RANDOM WAVES AND WAVE SPECTRUM

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SOME ACTUAL SAMPLE WAVE RECORDS

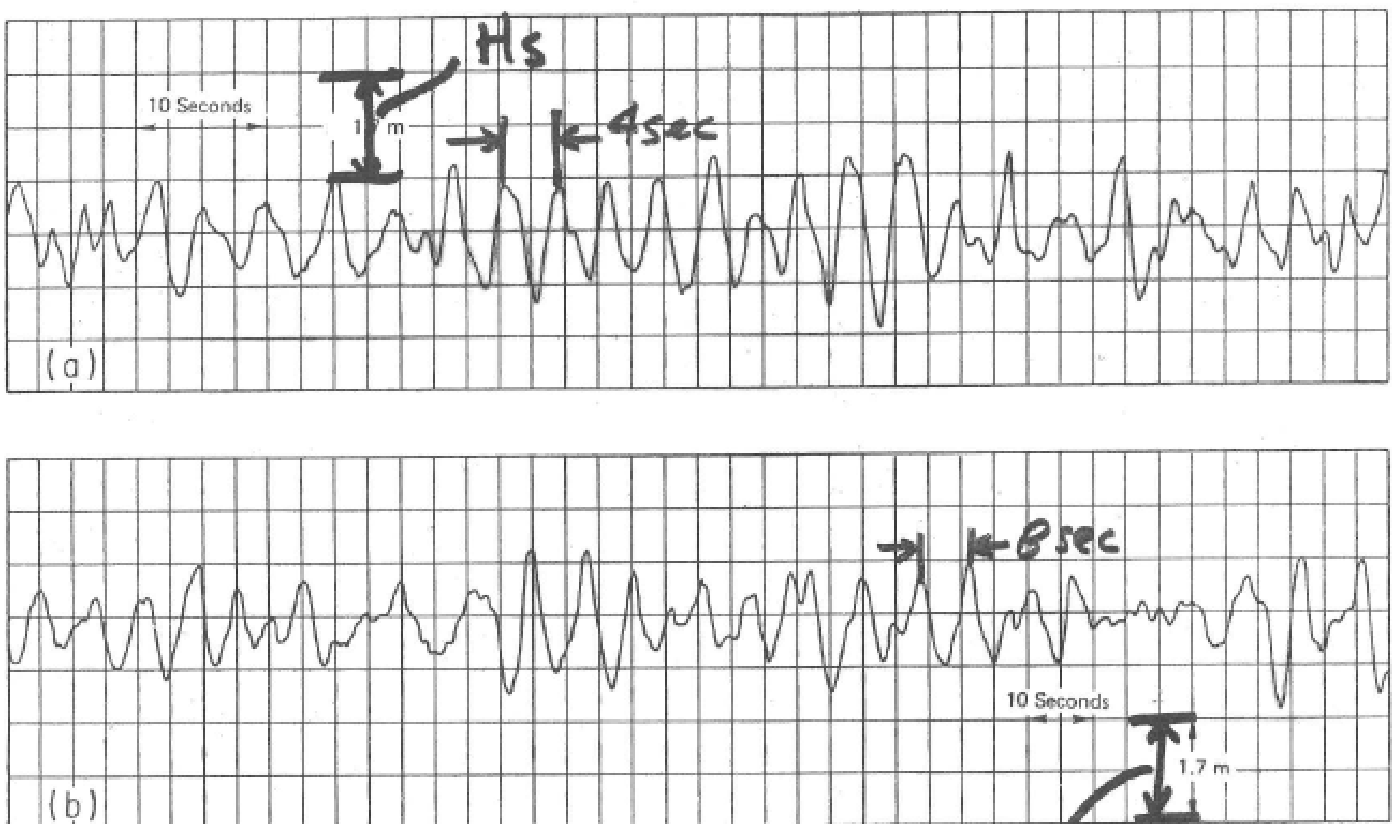
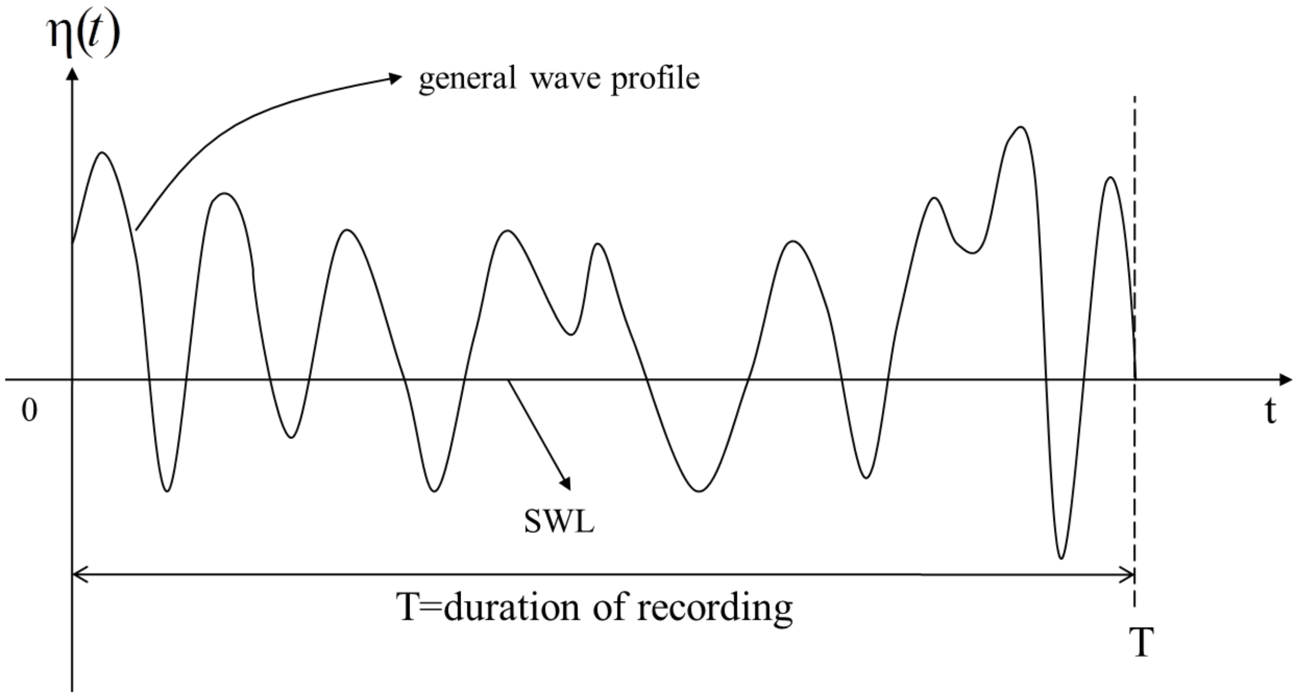


Figure 3-1. Sample wave records--(a) Chesapeake Bay Bridge Tunnel, Portal Island, significant height 1.7 meters (5.5 feet), period 4 seconds; (b) Huntington Beach, California, significant height 1.7 meters, period 8 seconds.

Hs

TWO WAYS TO ANALYZE RANDOM SEAS:

A: Via Fourier Analysis of Wave Record $\eta(t)$ at a location in the ocean:



$\eta(t)$ **can be expressed in terms of a series of sinusoidal waves with progressively decreasing periods:** $T, \frac{T}{2}, \frac{T}{3}, \frac{T}{4}, \cdots, \frac{T}{J}, \cdots$ **or progressively increasing (angular) frequencies:**

$\omega = \frac{2\pi}{T}, 2\omega, 3\omega, 4\omega, \cdots, J\omega, \cdots$

$$\eta(t) = \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t) \quad \textbf{(1)}$$

a_n, b_n**: Fourier coefficients (** n^{th} **harmonic)**

$$\text{or} \quad \eta(t) = \sum_{n=1}^{\infty} \frac{H_n^*}{2} \cos(n\omega t - \vartheta_n) \quad \textbf{(2)}$$

H_n^***: height of** n^{th} **harmonic**

ϑ_n**: phase of** n^{th} **harmonic**

$$H_n^* = 2\sqrt{a_n^2 + b_n^2}, \qquad \tan\vartheta_n = \frac{b_n}{a_n} \quad \textbf{(3)}$$

remember: $\cos(n\omega t - \vartheta_n) = \cos(n\omega t)\cos\vartheta_n + \sin(n\omega t)\sin\vartheta_n$

(H_n^*, ϑ_n**) or (**a_n, b_n**) depend on the form of** $\eta(t)$**.**

Other notations:

$\omega_1 = \omega = \omega_0$ **=fundamental frequency or** 1^{st} **harmonic**
$\omega_n = n \cdot \omega$ **= multiple of fundamental frequency or** n^{th} **harmonic**

The Fourier coefficients can be determined from the following integrals:

$$a_n = \frac{2}{T}\int_0^T \eta(t)\ \cos(n\omega t)dt \tag{4}$$

$$b_n = \frac{2}{T}\int_0^T \eta(t)\ \sin(n\omega t)dt \tag{5}$$

Note: In general there is a constant term [a_0**] too, i.e.:**

$$\eta(t) = a_\circ + \sum_{n=1}^{\infty} a_n\ \cos(n\omega t) + b_n \sin(n\omega t) \tag{6}$$

