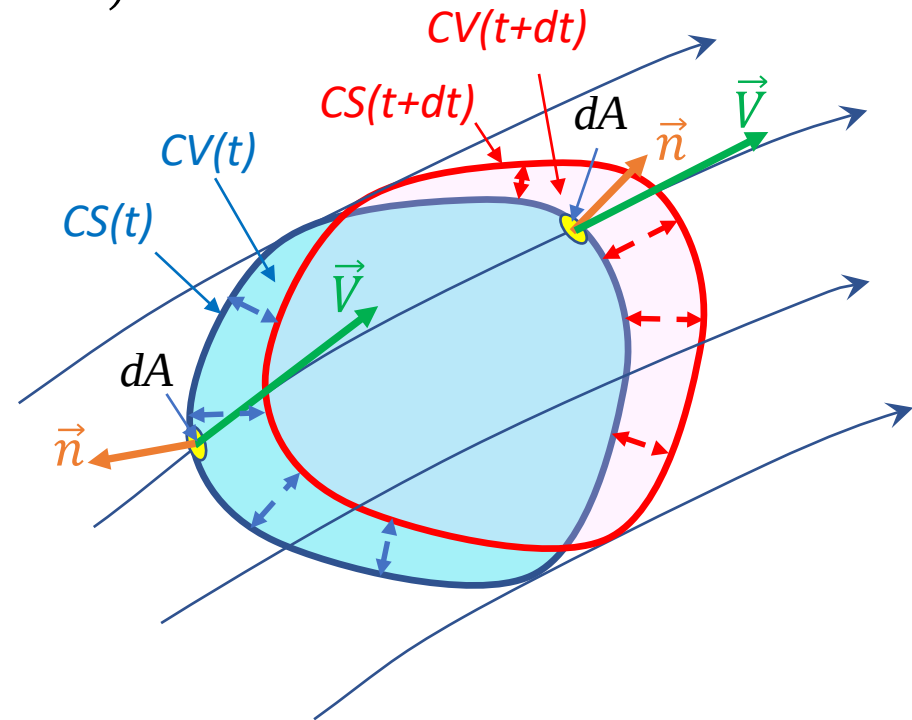
$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} b \rho d\forall + \int_{CS} b \rho \vec{V} \cdot \vec{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} d\forall + \int_{CS} b \rho \vec{V} \cdot \vec{dA}$$

$B = \text{energy } E \text{ (kinetic + potential)} \Rightarrow b = \left(\frac{V^2}{2} + gz \right)$

$$\frac{dE_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} \left(\frac{V^2}{2} + gz \right) \rho d\forall + \int_{CS} \left(\frac{V^2}{2} + gz \right) \rho \vec{V} \cdot \vec{dA}$$

Steady flow:

$$\frac{dE_{CV}}{dt} = \int_{CS} \left(\frac{V^2}{2} + gz \right) \rho \vec{V} \cdot \vec{dA}$$



Applications of Reynolds Transport Theorem

$$\frac{dB_{CV}}{dt} = \frac{\partial}{\partial t} \int_{CV} b \rho dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA} = \int_{CV} \frac{\partial(b\rho)}{\partial t} dV + \int_{CS} b \rho \vec{V} \cdot \vec{dA}$$

$B = \text{volume } V \Rightarrow b = 1/\rho$

$$\frac{dV_{CV}}{dt} = \int_{CV} \frac{\partial(1)}{\partial t} dV + \int_{CS} \vec{V} \cdot \vec{dA} = 0$$

$$\frac{dV_{CV}}{dt} = \int_{CS} \vec{V} \cdot \vec{dA}$$

$$\frac{dV_{CV}}{dt} = Q_{in} - Q_{out}$$

