Supplement to CE319F Lecture on Bernoulli Equation

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Bernoulli Equation

$$-\frac{\partial}{\partial s}(p+\gamma z)=\rho a_s=\rho V\frac{\partial V}{\partial s}=\frac{\partial}{\partial s}\left(\rho\frac{V^2}{2}\right)$$

$$\frac{\partial}{\partial s}\left(p+\gamma z+\rho\frac{V^2}{2}\right)=0$$

Dynamic pressure

$$p+\gamma z+\boxed{\rho\frac{V^2}{2}}=\text{Constant (along streamline)}$$

or (after dividing by γ)

Total head

$$\boxed{\boxed{\frac{p}{\gamma}+z}+\boxed{\frac{V^2}{2g}}}=\text{Constant}$$

So, for points 1 & 2 on the same streamline:

$$\frac{p_1}{\gamma}+z_1+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+z_2+\frac{V_2^2}{2g}$$

Assumptions:

- ***Along streamline, s***
- ***Steady flow***
- ***Incompressible fluid***
- ***Inviscid flow***

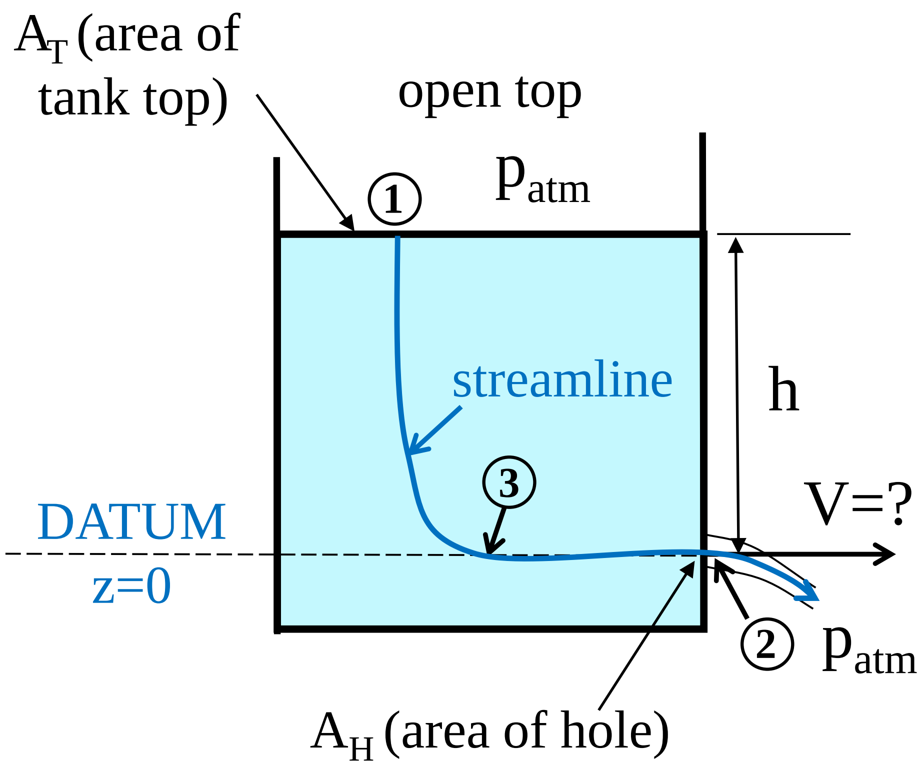
$$p+\rho\frac{V^2}{2}=\mathit{total\ pressure}$$

$$\frac{p}{\gamma}+z=\mathit{Piezometric\ head}$$

$$\frac{V^2}{2g}=\mathit{Velocity\ (dynamic)\ head}$$

Note for $V_1=V_2=0$ we recover law of hydrostatics for incompressible fluid.

Velocity of fluid out of hole in tank (same as Example 4.4) (all pressures are gage)



Q1) V=? If tank top open

$$\cancel{\frac{p_1}{\gamma}} + z_1 + \cancel{\frac{V_1^2}{2g}} = \cancel{\frac{p_2}{\gamma}} + \cancel{z_2} + \frac{V_2^2}{2g}$$

$$\boxed{V = V_2 = \sqrt{2gh}}$$

Q2) p_3=? If tank top open

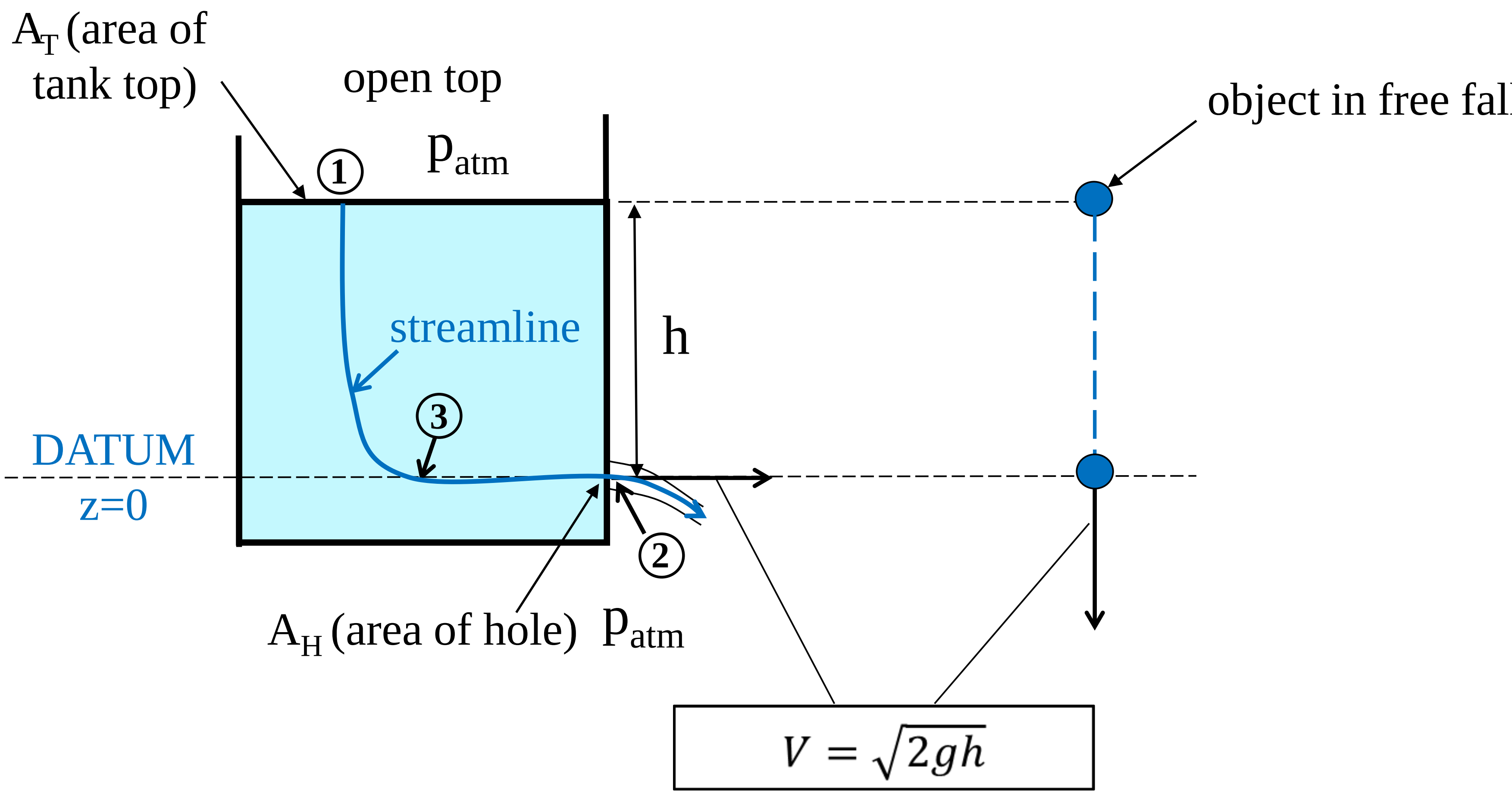
$$\cancel{\frac{p_1}{\gamma}} + z_1 + \cancel{\frac{V_1^2}{2g}} = \frac{p_3}{\gamma} + \cancel{z_3} + \cancel{\frac{V_3^2}{2g}}$$

$$\boxed{p_3 = \gamma h}$$

Q3) V=? If tank top closed

$$\frac{p_1}{\gamma} + z_1 + \cancel{\frac{V_1^2}{2g}} = \cancel{\frac{p_2}{\gamma}} + \cancel{z_2} + \frac{V_2^2}{2g} \Longrightarrow \boxed{V = V_2 = \sqrt{2g\frac{p_1}{\gamma} + 2gh} = \sqrt{\frac{2p_1}{\rho} + 2gh}}$$