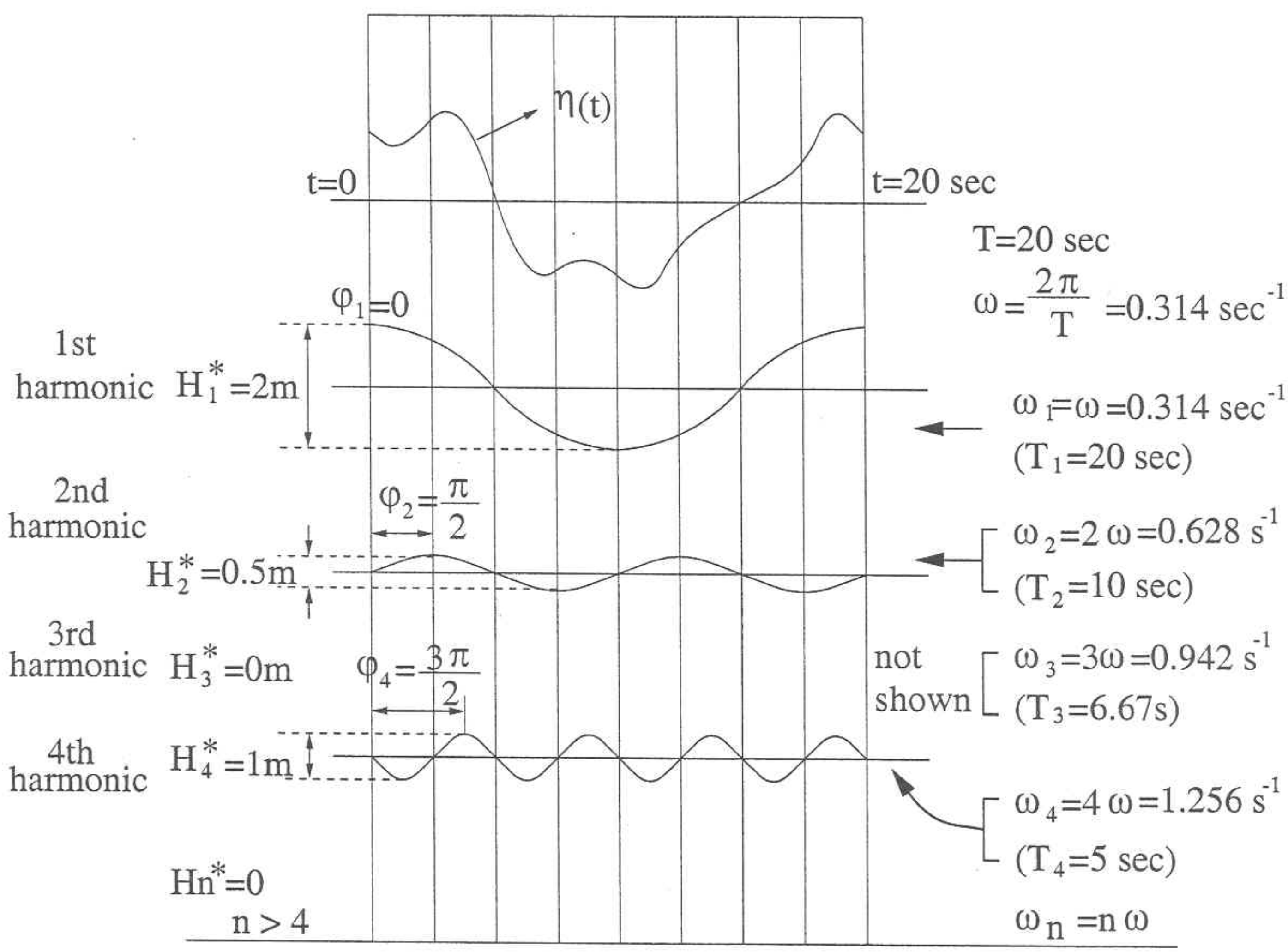
With $a_\circ = \frac{1}{T}\int_0^T \eta(t)\ dt = \overline{\eta}$ = mean value of η over T **(7)**

In the case of wave profiles η **is defined with respect to the SWL, thus:** $\underline{\overline{\eta} = a_0 = 0}$

Example:

(you may also use the MULTICOMPONENT WAVE APPLET)

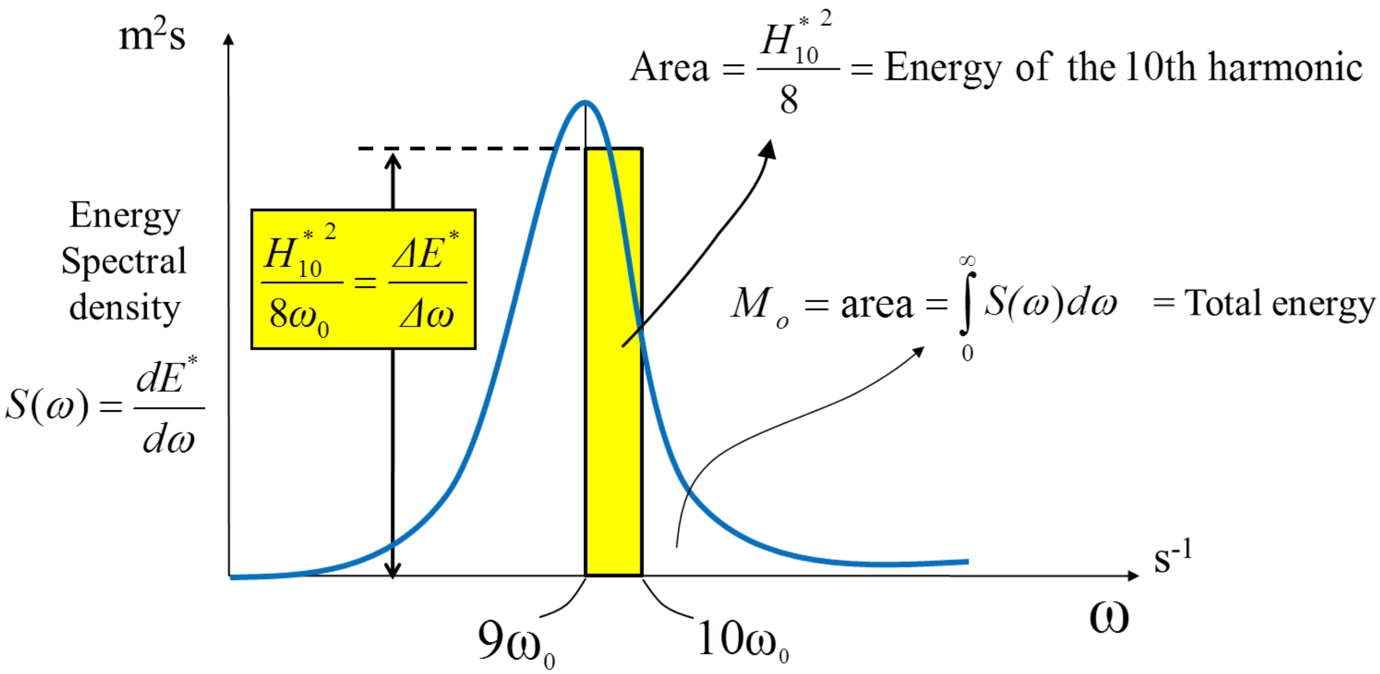
$$\eta(t) = \frac{H_1^*}{2}\cos(\omega_1 t - \vartheta_1) + \frac{H_2^*}{2}\cos(\omega_2 t - \vartheta_2) + \frac{H_4^*}{2}\cos(\omega_4 t - \vartheta_4) \quad \textbf{(8)}$$

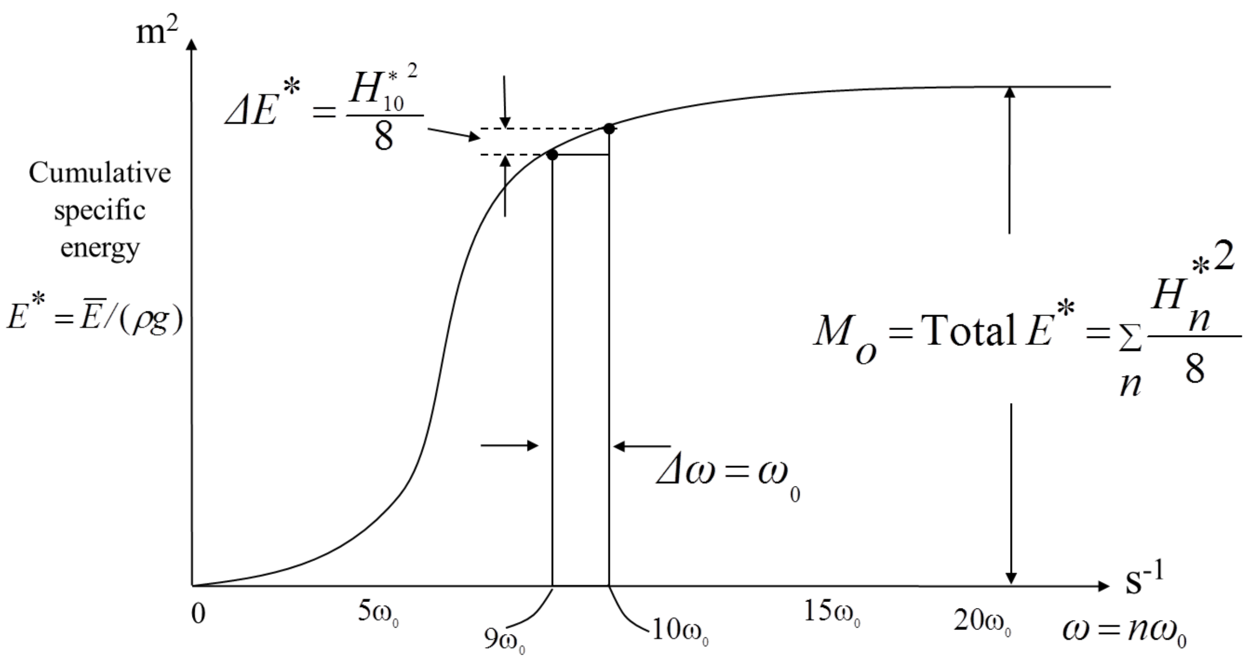


It can be shown that the total energy under a given wave record is equal to the sum of the energies of its components (based on Parseval's theorem)

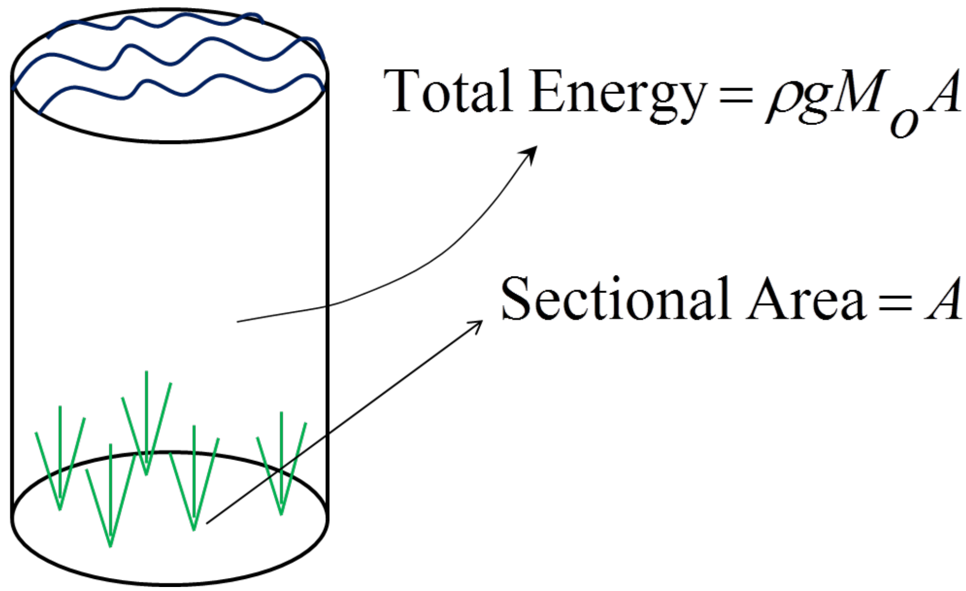
Remember: Specific Energy for each harmonic: $\overline{E} = \rho g \dfrac{(H^*)^2}{8}$ **or after dividing by** ρg

