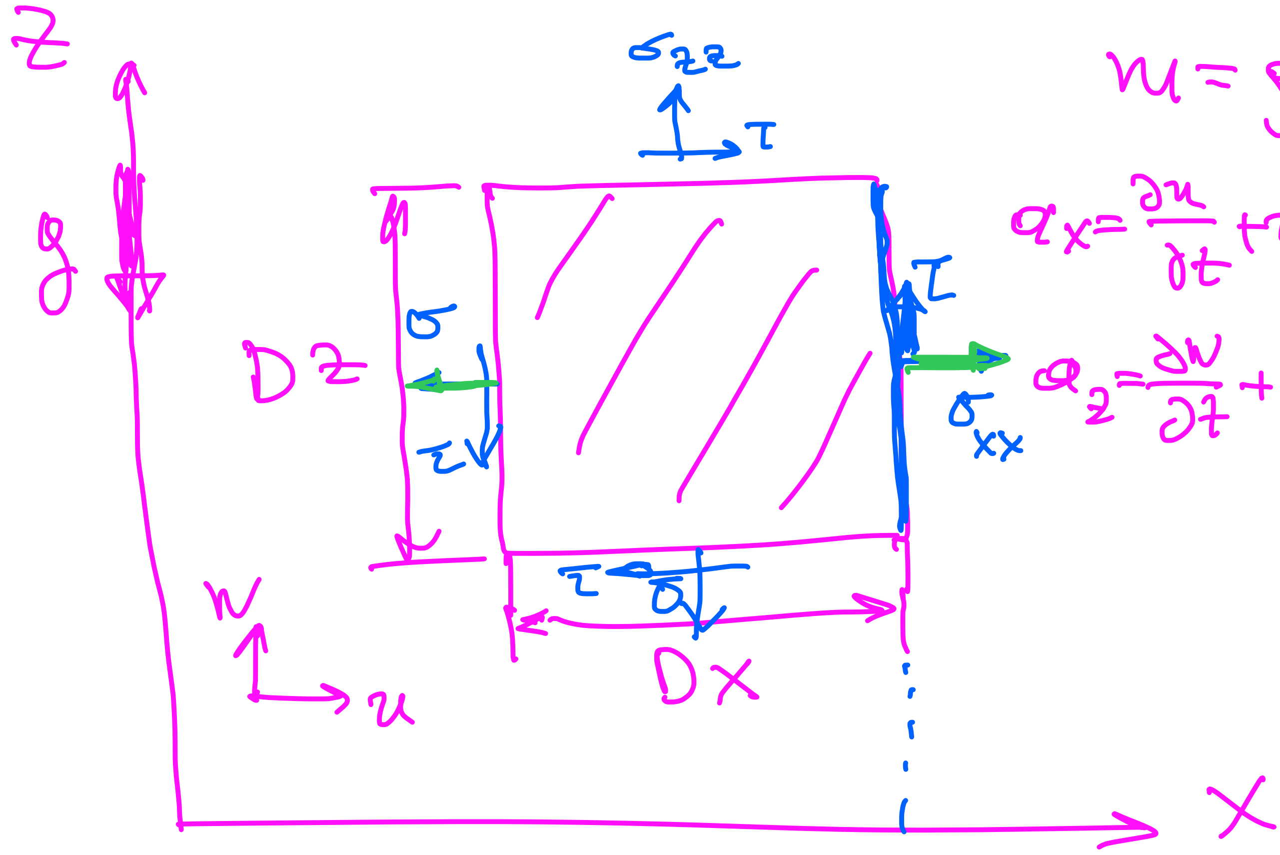
The Navier-Stokes equations

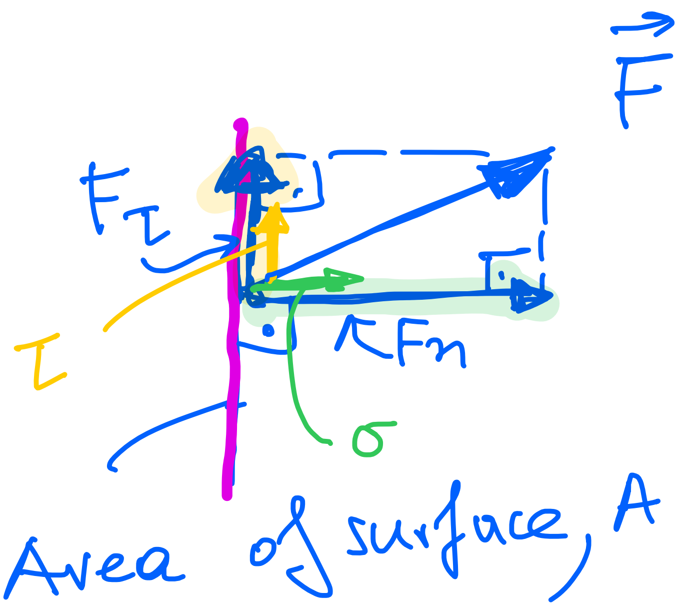
$$\sum F_x = m\, a_x$$
$$\sum F_z = m\, a_z$$



$$m = \rho\, Dx\, Dz$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}$$



Normal stress:

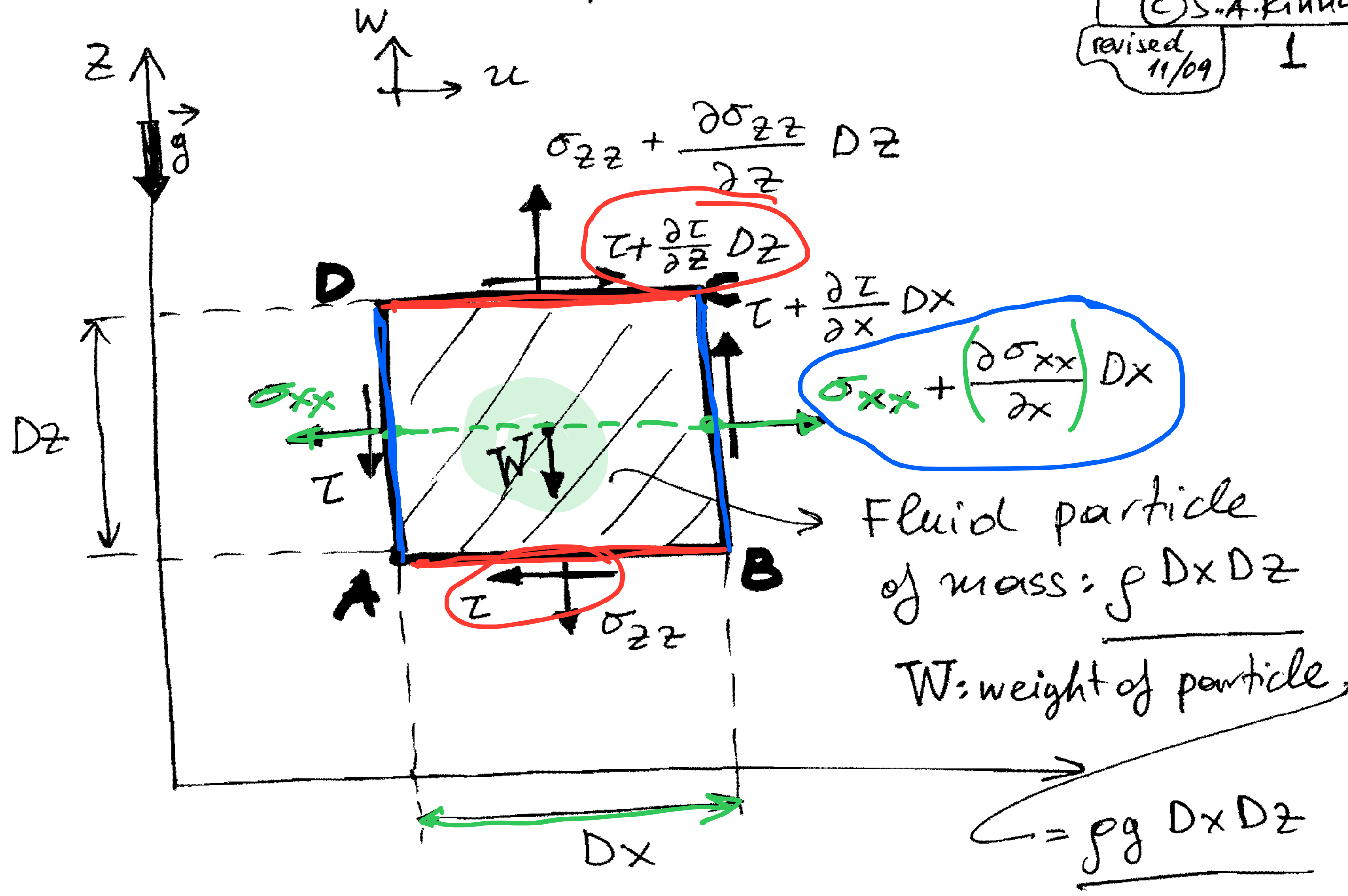
$$\sigma = \frac{F_n}{A}$$

$$\tau = \frac{F_\tau}{A}$$

Shear stress

$\vec{F}$: force acting from the fluid to the right on surface A. We decompose it into normal to the surface force F_n and tangential (shear) force F_τ.

Navier-Stokes equations in 2-D



$\tau = \sigma_{xz} = \sigma_{zx}$ (shear stress)

$$\sum F_x = (\rho\, Dx\, Dz) \cdot a_x$$

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \rightarrow \text{fluid particle acceleration along } x$$

$$\sum F_x = \left(\cancel{\sigma_{xx}} + \frac{\partial \sigma_{xx}}{\partial x} Dx\right) \cdot \underbrace{BC}_{Dz} - \left(\cancel{\sigma_{xx}}\right) \underbrace{AD}_{Dz}$$

$$+ \left(\cancel{\tau} + \frac{\partial \tau}{\partial z} Dz\right) \cdot \underbrace{DC}_{Dx} - \left(\cancel{\tau}\right) \underbrace{AB}_{Dx} =$$

$$= \frac{\partial \sigma_{xx}}{\partial x} Dx\, \cancel{Dz} + \frac{\partial \tau}{\partial z} Dz\, \cancel{Dx} = \rho\, \cancel{Dx\, Dz}\, a_x$$

$$\Rightarrow \boxed{\rho\, a_x = \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right] = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau}{\partial z}} \quad (1)$$

Similarly:

$$\sum F_z = (\rho \, Dx \, Dz) \, a_z$$

$$a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \rightarrow$$ fluid particle acceleration along z

$$\sum F_z = \left(\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial z} Dz\right) \underbrace{DC}_{Dx} - \sigma_{zz} \cdot \underbrace{AB}_{Dx} - W$$

$$+ \left(\tau + \frac{\partial \tau}{\partial x} Dx\right) \underbrace{BC}_{Dz} - \tau \cdot \underbrace{AD}_{Dz} =$$

$$= \frac{\partial \sigma_{zz}}{\partial z} \cancel{Dz \, Dx} + \frac{\partial \tau}{\partial x} \cancel{Dx \, Dz} - \rho g \cancel{Dx \, Dz} =$$

$$= \rho \cancel{Dx \, Dz} \, a_z$$

$$\Rightarrow \boxed{\rho a_z = \rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z}\right] = \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau}{\partial x} - \rho g} \quad (2)$$

Constitutive equations for Newtonian fluid:

$$\left.\begin{aligned} \sigma_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} \\ \sigma_{zz} &= -p + 2\mu \frac{\partial w}{\partial z} \\ \tau &= \mu \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \end{aligned}\right\} \quad (3)$$

p: fluid pressure

μ: dynamic (or absolute) viscosity

After the substitutions of equ (3) into eqs (1) & (2) and with some rearrangements and by using the continuity equ. $\rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0}$

we will end up with the Navier-Stokes equations!

$= 0$ in Euler equ.

Substituting σ_{xx}, σ_{zz}, and τ from ③ into ① and ② it can be shown that:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right] \quad (4)$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right] - \rho g$$

Navier-Stokes equations in 2-D

+

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

continuity equ.

The N-S equations can also be written in the following compact form:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u \quad (4a)$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu\nabla^2 w - \rho g$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ (Laplacian operator)

and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}$$ (substantial derivative operator)

To the above equations we need to add the continuity equation: $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (5)$$

Equations (4a) can be applied to laminar as well as turbulent flows.