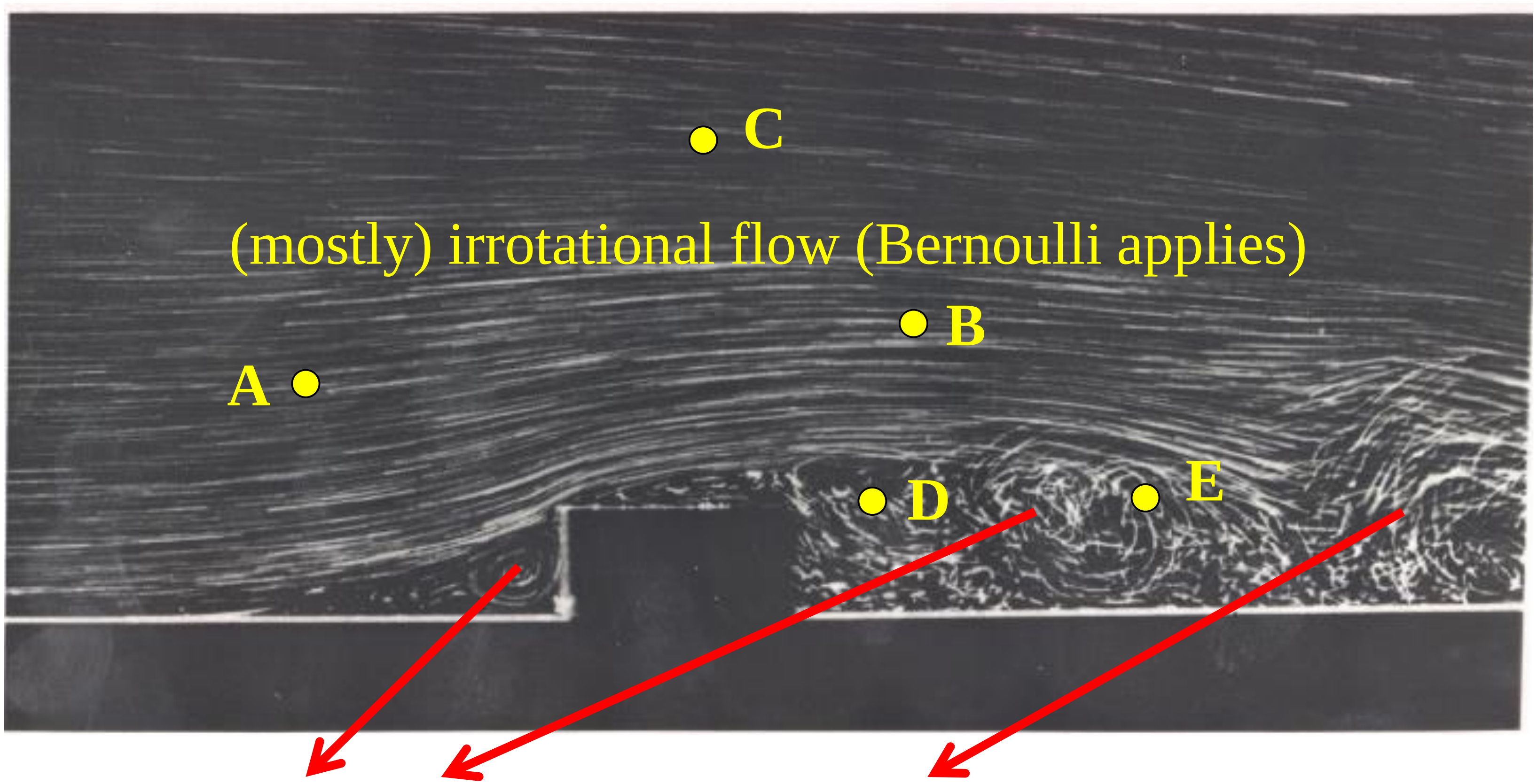
Velocity of fluid out of hole in tank (same as Example 4.4)



In both cases shown above: $V = \sqrt{2gh}$ ***WHY?***

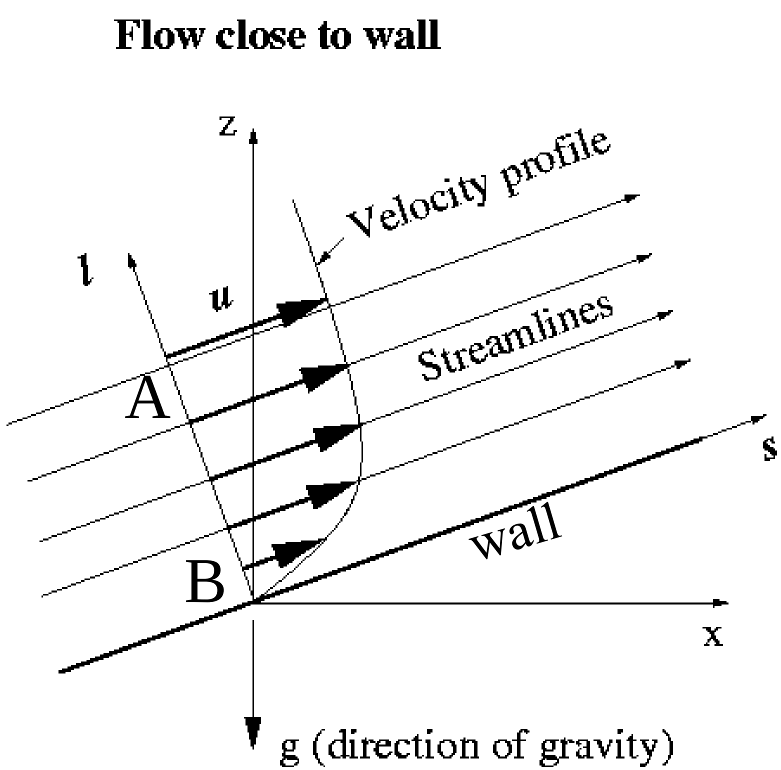
Q1: Can Bernoulli equation apply between points which do NOT belong to the same streamline?

- **Bernoulli applies among points A, B, C, where flow is IRROTATIONAL**
- **Bernoulli does NOT apply between points D & E, where flow is ROTATIONAL**



Highly **rotational** flow (where eddies, swirl, vorticity are formed)

Q2: How pressure varies along lines <u>normal to the direction of parallel flow?</u>



- **Bernoulli does NOT apply between points A & B (flow is rotational and viscous)**
- **Instead the hydrostatic law applies between points A & B:**

$$p_A + \gamma z_A = p_B + \gamma z_B$$

<u>WHY?</u>

Euler equation: $\dfrac{\partial}{\partial l}(p+\gamma z) = -\rho a_l$

In the shown case $a_l=0$ $\Rightarrow$ $\dfrac{\partial}{\partial l}(p+\gamma z) = 0$ $\Rightarrow$ $\boxed{(p+\gamma z) = cons.\, along\ l}$ *or* $\boxed{\dfrac{p}{\gamma} + z = cons.\, along\ l}$

Stagnation Tube (p. 129 of Textbook)