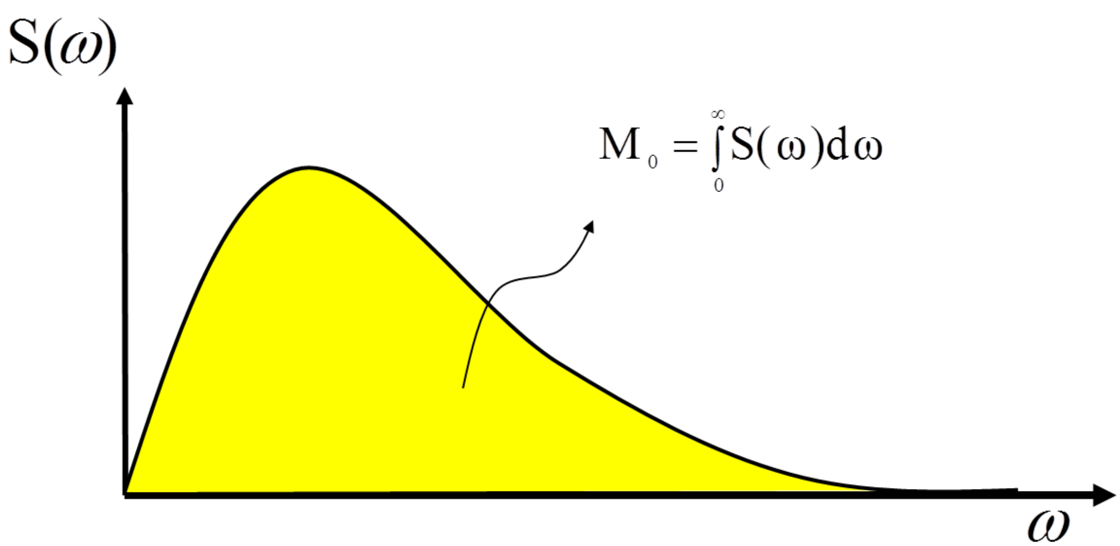
we define: $E^* = \dfrac{(H^*)^2}{8}$





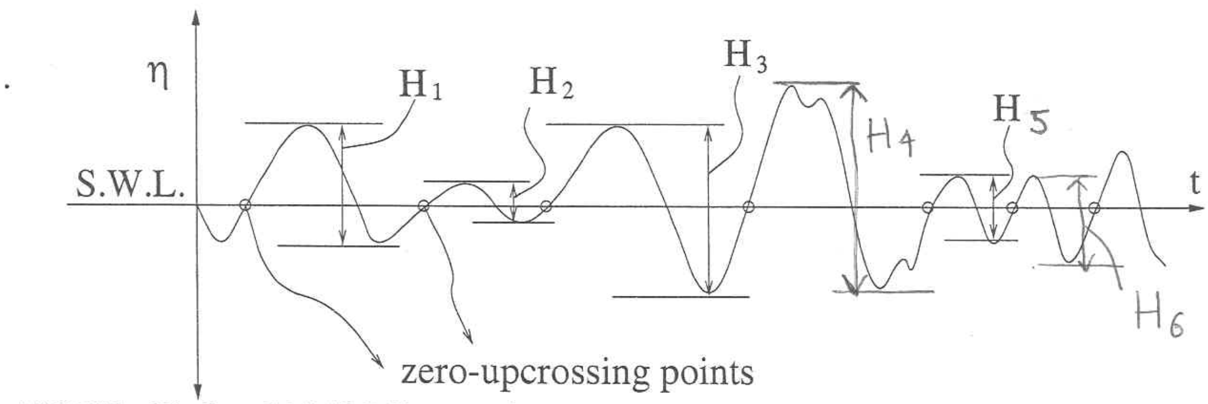
Total *wave energy* in a vertical column of water, under random waves, with sectional area *A*





B: Via Probabilistic Analysis of Wave Height Observations:

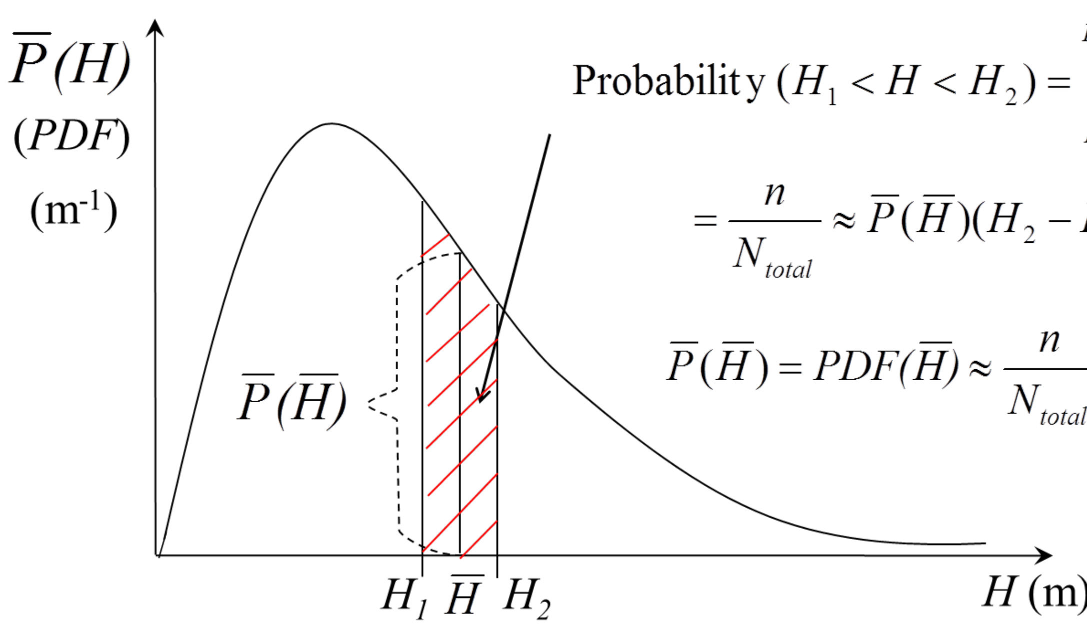
Wave Height Distribution:
Definition of Wave Height *Observation* (H)
(zero-up-crossing method of Pierson, 1954)



NOTE: H should NOT be confused with H^*. Use the WAVE SPECTRUM APPLET to understand the differences.

$n =$ number of wave heights between H_1 and H_2

$N_{total} =$ total number of wave heights

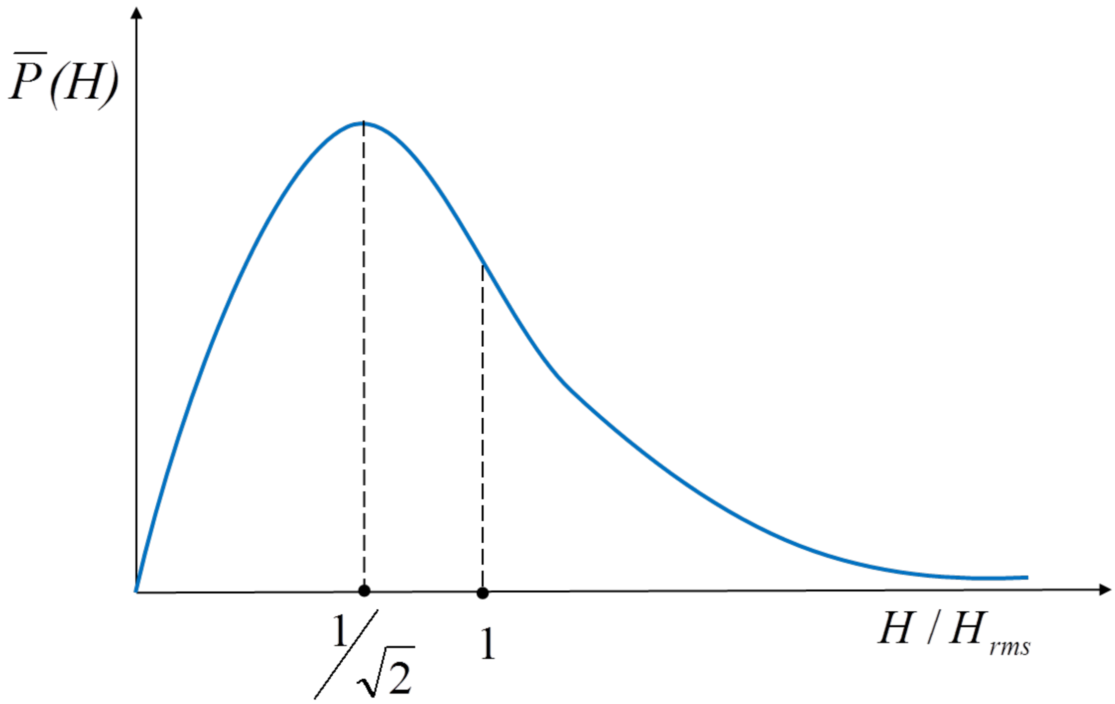


$$\text{Probability}\,(H_1 < H < H_2) = \int_{H_1}^{H_2} \overline{P}(H)\,dH =$$

$$= \frac{n}{N_{total}} \approx \overline{P}(\overline{H})(H_2 - H_1) \Rightarrow$$

$$\overline{P}(\overline{H}) = PDF(\overline{H}) \approx \frac{n}{N_{total}} \frac{1}{(H_2 - H_1)}$$

Rayleigh Probability Density Function (PDF) $\overline{P}$ (H) (Longuet-Higgins, 1952)

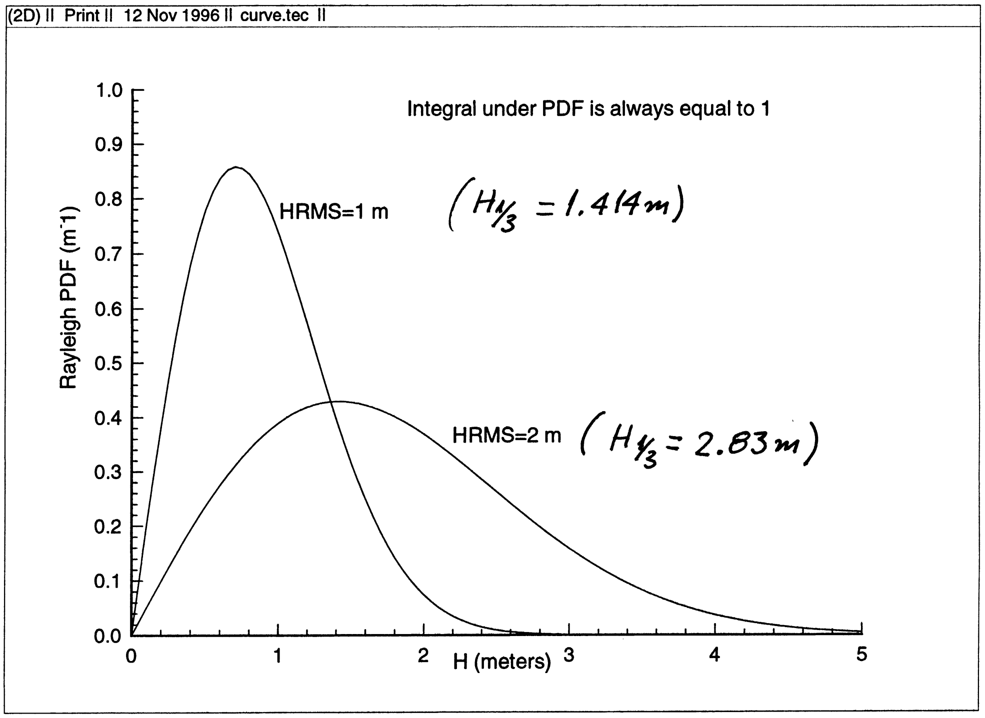


$$\overline{P}(H) = \frac{2H}{(H_{rms})^2} \cdot e^{-\left(\frac{H}{H_{rms}}\right)^2}$$

