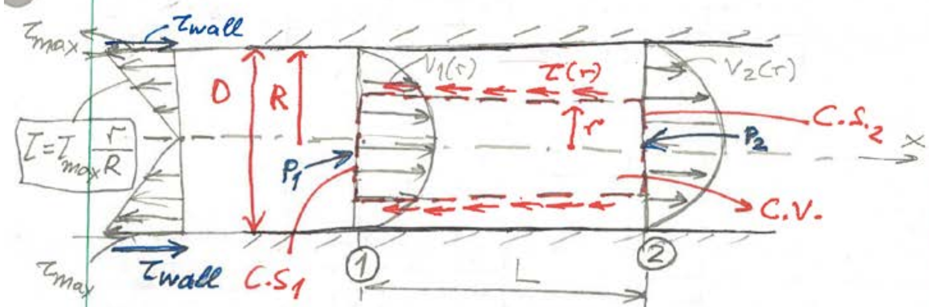
Developed Flows Inside Circular Pipes



Developed flow: Does NOT change along x.

Momentum equ. on shown C.V. (along x)

$$\sum F_x = \int_{C.S.} \rho V_x \vec{V}\cdot d\vec{A} \quad \text{(steady flow)}$$

$$P_1 A_1 - P_2 A_2 - \tau(2\pi r \cdot L) = \underbrace{\int_{C.S_2} \rho V_2 V_2 dA - \int_{C.S_1} \rho V_1 V_1 dA}_{=0 \text{ since } V_2(r)=V_1(r)}$$

$A_1 = A_2 = \pi r^2$

$$\Rightarrow (P_1-P_2)\cancel{\pi} r^{\cancel{2}} = \tau(2\cancel{\pi}\cancel{r}\cdot L) \Rightarrow \tau = \frac{P_1-P_2}{2L}\cdot r \quad (1) \quad \text{(shear stress on fluid)}$$

$$\underbrace{\tau_{max} = \tau_{wall}}_{\text{equal and opposite direction}} = \frac{P_1-P_2}{2L}\cdot R \quad (1')$$

$$\boxed{\tau = \tau_{max}\cdot\frac{r}{R} = \tau_{wall}\frac{r}{R}} \quad (2)$$

Valid for both LAMINAR & TURBULENT FLOWS

From (1') $\quad P_1 - P_2 = \tau_{max}\frac{2L}{R} \quad (3)$

Energy equ. between ① & ② $(h_p=0, h_t=0)$

$$\frac{P_1}{\gamma} + \cancel{z_1} + \cancel{\alpha_1\frac{\bar{V}_1^2}{2g}} = \frac{P_2}{\gamma} + \cancel{z_2} + \cancel{\alpha_2\frac{\bar{V}_2^2}{2g}} + h_L$$

$\alpha_1 = \alpha_2$ and $\bar{V}_1 = \bar{V}_2$ (WHY?)

(HINT: Developed Flow!)

Thus $\quad h_L = \frac{P_1 - P_2}{\gamma} \quad (4)$

$$(3)\,\&\,(4) \Rightarrow h_L = \frac{\tau_{max}}{\gamma}\frac{2L}{R} = \frac{\tau_{max}}{\gamma}\frac{4L}{D} \quad (D=2R)$$

$$\boxed{h_L = \frac{\tau_{max}}{\gamma}\frac{4L}{D} = \frac{\tau_{wall}}{\gamma}\frac{4L}{D}} \quad (5)$$

NOTE: $h_L \sim \tau_{wall}$

(For inviscid flow $\tau_{wall}=0 \Rightarrow h_L = 0$!)

Darcy-Weisbach equ: $\boxed{h_L = f\,\frac{L}{D}\cdot\frac{\bar{V}^2}{2g}} \quad (6)$

f → resistance coeff. or friction factor

$h_f \equiv h_L$ → another symbol for pipes

Applies to both LAMINAR & TURBULENT FLOWS

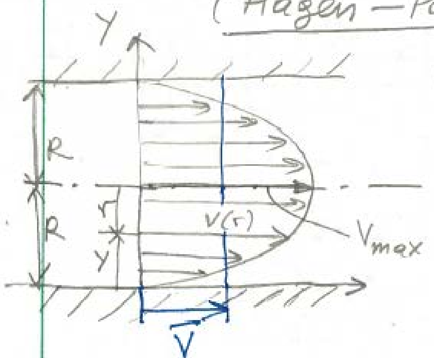
Combining (5) & (6)

$$\frac{\tau_{max}}{\underbrace{\gamma}_{=\rho \cancel{g}}}\,\frac{4\cancel{L}}{D} = f\,\frac{\cancel{L}}{D}\,\frac{\bar V^2}{2\cancel{g}}$$

$$\boxed{\tau_{wall} = \tau_{max} = f\rho\frac{\bar V^2}{8}} \quad (7)$$

← APPLIES TO BOTH LAMINAR & TURBULENT FLOWS!