Method for relating pressure measurement to velocity

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma}$$

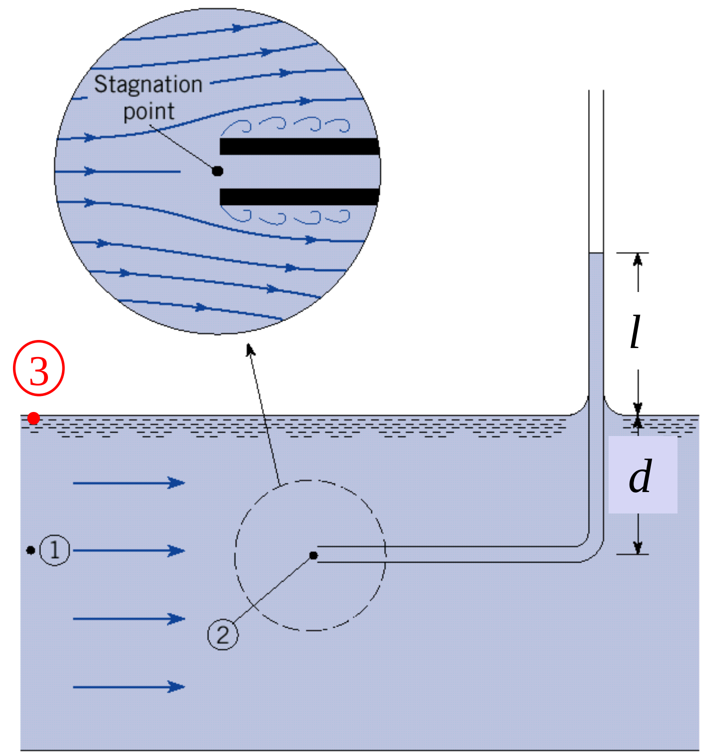
$$V_1^2 = \frac{2}{\rho}(p_2 - p_1)$$

$$= \frac{2}{\rho}(\gamma(l+d) - \gamma d)$$

$$V_1 = \sqrt{2gl}$$

$$p_2 = p_1 + \frac{\rho V_1^2}{2} = p_o = p_s =$$

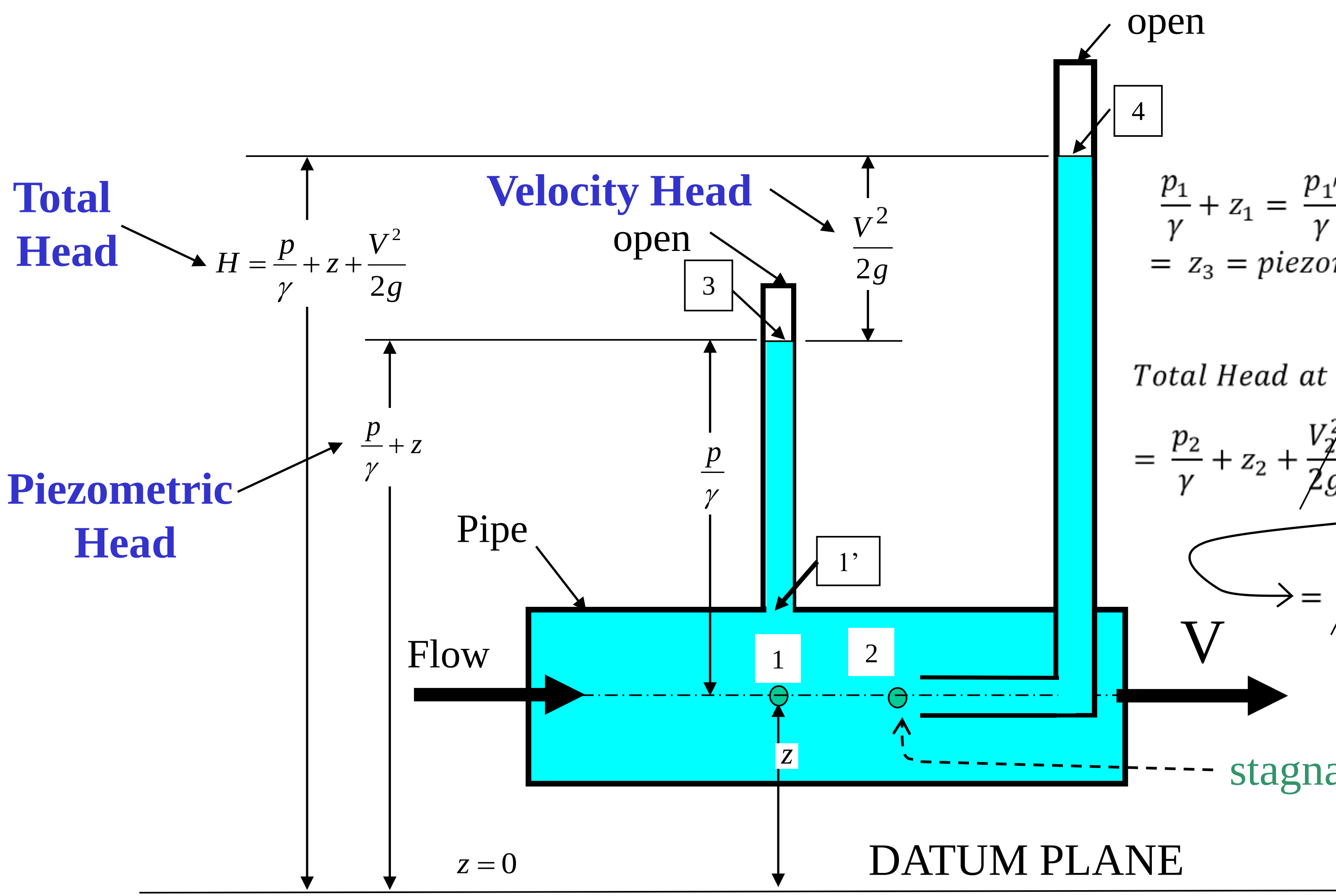
=stagnation or total pressure= pressure at stagnation point



Pressures between points 1 and 3 (free surface) are related via HYDROSTATIC LAW (WHY?)

Stagnation Tube in a Pipe

(presented for a more general case in the next slide)



$$\frac{p_1}{\gamma} + z_1 = \frac{p_{1'}}{\gamma} + z_{1'} = \cancel{\frac{p_3}{\gamma}} + z_3$$