Root Mean Square (RMS) of observed wave heights H_{rms}:

$$H_{rms} = \sqrt{\frac{\sum_{n=1}^{N} H_n^2}{N}}$$

Rayleigh PDFs for different H_{rms}:



Cumulative Distribution Function: *P(H)* **=probability that** $H' \leq H$

$$\text{Probability } (0<H'<H) = P(H) = \int_0^H \overline{P}(H')dH' = 1 - e^{-\left(\frac{H}{H_{rms}}\right)^2} \quad \textbf{(9)}$$

Define: $P^*(H) =$ **(probability that** $H' \geq H) = 1 - P(H) \Rightarrow$

$$P^*(H) = \text{Probability } (H' \geq H) = exc(\text{eedence})(H) = e^{-\left(\frac{H}{H_{rms}}\right)^2} \quad \textbf{(10)}$$

Note that: $\text{Probability } (H_1 \leq H' \leq H_2) = e^{-\left(\frac{H_1}{H_{rms}}\right)^2} - e^{-\left(\frac{H_2}{H_{rms}}\right)^2}$

<u>Example</u>: If we wish to design a structure for a design height H_d, for which:

Prob ($H>H_d$)=*exc*(H_d)=1%=0.01 then: $e^{-\left(\frac{H_d}{H_{rms}}\right)^2} = 0.01 \Rightarrow H_d = H_{rms}\sqrt{-\ln(0.01)} = 2.146 H_{rms}$
This structure will exceed the design loading 0.01*[24*365*3600 = # of secs in 1 year]/10=31,536 times (assuming an average period of 10 secs), and this information can be used to enter into a fatigue diagram to assess failure in fatigue.

Example

The following wave heights were recorded in a fixed location in the ocean over a 12 hour period:

Range of H (m)	0-1.5	1.5-3.0	3.0-4.5	4.5-6.0	6.0-7.5	7.5-10.5
Number of wave height observations (n)	4000	6000	2000	500	300	60

Determine the PDF for the above waves. Find the H_{rms} and compare the PDF with the Rayleigh PDF.

We need to plot $\overline{P}(\overline{H})$ that corresponds to our samples. To do that we take as:

$$\overline{H} = \frac{H_1 + H_2}{2} \tag{11}$$

where H_1 and H_2 are the boundaries of each interval

Then: $$\overline{P}(\overline{H}) = \frac{n}{N_{total}} \frac{1}{H_2 - H_1} \tag{12}$$

Because we want: $\overline{P}(\overline{H})\Delta H = \dfrac{n}{N_{total}}$

where n=number of heights such that: $H_1 < H < H_2$ **and N=total number of heights**

In our case:

$$N_{total} = 4{,}000 + 6{,}000 + 2{,}000 + 500 + 300 + 60 = 12{,}860$$

$H_1 \div H_2$	n/N_{total}	$\Delta H = H_2 - H_1$	$\overline{H} = \frac{(H_1 + H_2)}{2}$	$\overline{P}(\overline{H})$ **actual**	$\overline{P}(\overline{H})$ **Rayleigh**
0÷1.5	**0.311**	**1.5**	**0.75**	**0.207**	**0.192**
1.5÷3.0	**0.466**	**1.5**	**2.25**	**0.311**	**0.309**
3.0÷4.5	**0.155**	**1.5**	**3.75**	**0.104**	**0.148**
4.5÷6.0	**0.039**	**1.5**	**5.25**	**0.026**	**0.032**
6.0÷7.5	**0.023**	**1.5**	**6.75**	**0.015**	**0.0034**
7.5÷10.5	**0.005**	**3.0**	**9**	**0.0017**	**3.4×10^{-5}**

$$H_{rms}^2 = \sum \frac{n}{N_{total}} (H')^2$$
$$= 0.311\times(0.75)^2 + 0.466\times(2.25)^2 + 0.155\times(3.75)^2 + 0.039\times(5.25)^2 + 0.023\times(6.75)^2 + 0.005\times(9)^2 = 7.24m^2 \rightarrow \quad H_{rms} = 2.69m$$

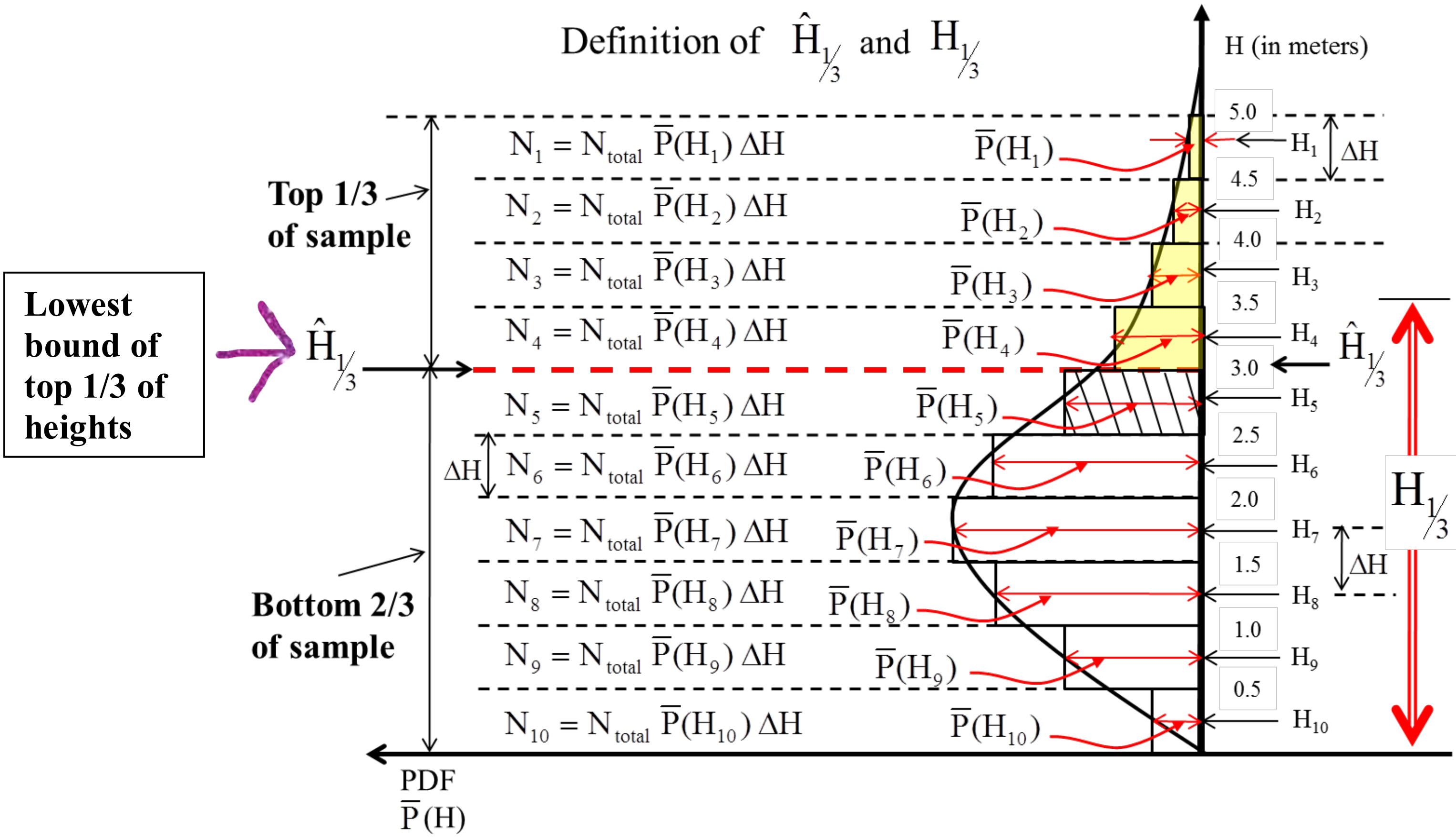
The Rayleigh PDF is then evaluated, using the following formula, with H_{rms}=2.69m and the values are shown on the Table above.

$$\overline{P}(\overline{H}) = \frac{2\overline{H}}{(H_{rms})^2} e^{-\left(\frac{\overline{H}}{H_{rms}}\right)^2} \qquad \textbf{(13)}$$

The Significant Height Hs

<u>Traditional Definition :</u> Average of the top 1/3 of wave heights

$$H_s = H_{1/3} = H_{33}$$



H1/3=[sum of top 1/3 heights]/
[# of top 1/3 heights]

$$H_{1/3} = \frac{N_1 H_1 + N_2 H_2 + N_3 H_3 + N_4 H_4}{N_1 + N_2 + N_3 + N_4}$$

$$H_{1/3} = \frac{[N_{total}\ \bar{P}(H_1)\ \Delta H]\, H_1 + [N_{total}\ \bar{P}(H_2)\ \Delta H]\, H_2 + [N_{total}\ \bar{P}(H_3)\ \Delta H]\, H_3 + [N_{total}\ \bar{P}(H_4)\ \Delta H]\, H_4}{[N_{total}\ \bar{P}(H_1)\ \Delta H] + [N_{total}\ \bar{P}(H_2)\ \Delta H] + [N_{total}\ \bar{P}(H_3)\ \Delta H] + [N_{total}\ \bar{P}(H_4)\ \Delta H]}$$

$$H_{1/3} = \frac{[\bar{P}(H_1)\ \Delta H]\, H_1 + [\bar{P}(H_2)\ \Delta H]\, H_2 + [\bar{P}(H_3)\ \Delta H]\, H_3 + [\bar{P}(H_4)\ \Delta H]\, H_4}{[\bar{P}(H_1)\ \Delta H] + [\bar{P}(H_2)\ \Delta H] + [\bar{P}(H_3)\ \Delta H] + [\bar{P}(H_4)\ \Delta H]}$$