LAMINAR FLOWS INSIDE CIRCULAR PIPES
(Hagen – Poiseuille flow)



μ: dynamic viscosity

$$\tau = \mu\frac{dV}{dy} \quad (y = R - r)$$

$$= \mu\frac{dV}{dr}\frac{dr}{dy} = -\mu\frac{dV}{dr}$$

$$\left(r = R - y \rightsquigarrow \frac{dr}{dy} = -1\right)$$

Thus $\tau = -\mu\dfrac{dV}{dr}$ (8)

But from (1) $\tau = \dfrac{P_1 - P_2}{2L}\cdot r = B\,r$ (9) DEF. (9)

Then $B\,r = -\mu\dfrac{dV}{dr}$ (10)

$$B r = -\mu\frac{dV}{dr} \quad (10)$$

$$\rightsquigarrow dV = -\frac{B}{\mu} r\,dr \rightsquigarrow \int_r^R dV = -\frac{B}{\mu}\int_r^R r\,dr \rightsquigarrow$$

$$\rightsquigarrow V\Big|_r^R = -\frac{B}{\mu}\frac{r^2}{2}\Big|_r^R \rightsquigarrow \underbrace{V(R)}_{\substack{\parallel \\ 0\ \text{(NO SLIP CONDITION!)}}} - V(r) = -\frac{B}{\mu}\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$\rightsquigarrow -V(r) = -\frac{BR^2}{2\mu}\left[1 - \left(\frac{r}{R}\right)^2\right] \rightsquigarrow V(r) = \frac{BR^2}{2\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$\rightsquigarrow \boxed{V(r) = V_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]} \quad (11)$$

Parabolic velocity profile for laminar flows inside pipes

Where: $\boxed{V_{max} = \dfrac{BR^2}{2\mu}}$ (12)

$$\bar V\ (\text{mean velocity}) = \frac{Q}{A} = \frac{\int_A V\,dA}{A} = \frac{V_{max}}{2}$$

using V(r) from (11)

$$\boxed{\bar V = \frac{V_{max}}{2}} \quad (13)$$

← For LAMINAR flows inside pipes

From (9) $\underline{\tau_{max} = BR}$ (14)

Then from: $\tau_{max} = f \rho \frac{\bar{V}^2}{8}$ (7) and (14)

we get: $f \rho \frac{\bar{V}^2}{8} = BR$ (15)

But from (13) & (12): $\bar{V} = \frac{V_{max}}{2} = \frac{BR^2}{4\mu}$ (16)

(15) $\Rightarrow f \rho \frac{\bar{V}}{8} \bar{V} = BR$

or $f \rho \frac{\bar{V}}{8} \frac{\cancel{B}R^{\cancel{2}}}{4\mu} = \cancel{B}\cancel{R}$ or $f \rho \frac{\bar{V} R}{32\mu} = 1$

or $f \frac{\bar{V} \cdot \frac{D}{2}}{32 \left(\frac{\mu}{\rho}\right)} = 1$ or $f \frac{\bar{V} D}{\nu} \frac{1}{64} = 1$

$\left(\frac{\mu}{\rho}\right) = \nu$ (Kinematic viscosity)

or $f \cdot \frac{Re}{64} = 1$ or $\boxed{f = \frac{64}{Re}}$ (17) ← FOR LAMINAR FLOWS IN PIPES

$\boxed{Re \underset{\substack{\uparrow \\ \text{DEF.}}}{=} \text{Reynolds \#} = \frac{\bar{V} D}{\nu}}$ (18) ← dimension-less number

Re_D or R other symbols

(remember units of ν is $\frac{m^2}{s}$ in S.I.)

very important number characterizing effects of viscosity in pipe flows!

For Laminar flows equ. (6) can also be written as:

$$h_L = f \frac{L}{D} \frac{\bar{V}^2}{2g} = \frac{64}{Re} \cdot \frac{L}{D} \cdot \frac{\bar{V}^2}{2g} =$$

$$= \frac{\cancel{64}^{32}}{\left(\frac{\bar{V} D}{\nu}\right)} \cdot \frac{L}{D} \cdot \frac{\bar{V}^{\cancel{2}}}{\cancel{2}g} = \frac{32\nu}{D} \frac{L}{D} \frac{\bar{V}}{g} =$$

$$= \frac{32\mu}{\rho} \frac{L}{D^2} \frac{\bar{V}}{g} \Longrightarrow$$

$$\boxed{h_f \equiv h_L = \frac{32 \mu L \bar{V}}{\gamma D^2}} \quad (19)$$

← For LAMINAR FLOWS INSIDE PIPES

For TURBULENT FLOWS INSIDE PIPES:

Darcy–Weisbach equ. still applies

$$h_f \equiv h_L = f \frac{L}{D} \frac{\bar{V}^2}{2g} \quad (6)$$

where f is given from Prandtl equ.

$$\boxed{\frac{1}{\sqrt{f}} = 2 \log_{10}\left(Re \sqrt{f}\right) - 0.8} \quad (20)$$

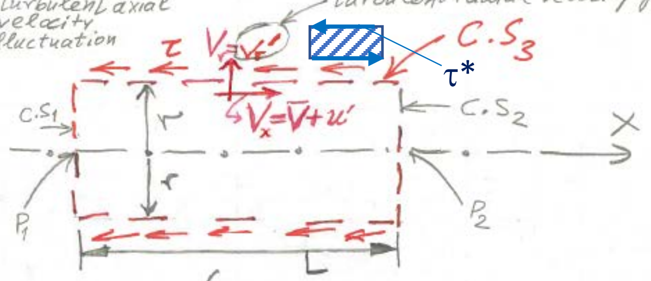
FOR TURBULENT FLOWS INSIDE SMOOTH PIPES

For Rough pipes (or smooth) $\Rightarrow$ **Moody Chart (or Diagram)!**

In the case of turbulent flows:

$V_x = \bar{V} + u'$ ← turbulent axial velocity fluctuation

time averaged axial velocity



Note:

$\overline{u'} = 0$

$\overline{v_r'} = 0$

Momentum equ.: $\sum F_x = \int_{C.S._3} \rho V_x V_r \, dA$ ← no longer zero!

Note: See equ. (25d) on next page regarding $\int_{C.S_1}$ & $\int_{C.S_2}$

(For Laminar flows the radial velocity $V_r = 0$)

$$P_1 A_1 - P_2 A_2 - \tau (2\pi r) \cdot L = \int_{C.S._3} \rho (\bar{V} + u') v_r' \, dA \quad (21)$$

due to equ. (8)

$$\left(P_1 - P_2\right) \pi r^2 + \mu \frac{dV}{dr} (2\pi r) L = \int_{C.S_3} \rho (\bar{V} + u') v_r' \, dA \quad (22)$$

Note: $p = \bar{p} + p'$ ← turbulent fluctuation: $\overline{p'} = 0$

time averaged

after averaging in time

$$(\bar{P}_1 - \bar{P}_2) \pi r^2 + \mu \frac{d\bar{V}}{dr} (2\pi r) L = \int_{C.S_3} \rho \overline{u' v_r'} \, dA \quad (23)$$

$$(\bar{P}_1 - \bar{P}_2) \pi r^2 + \mu \frac{d\bar{V}}{dr} (2\pi r) L = \rho \overline{u' v_r'} (2\pi r) L \quad (24)$$

Assumptions we have made to get to equ. (24)

- $\overline{u' v_r'}$ only depends on r (25a)
- $\overline{\bar{V} v_r'} = \bar{V} \, \overline{v_r'} = \bar{V} \cdot 0 = 0$ (proved, since $\overline{v_r'} = 0$) (25b)
- $\frac{d}{dt} \int_{C.V.} \rho (\bar{\vec{q}}) \, d\forall = 0$ (25c) (can be proved) (with $\vec{q} = (\bar{V} + u')\vec{i} + v_r' \vec{j}$)
- $\int_{C.S_1} \rho (\bar{V} + u')^2 dA = \int_{C.S_2} \rho (\bar{V} + u')^2 dA$ (can be proved) (25d)

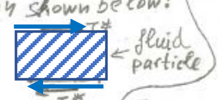
We finally get:

$$(\bar{P}_1 - \bar{P}_2) \pi r^2 + \underbrace{\left[\mu \frac{d\bar{V}}{dr} - \rho \overline{u' v_r'}\right]}_{\text{New } \tau^*} (2\pi r) L = 0 \quad (24a)$$

Note 1: $\boxed{\tau^* = -\tau}$

Actually τ^* is positive in the direction shown below:

Note 2: $\frac{d\bar{V}}{dr} < 0$



$$\boxed{\tau^* = \mu \frac{d\bar{V}}{dr} - \rho \overline{u' v_r'}} \quad (26)$$

Reynolds turbulent stress!

$$\Rightarrow \quad -\left[\mu \frac{d\bar{V}}{dr} - \rho \overline{u' v_r'}\right] = \frac{\bar{P}_1 - \bar{P}_2}{2L} r \quad \text{or} \quad \boxed{-\tau^* = \frac{\bar{P}_1 - \bar{P}_2}{2L} r} \quad (27)$$

Compare equ. (27) with equ. (1) (remember $\tau^* = -\tau$)