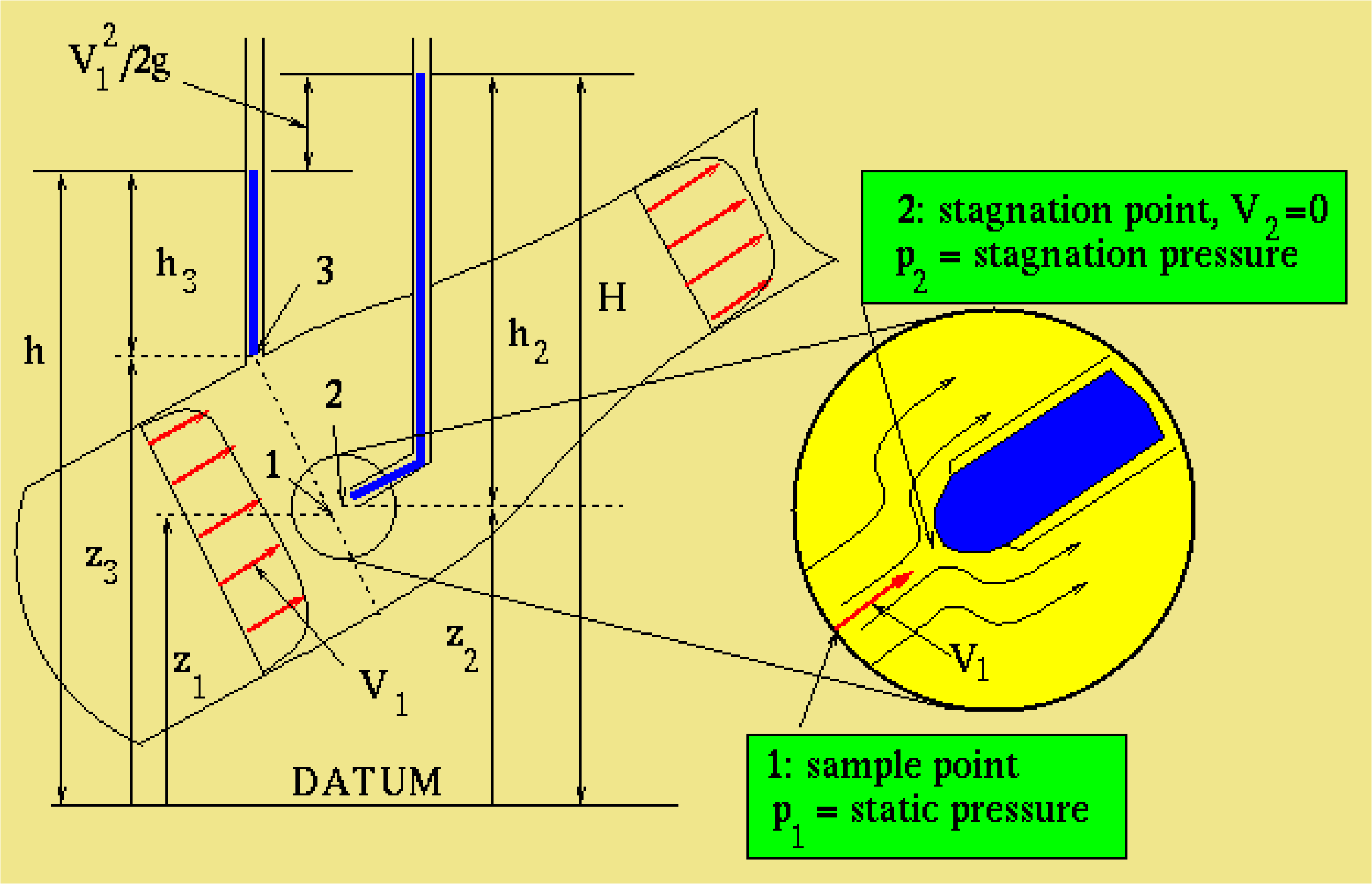
$$= z_3 = piezometric\ head\ at\ 1$$

$$Total\ Head\ at\ 1 = \frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g}$$

$$= \frac{p_2}{\gamma} + z_2 + \cancel{\frac{V_2^2}{2g}} = \frac{p_2}{\gamma} + z_2 =$$

$$= \cancel{\frac{p_4}{\gamma}} + z_4 = z_4$$

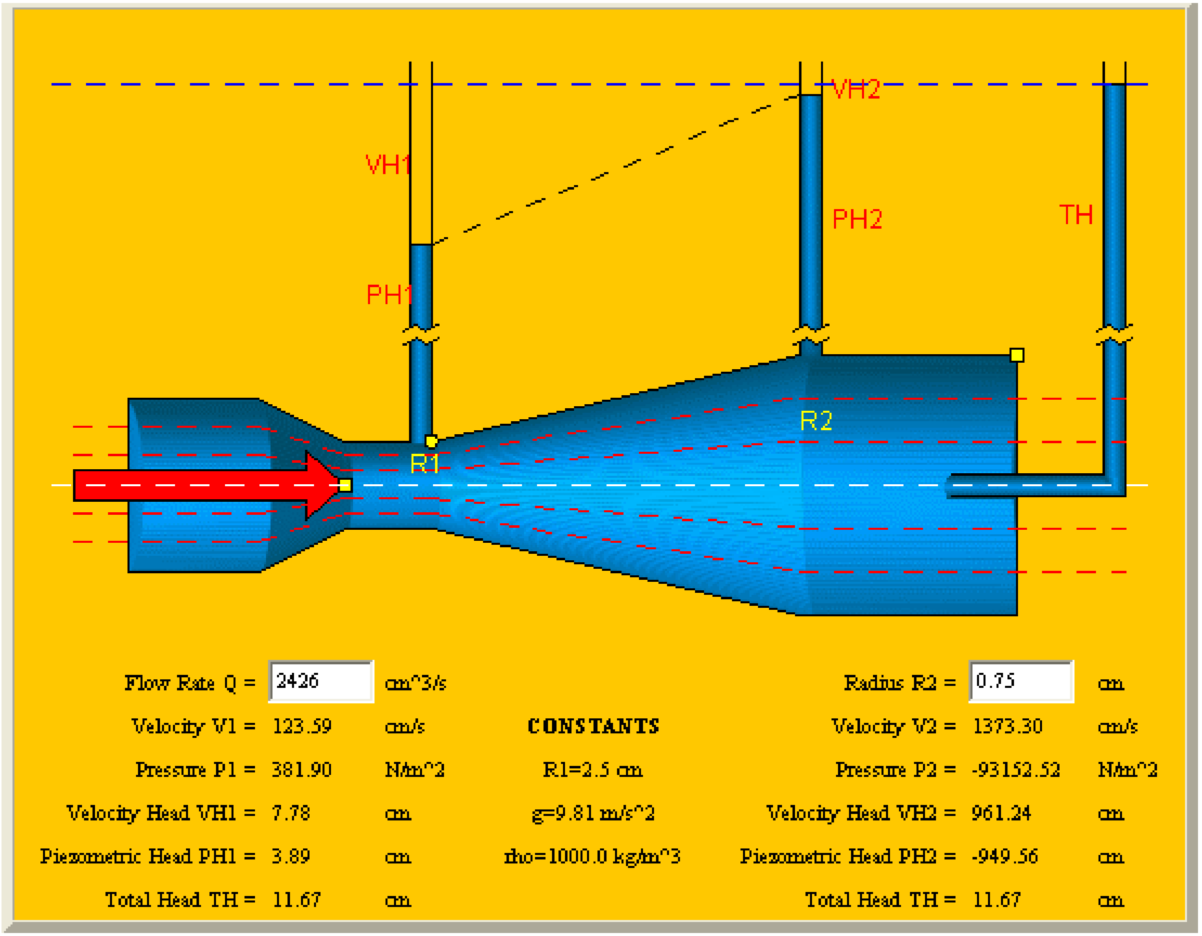
Measurement and physical meaning of total head (H) and piezometric head (h)



For details see:

http://cavity.caee.utexas.edu/kinnas/COURSES/ce319/ebook/head/head.html

The Venturi Applet



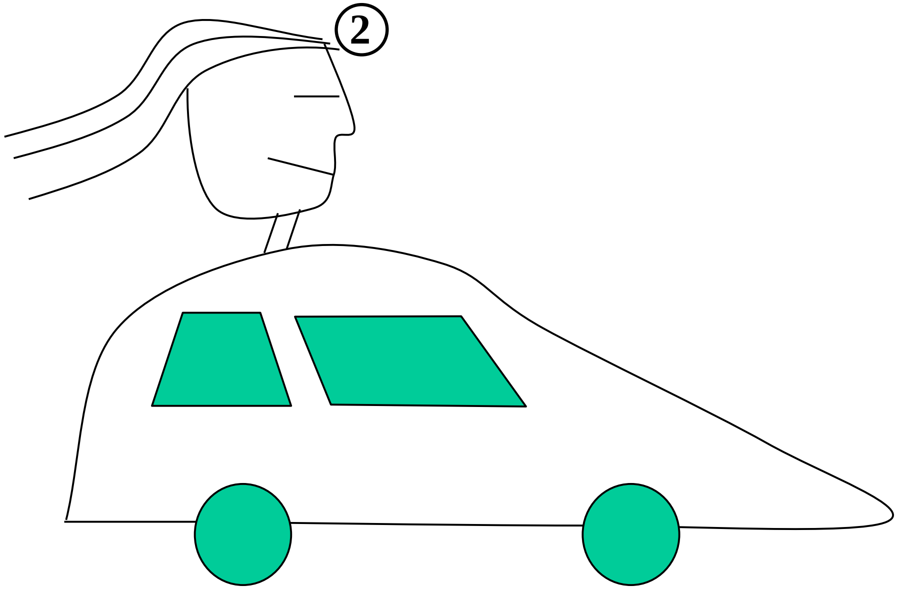
http://www.caee.utexas.edu/prof/kinnas/319LAB/fr_tool.html

Example problem

Q: What pressure would you feel on your palm (or forehead!) if you stick your hand (or head!) out of the window of a car moving at 65 miles/hour, and place it vertical to the direction of motion?