As $\Delta H \to 0$, $H_{1/3}$ becomes (note the denominator goes to 1/3, due to the definition of $\hat{H}_{1/3}$):

$$H_{1/3} = \frac{\int_{\hat{H}_{1/3}}^{\infty} \overline{P}(H) H dH}{\int_{\hat{H}_{1/3}}^{\infty} \overline{P}(H) dH} = \text{centroid of 1/3 top area of PDF}$$

and for Rayleigh PDF:

$$H_s = H_{1/3} = 1.416 H_{rms} = \sqrt{2} H_{rms}$$

<u>Example:</u>

a) Determine, in the case of Raleigh PDF, an expression for *exc(H)* in terms of H_s

$$\text{Probability } (H' \geq H) = exc(H) = e^{-(H/H_{rms})^2} = e^{-2(H/H_s)^2}$$

Given that: $H_s = \sqrt{2} H_{rms}$

Note the above equation can be generalized as follows:

$$\text{exc}(H) = e^{-a(H/Hs)^b}$$

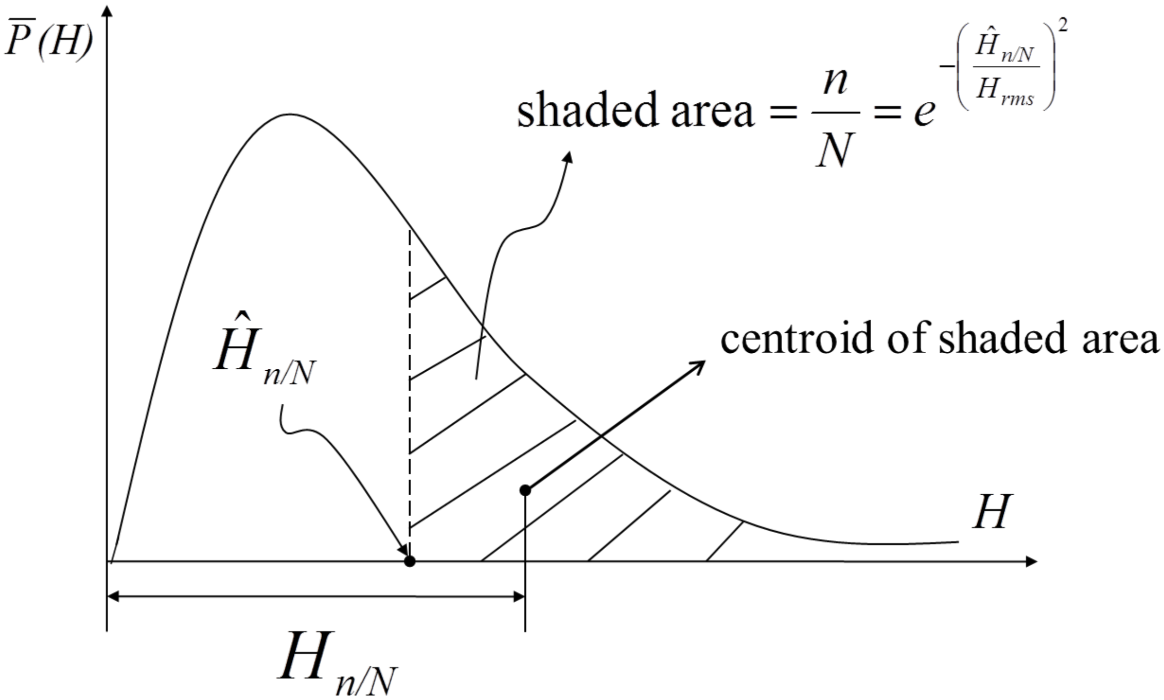
Where a and b are appropriate parameters to match measurements.

a=2, b=2 for Rayleigh; a=2.28 & b=2.13 (Krogstad); a=2.26 & b=2.126 (Forristall)

b) Determine *exc(H_s)* for Rayleigh PDF:

$$\text{Probability } (H' \geq H_s) = exc(H_s) = e^{-2} = 0.135 = 13.5\%$$
$$\text{or, based on Forristall}: exc(H_s) = e^{-2.26} = 0.104 = 10.4\%$$

Generalization of $H_{1/3}$



$H_{n/N}$ **= average of the highest n heights out of the total N heights.**

$$H_{n/N} = \frac{\int_{\hat{H}_{n/N}}^{\infty} \overline{P}(H')H'dH'}{\int_{\hat{H}_{n/N}}^{\infty} \overline{P}(H')dH'} \tag{14}$$

where: $$\frac{n}{N} = e^{-\left(\frac{\hat{H}_{n/N}}{H_{rms}}\right)^2} \Rightarrow \frac{\hat{H}_{n/N}}{H_{rms}} = \left[-\ln\left(\frac{n}{N}\right)\right]^{\frac{1}{2}} \tag{15}$$

$\hat{H}_{n/N}$ and $H_{n/N}$ are given in Fig. 3-5 of SPM as a function of n/N (assuming Rayleigh PDF)

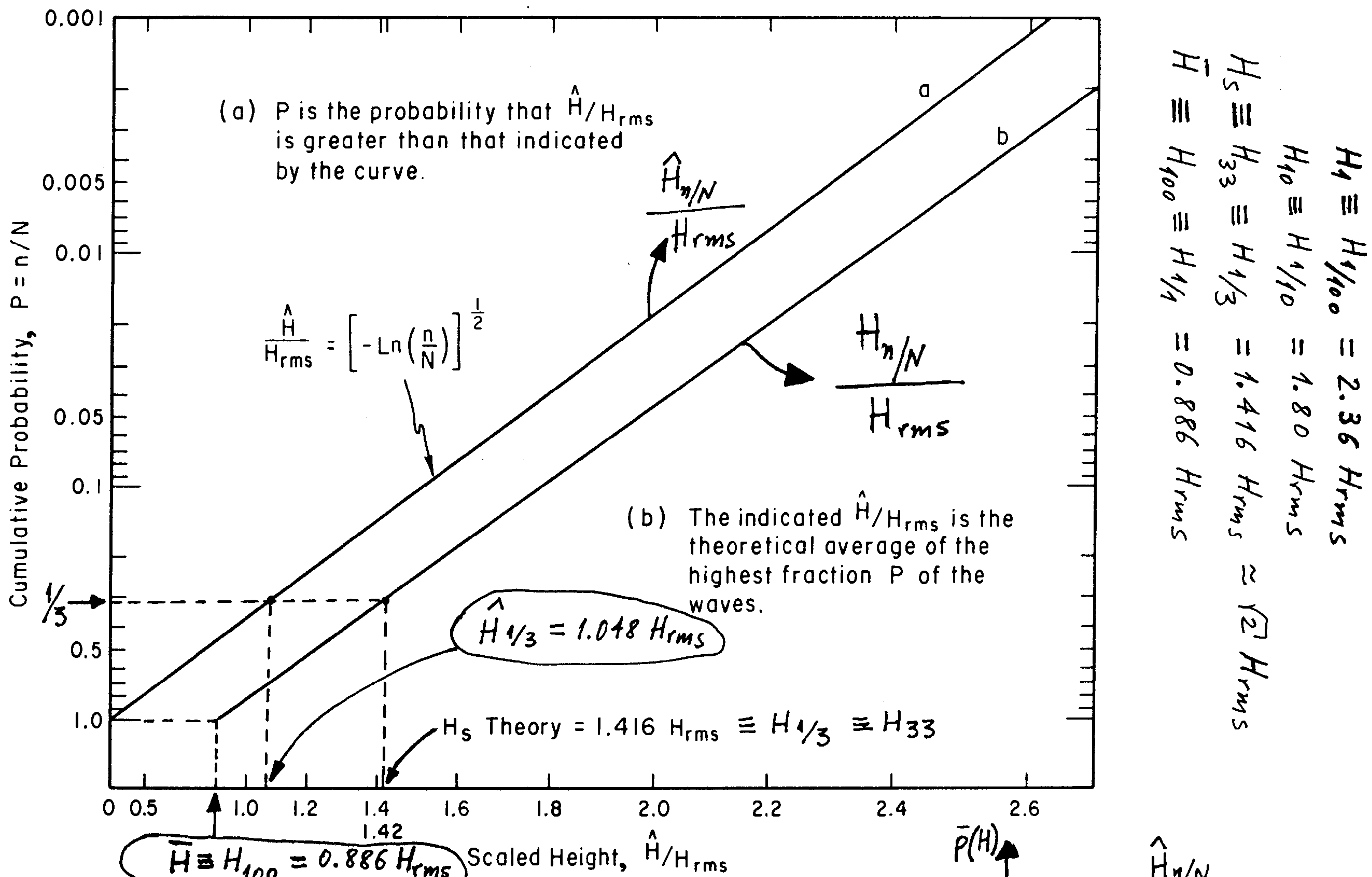
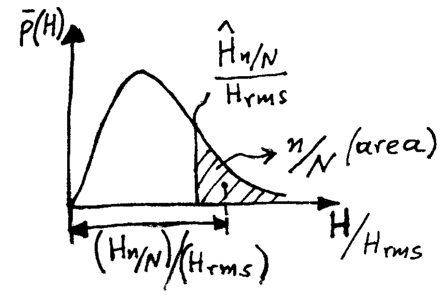


Figure 3-5. Theoretical wave height distributions.