Consider flow relative to you:

$$V_1 = 65mph = 65\,\frac{miles}{hour} = 65\,\frac{1{,}610\ m}{3{,}600\ sec} = 29.07\ m/sec$$



1

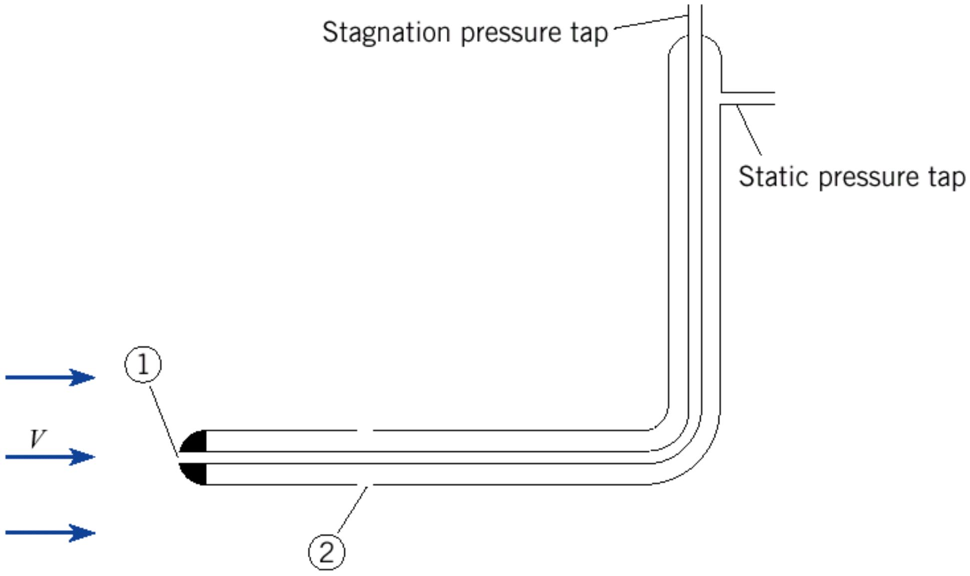
$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$p_2 = \rho\frac{V_1^2}{2} = \textit{stagnation pressure}$$

$$\rho = 1.2\ kg/m^3 \text{ (assume } 20^\circ\text{C)}$$

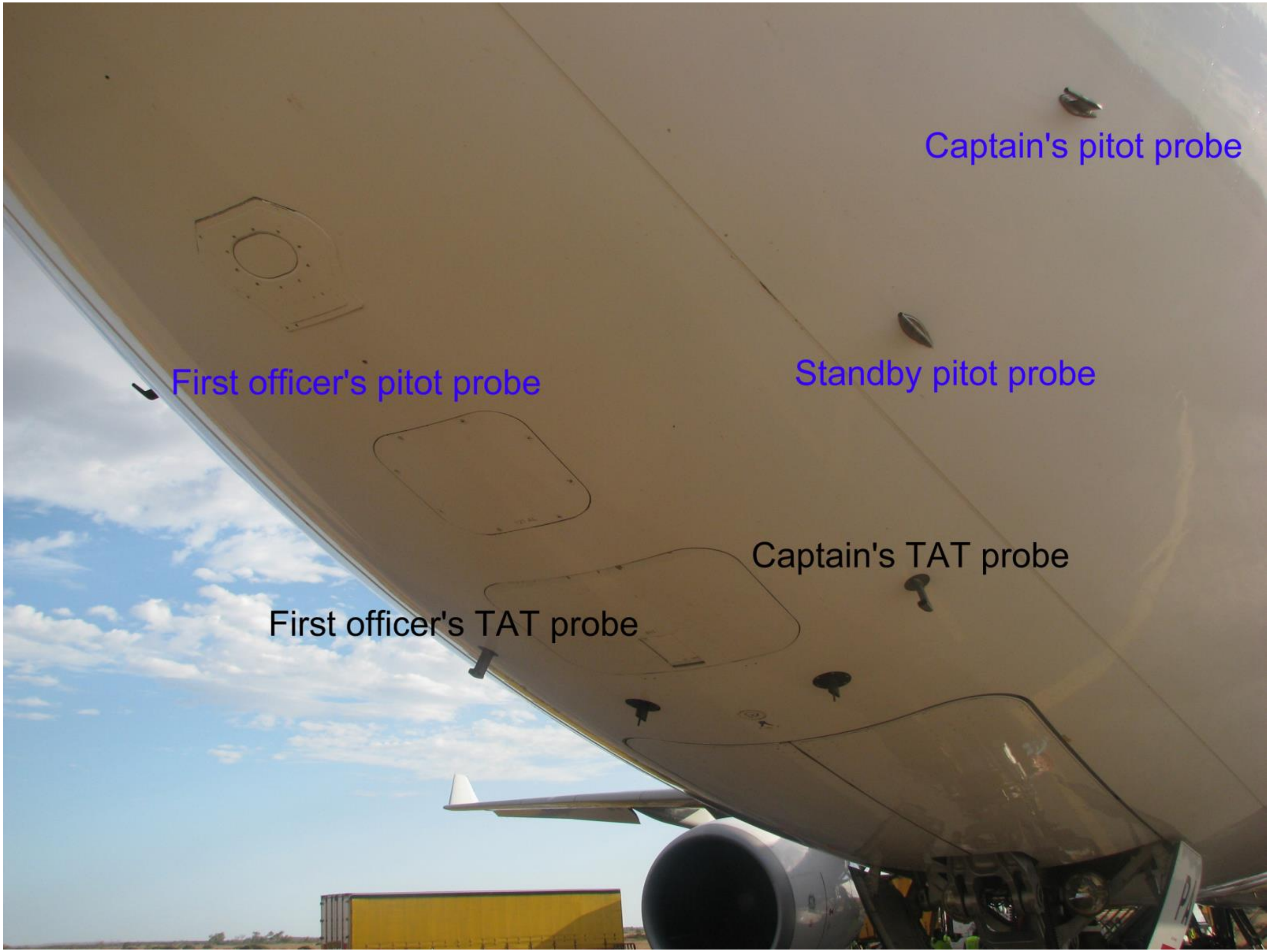
$$p_2 = 507\ Pa$$

Pitot Tube or Pitot-Static Tube



- Very important, easy to use (and cheap!) velocity measurement device.
- Derivation to be done on the board (also given in web-out)

Pitot and temperature probes on the "belly" of an A330

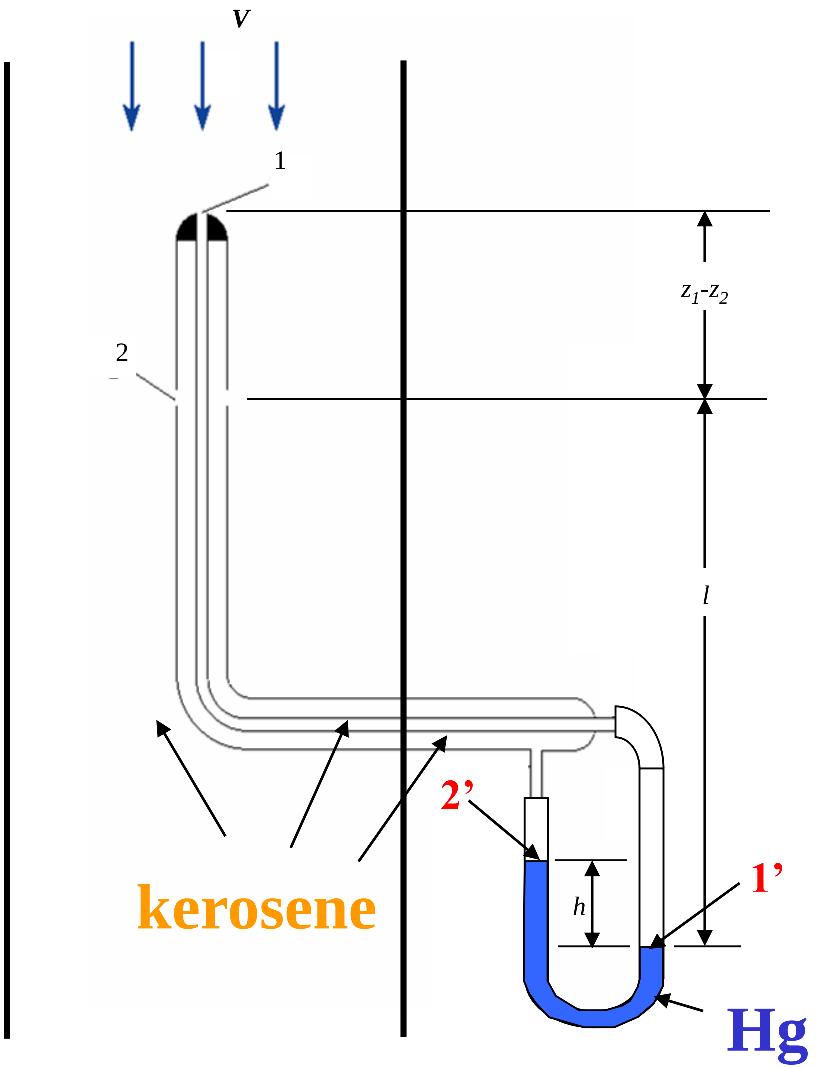


Example 4.7 of Textbook

A **mercury**-**kerosene** manometer is connected to the Pitot tube as shown. If the deflection, h, on the manometer is 7 inches, what is the kerosene velocity in the pipe? Assume that the specific gravity of kerosene is 0.81.

$$V = \sqrt{2gh\left[\frac{\gamma_{Ma}}{\gamma} - 1\right]} = \sqrt{2gh[\gamma_{Ma} - \gamma]/\gamma} =$$
$$\sqrt{2h[\gamma_{Ma} - \gamma]/\rho} = \sqrt{2\Delta p_z/\rho}$$

$$\begin{aligned}
\Delta p_z &= p_{z,1} - p_{z,2} \\
&= (p_1 + \gamma z_1) - (p_2 + \gamma z_2) \\
&= (p_{1'} + \gamma z_{1'}) - (p_{2'} + \gamma z_{2'}) \\
&= (p_{1'} - p_{2'}) + (\gamma z_{1'} - \gamma z_{2'}) \\
&= \gamma_{Ma}\, h - \gamma h = (\gamma_{Ma} - \gamma)\, h
\end{aligned}$$

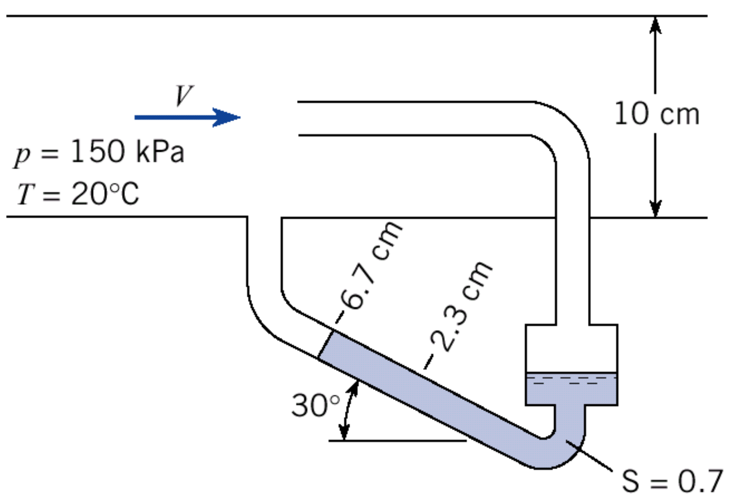


Example

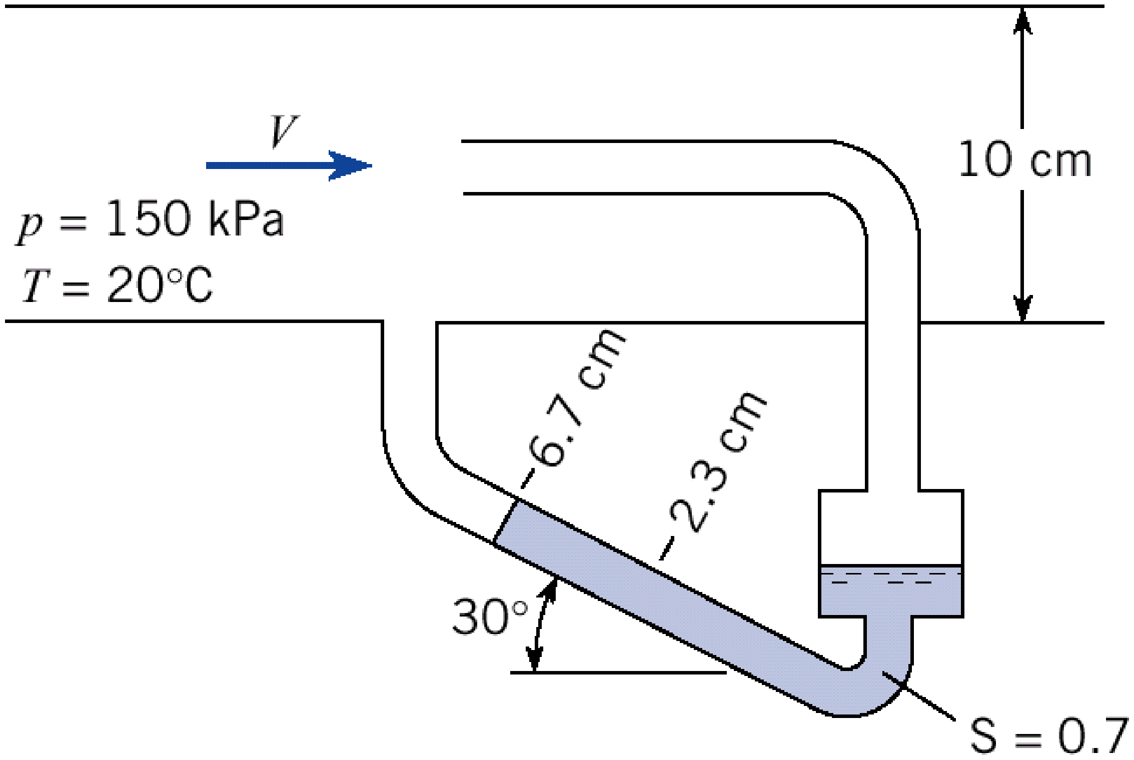
A tube with a 2 mm diameter is mounted at the center of a duct conveying air. The well of manometer fluid is large enough so that level changes in the well are negligible. With no flow in the duct, the level of the slant manometer is 2.3 cm. With flow in the duct it moves to 6.7 cm on the slant scale.

Find the velocity of air in the duct.

IMPORTANT NOTE: The given pressure of air inside duct is ABSOLUTE. You must use absolute pressure of gas when you apply ideal gas law!

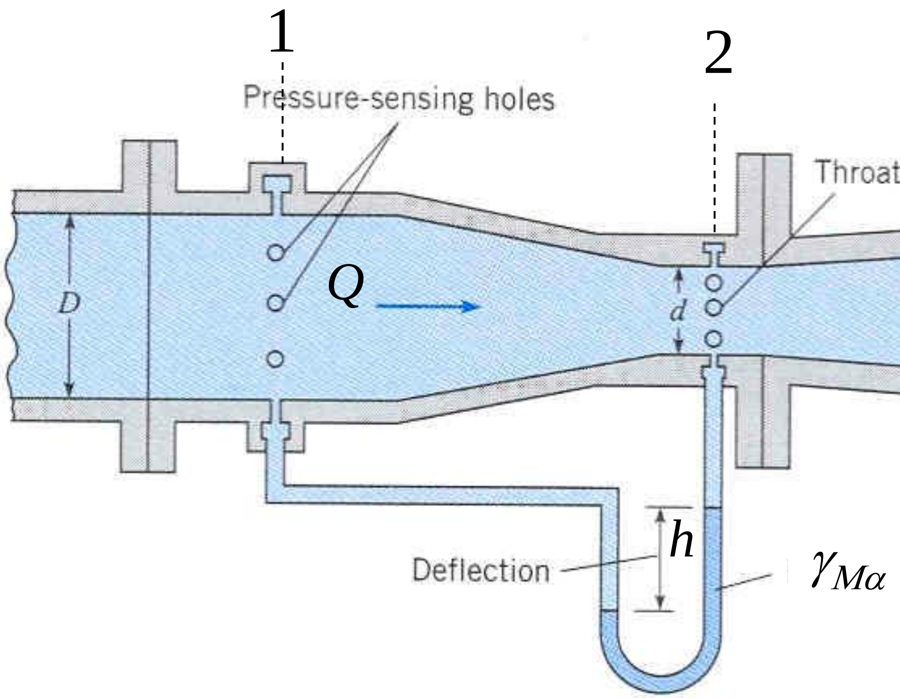


Solution



The Venturi meter (1/2)

(device for measuring flow-rate)



Continuity equation:

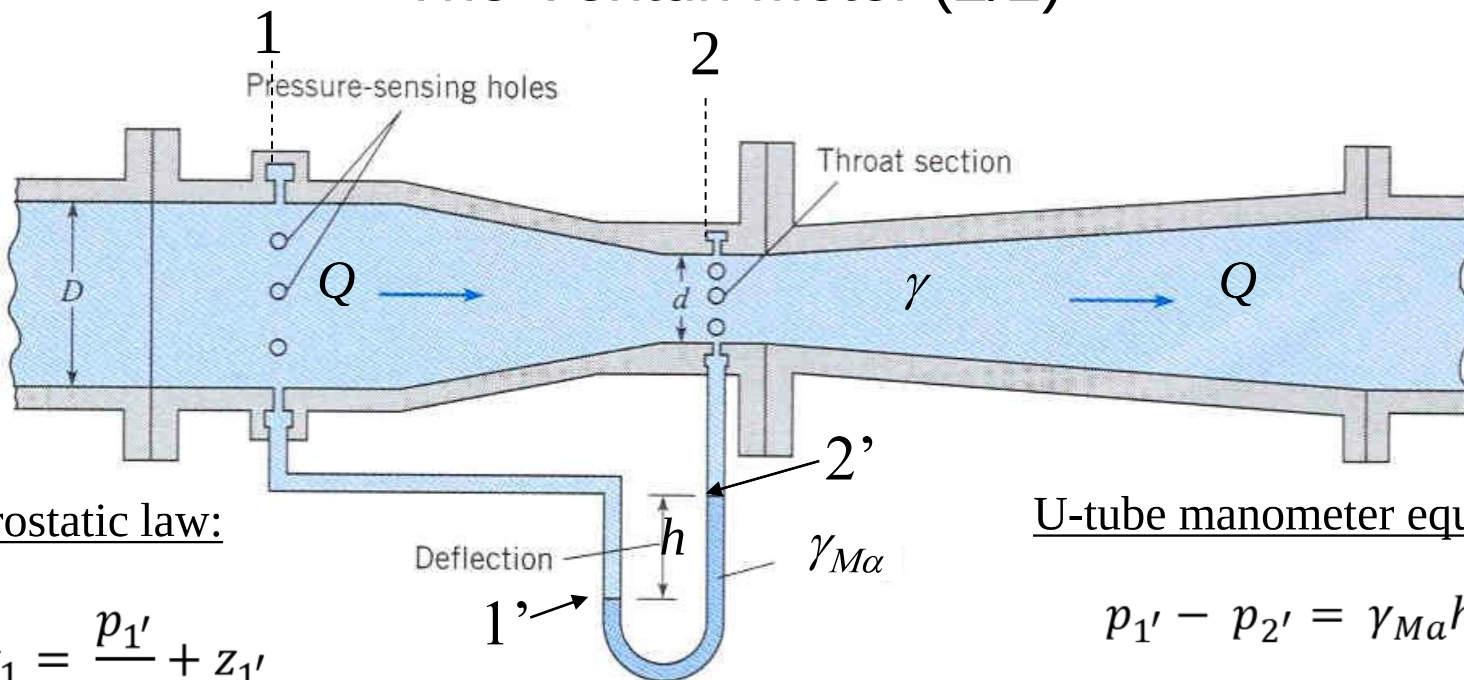
$$V_1 = \frac{Q}{A_1} \ \& \ V_2 = \frac{Q}{A_2}$$

Bernoulli equation:

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} \quad \text{or} \quad \frac{p_1}{\gamma} + z_1 + \frac{Q^2}{2gA_1^2} = \frac{p_2}{\gamma} + z_2 + \frac{Q^2}{2gA_2^2}$$

Venturi's Idea: If I measure (p_1-p_2), then I can find Q!

The Venturi meter (2/2)



<u>Hydrostatic law:</u>

$$\frac{p_1}{\gamma} + z_1 = \frac{p_{1'}}{\gamma} + z_{1'}$$

$$\frac{p_2}{\gamma} + z_2 = \frac{p_{2'}}{\gamma} + z_{2'}$$

<u>U-tube manometer equation:</u>

$$p_{1'} - p_{2'} = \gamma_{Ma} h$$

$$\Delta h_z = \left(\frac{p_{1'}}{\gamma} + z_{1'}\right) - \left(\frac{p_{2'}}{\gamma} + z_{2'}\right) = \frac{\gamma_{Ma} h}{\gamma} - h = \left(\frac{\gamma_{Ma}}{\gamma} - 1\right) h$$

$$\Delta p_z = p_{z,1} - p_{z,2} = (p_1 + \gamma z_1) - (p_2 + \gamma z_2)$$
$$= (p_{1'} + \gamma z_{1'}) - (p_{2'} + \gamma z_{2'}) = \gamma\, \Delta h_z = (\gamma_{Ma} - \gamma)\, h$$

<u>Bernoulli equation:</u>

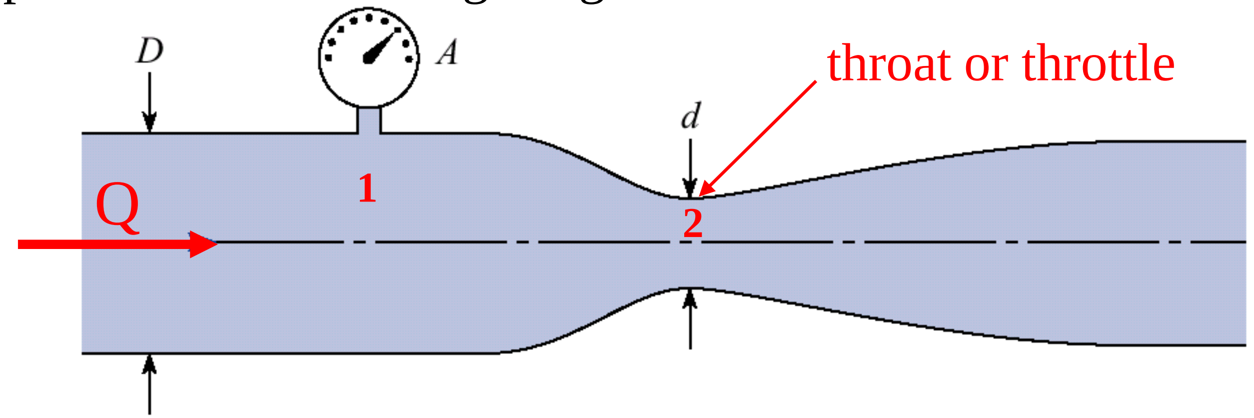
$$\frac{p_{1'}}{\gamma} + z_{1'} + \frac{Q^2}{2gA_1^2} = \frac{p_{2'}}{\gamma} + z_{2'} + \frac{Q^2}{2gA_2^2}$$

Then:

$$\boxed{Q = \frac{\sqrt{2gh\left[\frac{\gamma_{Ma}}{\gamma} - 1\right]}\, A_2}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} = \frac{\sqrt{\frac{2\Delta p_z}{\rho}}\, A_2}{\sqrt{1 - \left(\frac{d}{D}\right)^4}}}$$

Example – Venturi & Cavitation

When gage A reads 120 kPa gage, cavitation just starts to occur in the venturi meter. If D = 40 cm and d = 10 cm, what is the water discharge in the system for a condition of incipient cavitation? The atmospheric pressure is 100 kPa. The water temperature is 10 ^{o}C. Neglect gravitational effects.



$$\frac{p_1}{\gamma} + z_1 + \frac{Q^2}{2gA_1^2} = \frac{p_2}{\gamma} + z_2 + \frac{Q^2}{2gA_2^2}$$

< 0

$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + z_1 - z_2 + \frac{Q^2}{2gA_1^2} - \frac{Q^2}{2gA_2^2} = \frac{p_1}{\gamma} + z_1 - z_2 + \frac{Q^2}{2g}\boxed{\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right)}$$

NOTE: As Q $\uparrow$ $p_2 \downarrow$ and cavitation occurs when the ABSOLUTE pressure at 2 reaches vapor pressure, p_v , for the given temperature of operation