How well the Rayleigh PDF agrees with collected data? (Fig. 3-4 from SPM)

(the *exc(H)* is shown on the vertical axis)

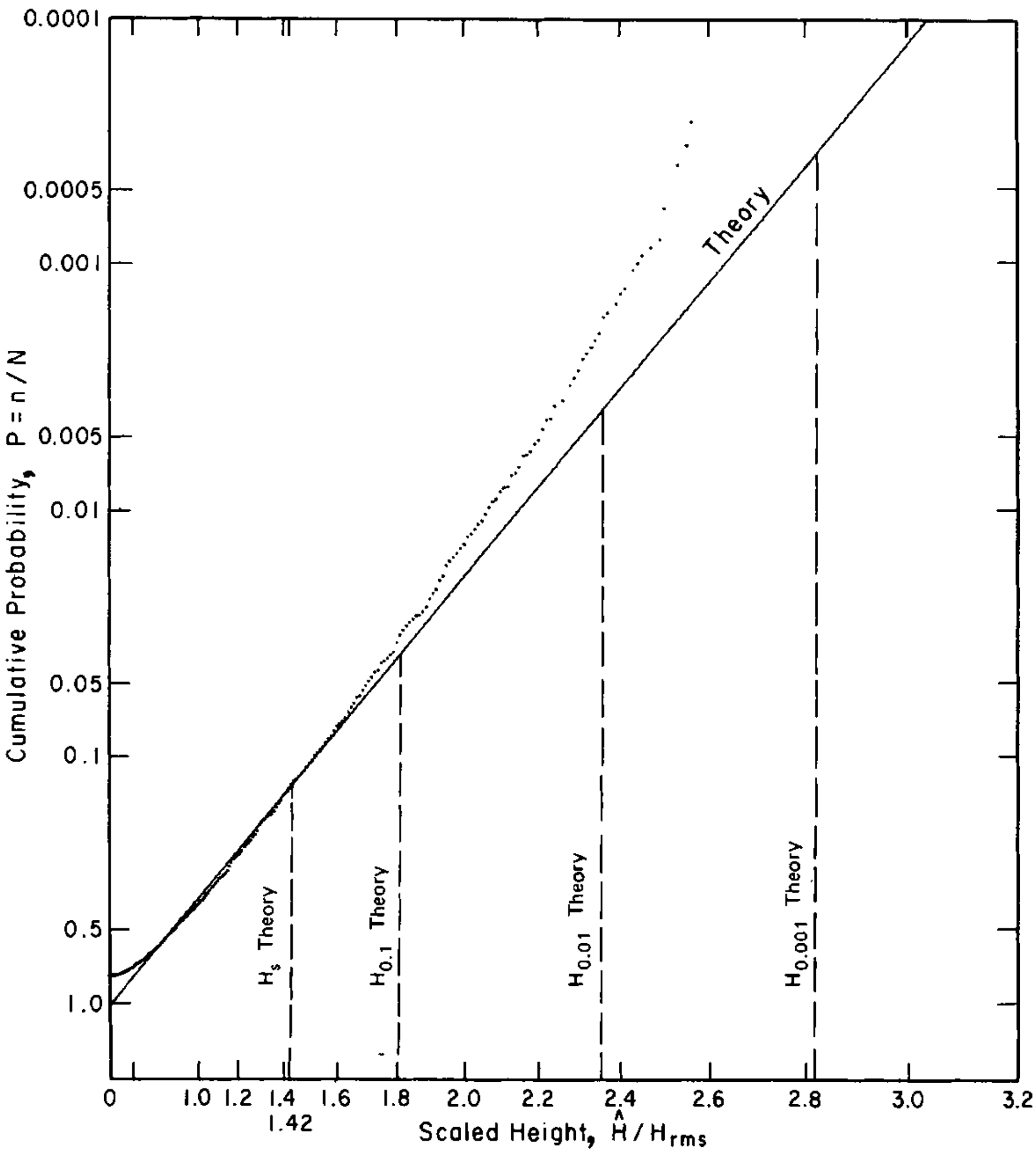


Figure 3-4. Theoretical and observed wave height distributions. (Observed waves from 72 individual 15-minute observations from several Atlantic coast wave gages are superimposed on the Rayleigh distribution curve.)

Relationship between H_s with M_0

$$M_0 = \int_0^\infty S(\omega)d\omega = \frac{Total\ \ Energy}{\rho g \times Area} \quad \textbf{(16)}$$

$$However: \quad M_0 = \frac{1}{N}\sum_{n=1}^{N}\frac{H_n^2}{8} \quad \textbf{(17)}$$

$$\equiv \text{Average}\left[\text{energy}/(\rho g \times Area)\right] \text{over all consecutive heights}$$

From definition of H_{rms} :

$$H_{rms} \equiv \sqrt{\frac{1}{N}\sum_{n=1}^{N} H_n^2} \quad \textbf{(18)}$$

$$\Rightarrow M_0 = \frac{H_{rms}^2}{8} \rightarrow H_{rms} = 2\sqrt{2}\sqrt{M_0} \quad \textbf{(19)}$$

(remember: $H_s = \sqrt{2}H_{rms}$ for Raleigh PDF)

$$\boxed{H_s = \sqrt{2}H_{rms} = 4\sqrt{M_0} = 4\sqrt{\int_0^\infty S(\omega)d\omega}} \quad \textbf{(20)}$$

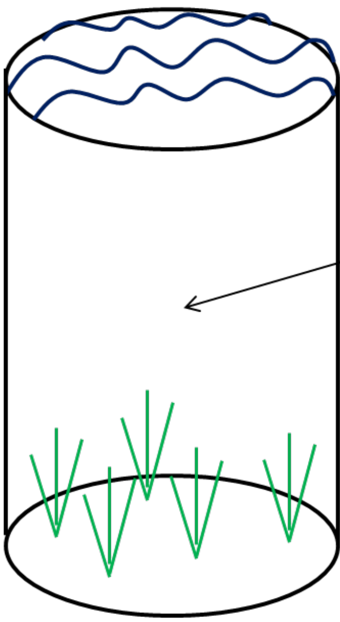
<u>The above definition of H_s is more general and does NOT depend on the probability density function</u> that the wave height observations might follow.

However, <u>in the event the Rayleigh PDF applies, then the traditional definition of $H_s = H_{1/3}$,</u> and the one above will define the same significant height.

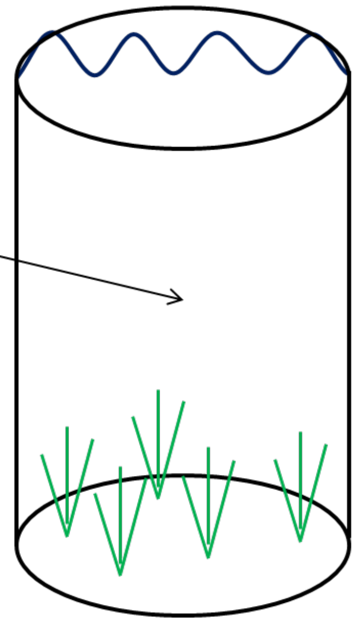
The above is also shown in the next page:

General definition of significant H_s

Random seas

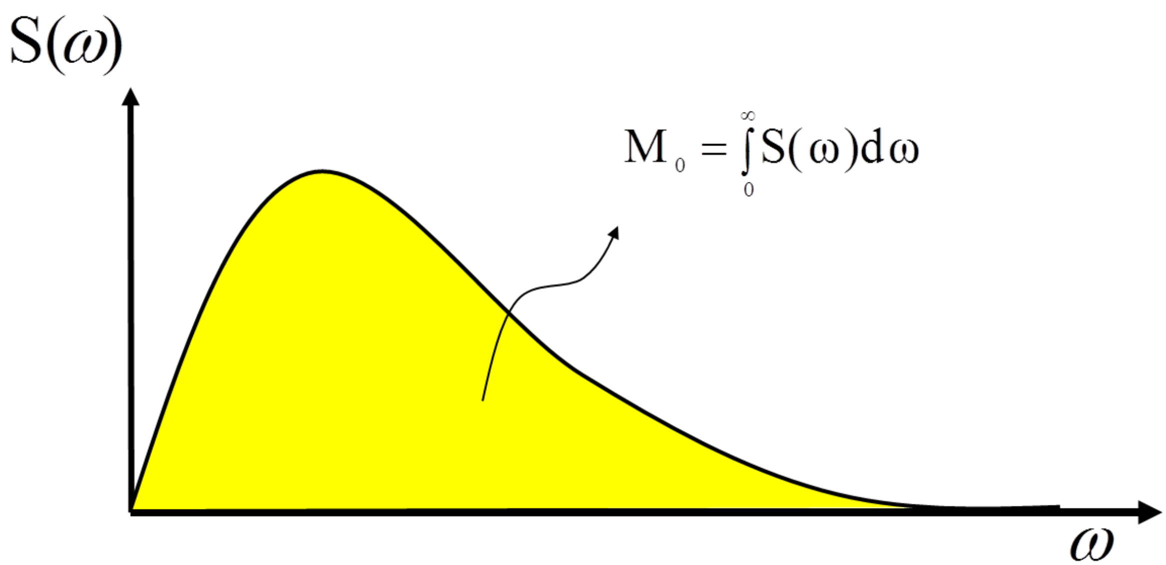


Single wave $H=H_{rms}$

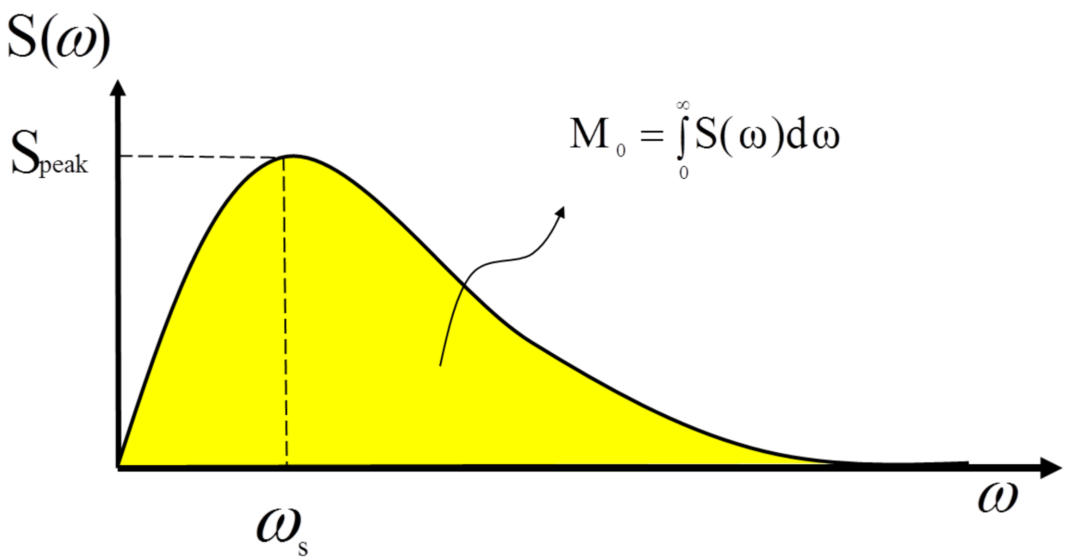


$$\boxed{\text{Same Total Energy} = \rho g M_o A = \rho g \frac{H_{rms}^2}{8} A}$$

$$\boxed{H_s = \sqrt{2} H_{rms} = 4\sqrt{M_o}}$$



The Significant Frequency and Period:

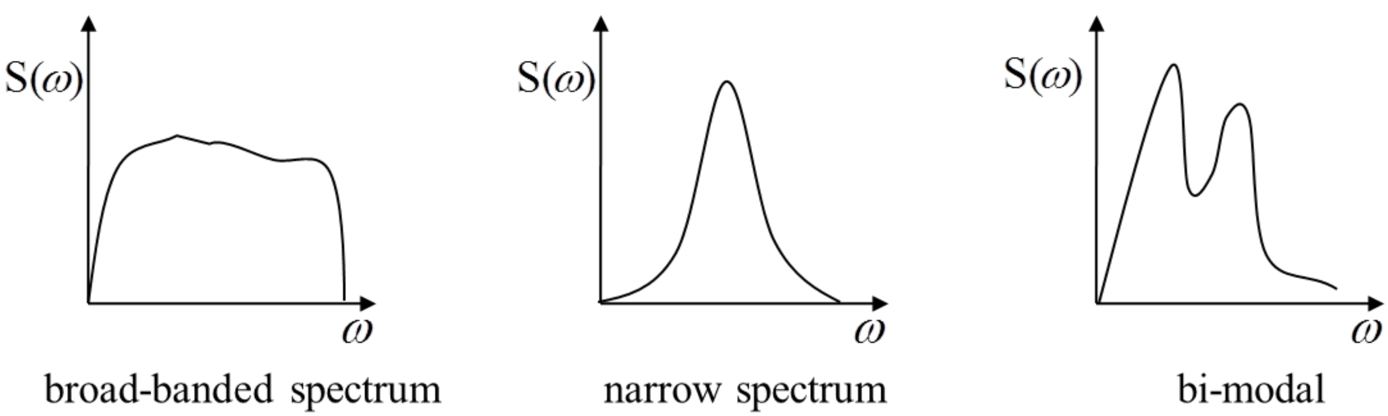


$T_s =$ **significant period (also called peak period T_P);** $T_s = T_P = \dfrac{2\pi}{\omega_s}$

ω_s **= significant (or dominant) frequency (also called peak frequency ω_P)**
(corresponds to angular frequency of the wave harmonic with the largest amplitude)

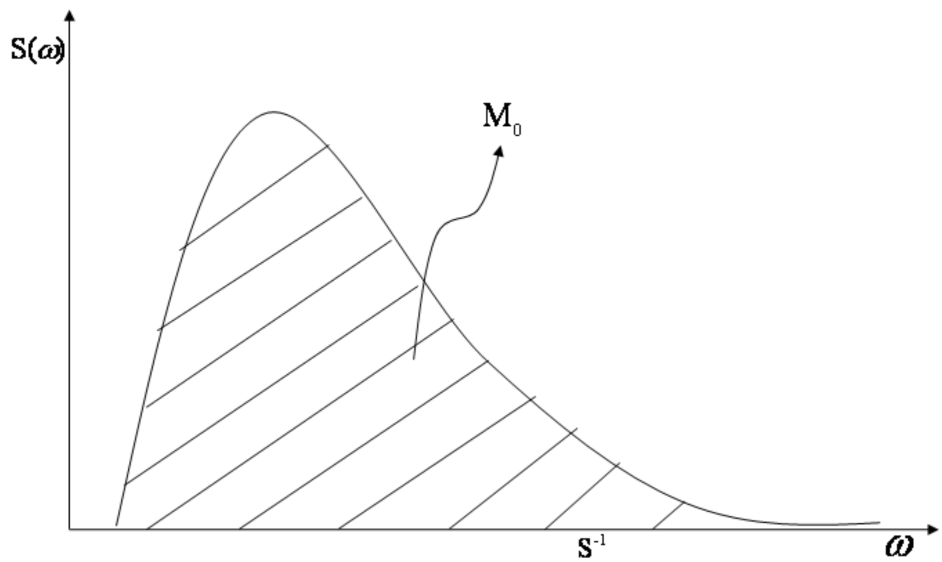
H_s **= significant height =** $4\sqrt{M_0}$

Typical Wave Spectra:



NOTE: Some wave spectra use f=ω/2π instead of ω. In that case: S(f)=2πS(ω), so that the area under S(f) represents the same amount of energy.

Pierson-Moskowitz Spectrum:
(For Fully Developed Seas – FDS Equilibrium Spectrum)
(see also p. 3-42 of SPM)



$$S(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\beta\left(g / U\omega\right)^4\right] \qquad \textbf{(21)}$$

$\alpha = 8.1 \times 10^{-3}$

$\beta = 0.74$

$g = 9.81 m/s^2$

$U =$ **wind speed in** m/s

$\omega =$ **frequency in** s^{-1}

It can be shown that:

$$M_0 = \int_0^\infty S(\omega) d\omega = \frac{\alpha U^4}{4\beta g^2} (m^2) \qquad \textbf{(22)}$$

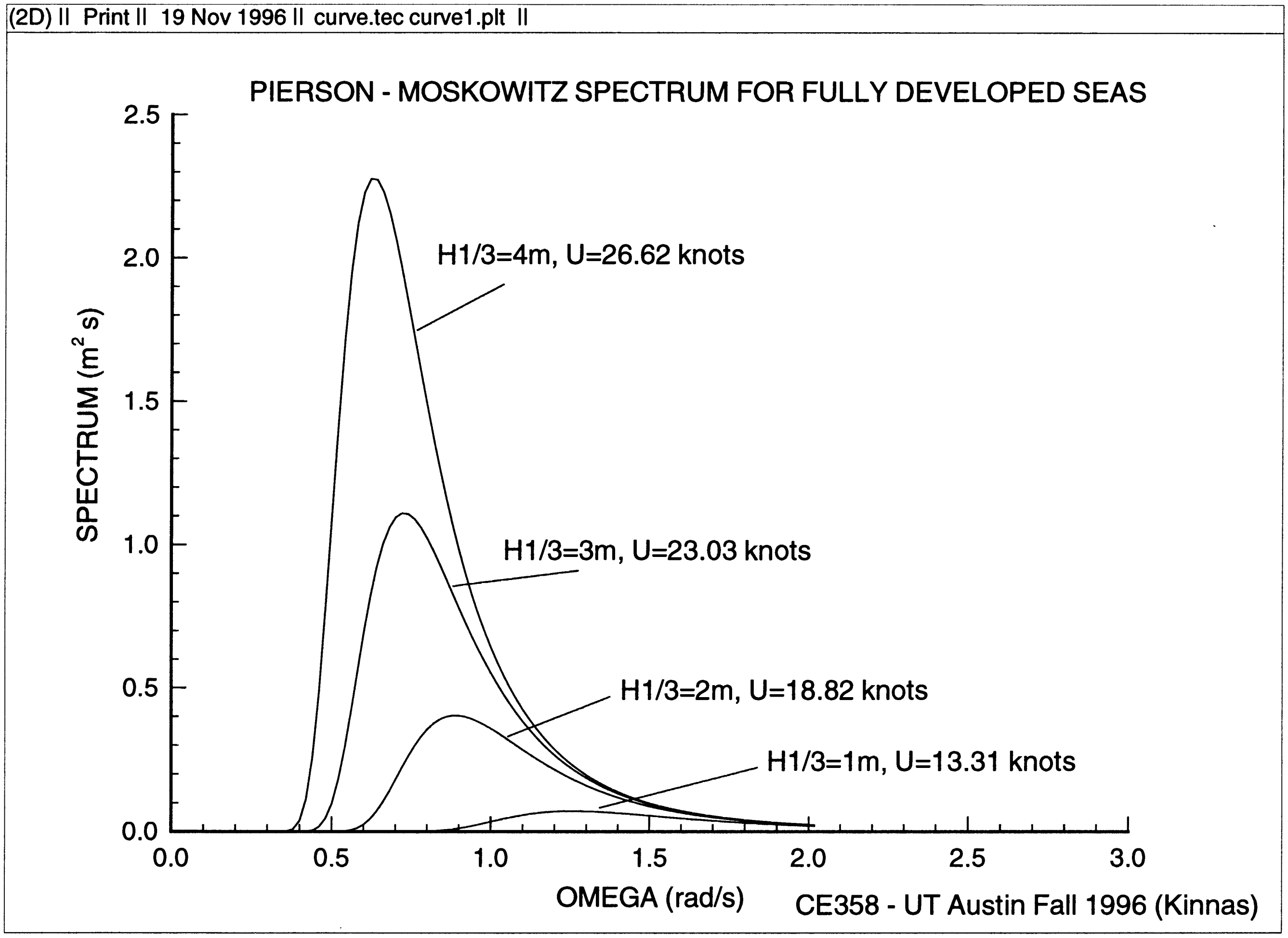
$$\frac{gH_s}{U^2} = 0.209 \qquad \text{Note: } H_s \sim U^2 \qquad \textbf{(23)}$$

$$\frac{gT_s}{U} = 7.16 \qquad \text{Note: } T_s \sim U \qquad \textbf{(24)}$$

where H_s and T_s are the significant height and period, respectively.

FDS Wave Spectra for various wind speeds

Note Area under Spectrum ~ H_s^2 ~ U^4

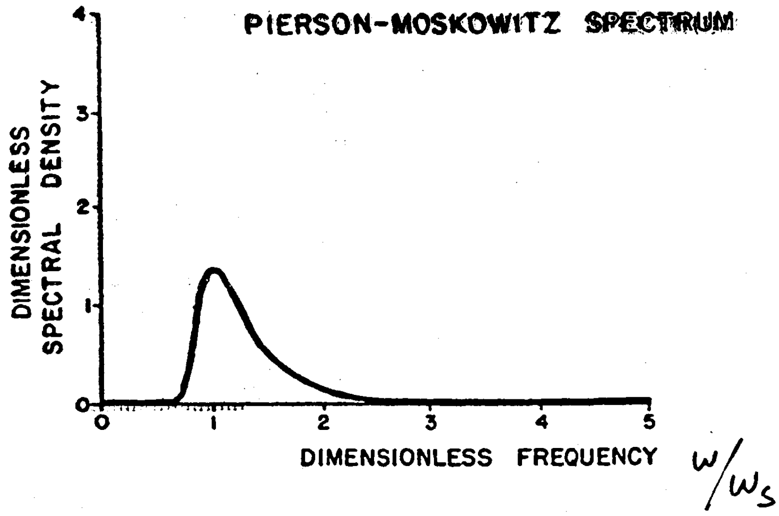


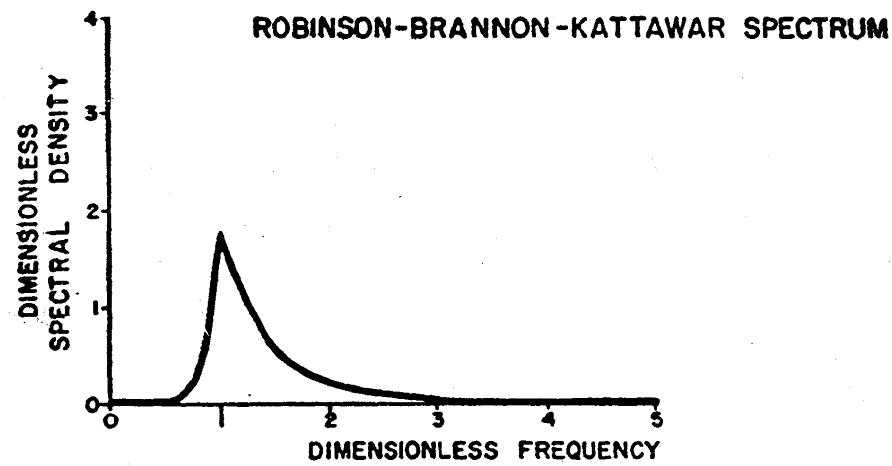
1 knot=1 nautical mile/hour = 1.151 mph = 0.514 m/s
1 nautical mile = 1,852 m
1 (statute) mile = 1,609.35 m

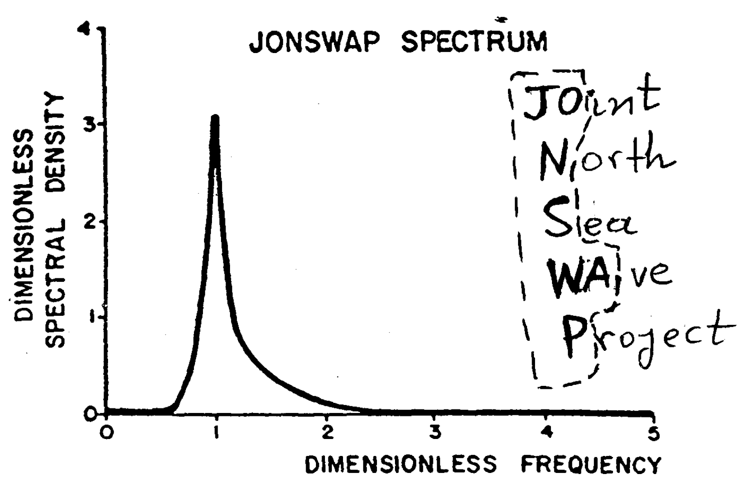
Q: Can you explain why the peak of the wave spectrum moves to the "left" as the significant height (or the wind speed) increases?