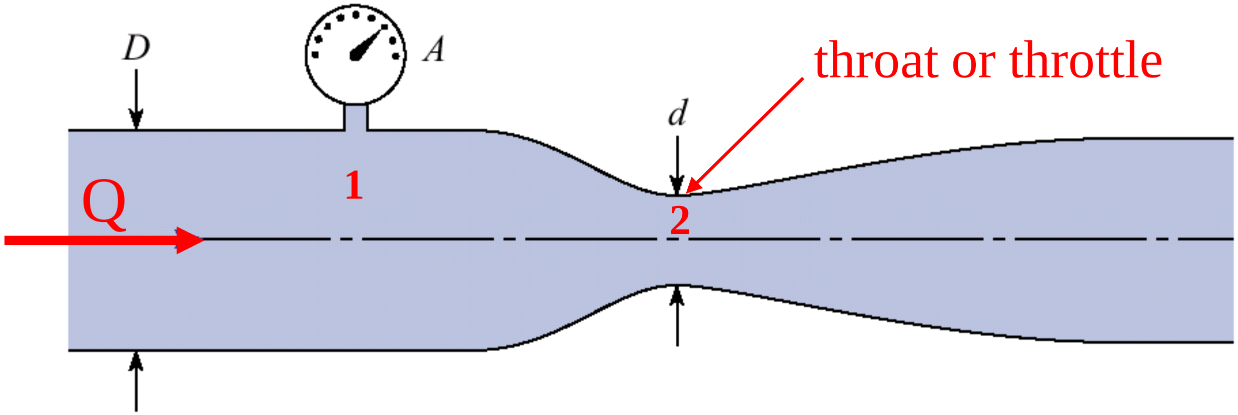
Excerpt from the Table for Water

TABLE A.5 Approximate Physical Properties of Water* at Atmospheric Pressure

Temperature	Density	Specific Weight	Dynamic Viscosity	Kinematic Viscosity	Vapor Pressure
	kg/m^3	N/m^3	$N \cdot s/m^2$	m^2/s	N/m^2 abs
0°C	1000	9810	1.79×10^{-3}	1.79×10^{-6}	611
5°C	1000	9810	1.51×10^{-3}	1.51×10^{-6}	872
10°C	1000	9810	1.31×10^{-3}	1.31×10^{-6}	1,230
15°C	999	9800	1.14×10^{-3}	1.14×10^{-6}	1,700
20°C	998	9790	1.00×10^{-3}	1.00×10^{-6}	2,340
25°C	997	9781	8.91×10^{-4}	8.94×10^{-7}	3,170
30°C	996	9771	7.97×10^{-4}	8.00×10^{-7}	4,250
35°C	994	9751	7.20×10^{-4}	7.24×10^{-7}	5,630
40°C	992	9732	6.53×10^{-4}	6.58×10^{-7}	7,380
50°C	988	9693	5.47×10^{-4}	5.53×10^{-7}	12,300
60°C	983	9643	4.66×10^{-4}	4.74×10^{-7}	20,000
70°C	978	9594	4.04×10^{-4}	4.13×10^{-7}	31,200
80°C	972	9535	3.54×10^{-4}	3.64×10^{-7}	47,400
90°C	965	9467	3.15×10^{-4}	3.26×10^{-7}	70,100
100°C	958	9398	2.82×10^{-4}	2.94×10^{-7}	101,300

Example – Venturi & Cavitation (solution)

When gage A reads 120 kPa gage, cavitation just starts to occur in the venturi meter. If D = 40 cm and d = 10 cm, what is the water discharge in the system for a condition of incipient cavitation? The atmospheric pressure is 100 kPa. The water temperature is 10 °C. Neglect gravitational effects.



$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + \cancel{z_1 - z_2} + \frac{Q^2}{2g}\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right) \Longrightarrow p_2 = p_1 + \rho\frac{Q^2}{2}\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right)$$

$$A_1 = \frac{\pi D^2}{4}$$

$$A_2 = \frac{\pi d^2}{4}$$

p_2 (absolute)=1,230 Pa

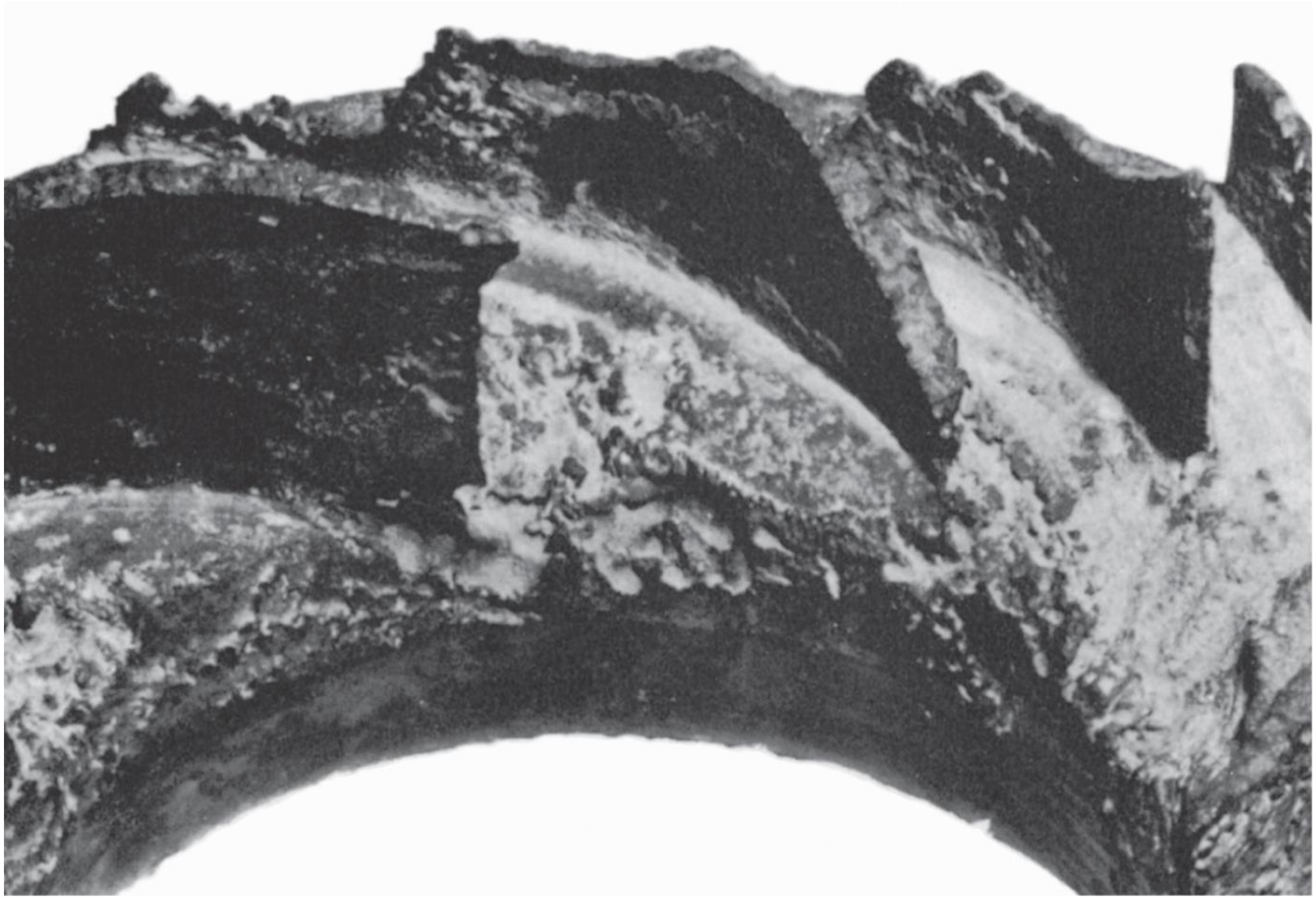
p_1 (absolute) = p_1 (gage) + p_{atm} = 120,000 + 100,000 = 220,000 Pa

$$with\ D = 0.4\,m, d = 0.1\,m\ \&\ \rho = 1{,}000\,kg/m^3 \Longrightarrow \boxed{Q = 0.1646\ m^3/s}$$

Picture of **cavitation** on the upper (suction) side of a hydrofoil placed inside a cavitation tunnel (flow goes from left to right)

For more pictures/theory check UT's Cavitation Home Page at
http://cavity.caee.utexas.edu

Cavitation damage on impeller of a pump (taken from 9th edition of textbook)



Cavitation damage on dam spillway tunnel (taken from 10th edition of textbook)