Some other common wave spectra:

(made non-dimensional so that area under wave spectrum is equal to 1)







As we will see in the lecture on Wave Forces, an offshore structure must be analyzed subject to an incoming mono-chromatic wave of height H_{max} and period T_m corresponding to the 100-year storm. We assume that this is the worst-case scenario.

Big Question:

How can I determine H_{max} (height of 100-year storm) and T_m (period of the 100-year storm) for a certain location in the ocean?

Answer:

If the location has been already known and developed (e.g. GOM, those parameters are given in API, as we will see on the section on Wave Forces). IMPORTANT NOTE: These parameters can be revised, based on information from recent STRONGER storms (e.g. *Ivan, Rita, Katrina, etc.*)

If it is a new location, then measurements, hindcasting can provide us with estimates for H_s (significant height) and T_p (peak period) over many past years. These collected data from the past years, theory (usually adjusted empirically), regression analysis, and extrapolations, can provide us estimates for H_{max} and T_m, as described in the lectures on Design Parameter Specifications in this course.

Problem 5 from the next list of problems and solutions presents an attempt to relate the maximum wave height to the significant wave height.

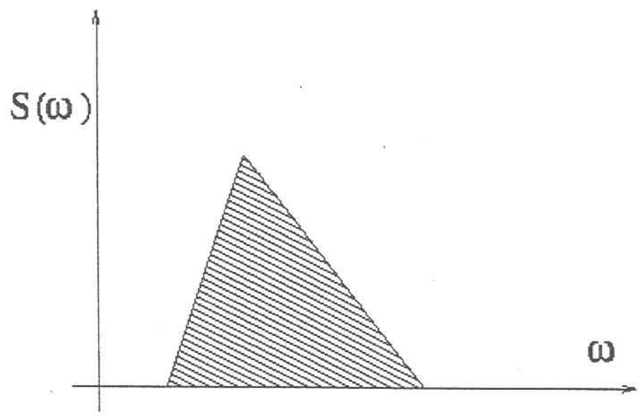
Some Example Problems on Random Waves and Their Solutions:

3. A wave spectrum in *deep* water may be approximated by the shown triangle. Wave lengths larger than 80 m or less than 8 m are known to contribute *negligible* amounts of energy. It is known that the average value of the highest 4,000 out of 20,000 measured wave heights is equal to 1.3 m. (40 points)

   a) Determine the significant wave height and the peak value of the spectral energy density

   b) It is also known that the significant period of the wave spectrum is such that monochromatic waves with this period and wave height equal to *twice* the significant wave height, they would be just about to break (in deep water). Determine the significant period of the waves

   c) Determine the percentage of the total energy contained in all waves with periods larger than the significant period



4. A sample of 15,000 consecutive wave height observations at a certain location in the ocean is considered. The average of the 1,000 highest of these waves has been found to be equal to 2.8 m. Determine the following: (25 points)

   a) The significant wave height

   b) The average of *all* the wave heights

   c) The *lowest* bound of the 20% highest waves

5. Prove that the maximum wave height, H_{max}, out of a sample on N wave height observations can be estimated by the following formula (H_s is the significant height of the sample): (20 points)

$$H_{max} = H_s(0.5 \ln N)^{0.5} \qquad (2)$$

…and their solutions:

Problem 3

$$L_{max} = 80\,m \quad , \quad L_{min.} = 8\,m$$

$$L = \frac{gT^2}{2\pi} \longrightarrow T = \sqrt{\frac{2\pi L}{g}} \longrightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{\sqrt{\frac{2\pi L}{g}}} = \sqrt{\frac{2\pi g}{L}}$$

$$\omega_1 = \omega_{min} = \sqrt{\frac{2\pi g}{80}} = 0.877\ sec^{-1}$$

$$\omega_2 = \omega_{max.} = \sqrt{\frac{2\pi g}{8}} = 2.77\ sec^{-1}$$

a) $H_{1/5} = 1.3\ m = 1.56\ H_{rms}$ (Fig 3-5, Theoretical wave height distributions

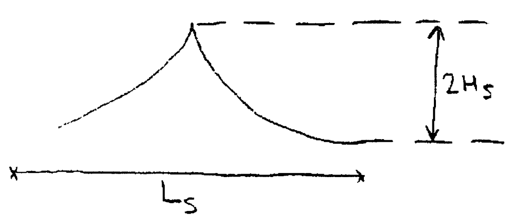
$$\Rightarrow H_{rms} = 0.833\ m$$

$$H_s = \sqrt{2}\ H_{rms} \Longrightarrow \boxed{H_s = 1.18\ m}$$

$$S_{max} \times \frac{1}{2} \times (2.77 - 0.877) = M_0 = \frac{H_s^2}{16}$$

$$\Longrightarrow \boxed{S_{max} = 0.0919\ m^2 s}$$

b) $L_s = \dfrac{gT_s^2}{2\pi} \qquad \dfrac{2H_s}{L_s} = \dfrac{1}{7} \rightarrow L_s = 14H_s = 16.52\,m$



$$\Longrightarrow \omega_s = \sqrt{\frac{2\pi g}{L_s}} = 1.93\ sec^{-1}$$

$$T_s = \frac{2\pi}{\omega_s} = \boxed{3.25\ sec}$$

c) $$X = \frac{\text{Area (ABD)}}{\text{Total (Area)}} = \frac{BD}{BC} = \frac{\omega_s - \omega_1}{\omega_2 - \omega_1} = \frac{1.93 - 0.877}{2.77 - 0.877} = 0.556$$

$$\Longrightarrow \boxed{55.6\ \%}$$

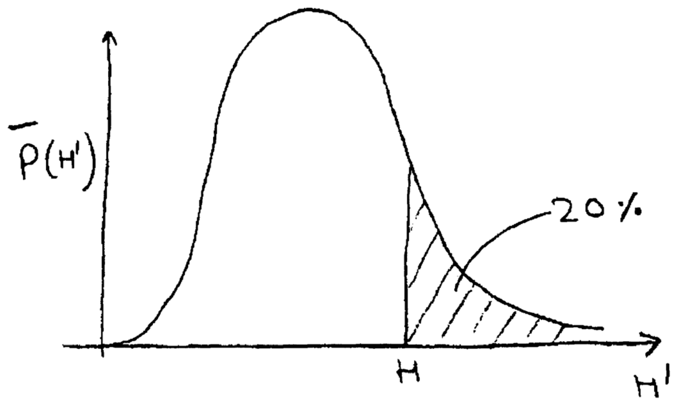
Problem 4

a) $H_{1/15} = 2.8\,m = 1.9\,H_{rms} \rightarrow H_{rms} = \frac{2.8}{1.9} = 1.473\,m$

$$H_s = \sqrt{2}\,H_{rms} = \boxed{2.08\,m}$$

b) $\bar{H} = 0.886\,H_{rms} = \boxed{1.846\,m}$

c)



$$P(H' > H) = e^{-(H/H_{rms})^2}$$
$$= 0.2$$
$$-\left(\frac{H}{H_{rms}}\right)^2 = \ln(0.2)$$

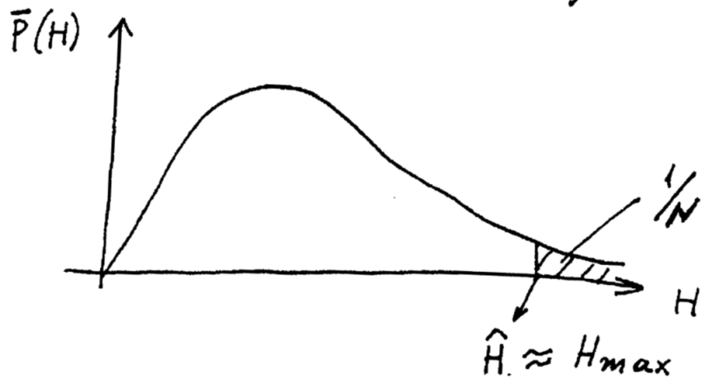
$$\frac{H}{H_{rms}} = \sqrt{-\ln(0.2)}$$

$$H = H_{rms}\sqrt{-\ln(0.2)} = \boxed{1.868\,m}$$

Solution for Prob. 5, HW#8 200

Proof of equation:

$$H_{max} = H_{1/3} \left(\frac{1}{2} \ln N \right)^{1/2}$$



$$\text{Probability}\left[H > \hat{H} \right] = \frac{1}{N} = e^{-(\hat{H}/H_{rms})^2} \Rightarrow$$

$$\ln\left(\frac{1}{N}\right) = -\left(\hat{H}/H_{rms}\right)^2 \Rightarrow \sqrt{-\ln\left(\frac{1}{N}\right)} = \hat{H}/H_{rms} \Rightarrow$$

$$\Rightarrow \hat{H} = H_{rms}\sqrt{\ln(N)} \quad \left(\text{remember } \ln\frac{1}{x} = -\ln x\right)$$

From formulas $H_{rms} = H_{1/3}/\sqrt{2} \Rightarrow$

$$\Rightarrow \hat{H} \approx H_{max} = H_{1/3}\left(\frac{1}{2}\ln N\right)^{1/2}$$

Note in the case of Forristall, it can be shown that: $H_{\max} = H_s \left[\frac{\ln N}{2.26} \right]^{0.47}$

For example if N=1,000, then *H_{max}=1.86H_s* from the former, and *H_{max}=1.69H_s* , from the latter formula.