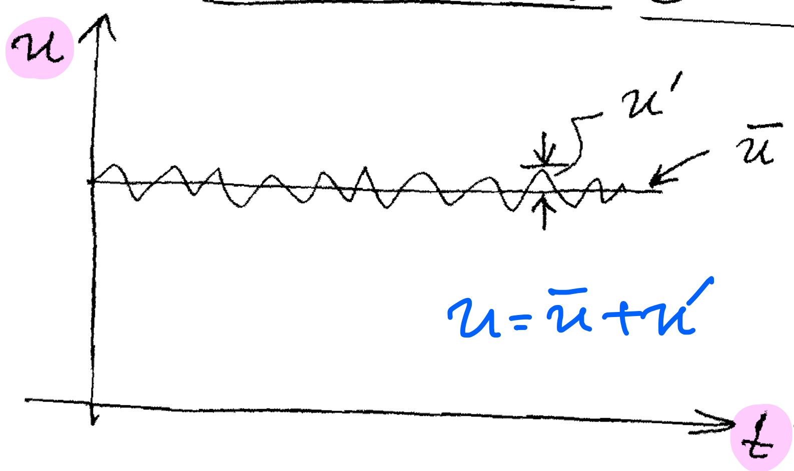
However in the case of turbulent flows the required spatial and temporal discretization must be very fine (very small $\Delta x, \Delta y, \Delta t$) and that requires powerful (parallel) computer with very large RAM and makes the computational method very slow (CPU Time ↑↑↑), even for simple 2-D geometries. We call methods that use equ. (4a) for modeling of turbulent flows: DNS which stands for Direct (Navier Stokes) Numerical Simulations

Turbulent flows can be dealt easier by averaging of the quantities of interest:

$$u = \bar{u} + u'$$
$$w = \bar{w} + w'$$
$$p = \bar{p} + p'$$
$$\ldots$$

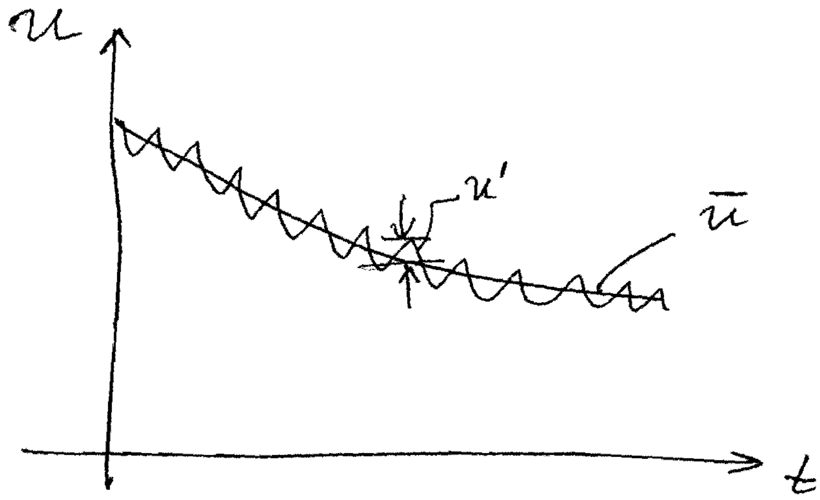
$\bar{u}$ = average value

u' = turbulent fluctuation



$\overline{u'}=0$

steady turbulent flow



unsteady turbulent flow

It can be shown that equs ① and ② become:

$$\rho\left[\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=\frac{\partial \bar{\sigma}_{xx}}{\partial x}+\frac{\partial \bar{\tau}}{\partial z} \quad (1t)$$

and

$$\rho\left[\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right]=\frac{\partial \bar{\sigma}_{zz}}{\partial z}+\frac{\partial \bar{\tau}}{\partial x}-\rho g \quad (2t)$$

and the continuity equation

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{w}}{\partial z}=0 \quad (5t)$$

The derivation is provided below (after the next slide)

In other words the equations apply for the averaged flow as well

However, the constitutive equations become:

$$\bar{\sigma}_{xx} = -\bar{p} + 2\mu\frac{\partial \bar{u}}{\partial x} - \rho\overline{(u')^2}$$
$$\bar{\sigma}_{zz} = -\bar{p} + 2\mu\frac{\partial \bar{w}}{\partial z} - \rho\overline{(w')^2} \qquad (3t)$$
$$\bar{\tau} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) - \rho\overline{(u'w')}$$

The additional terms in the RHS of (3t) are the Reynolds (turbulent) stresses.

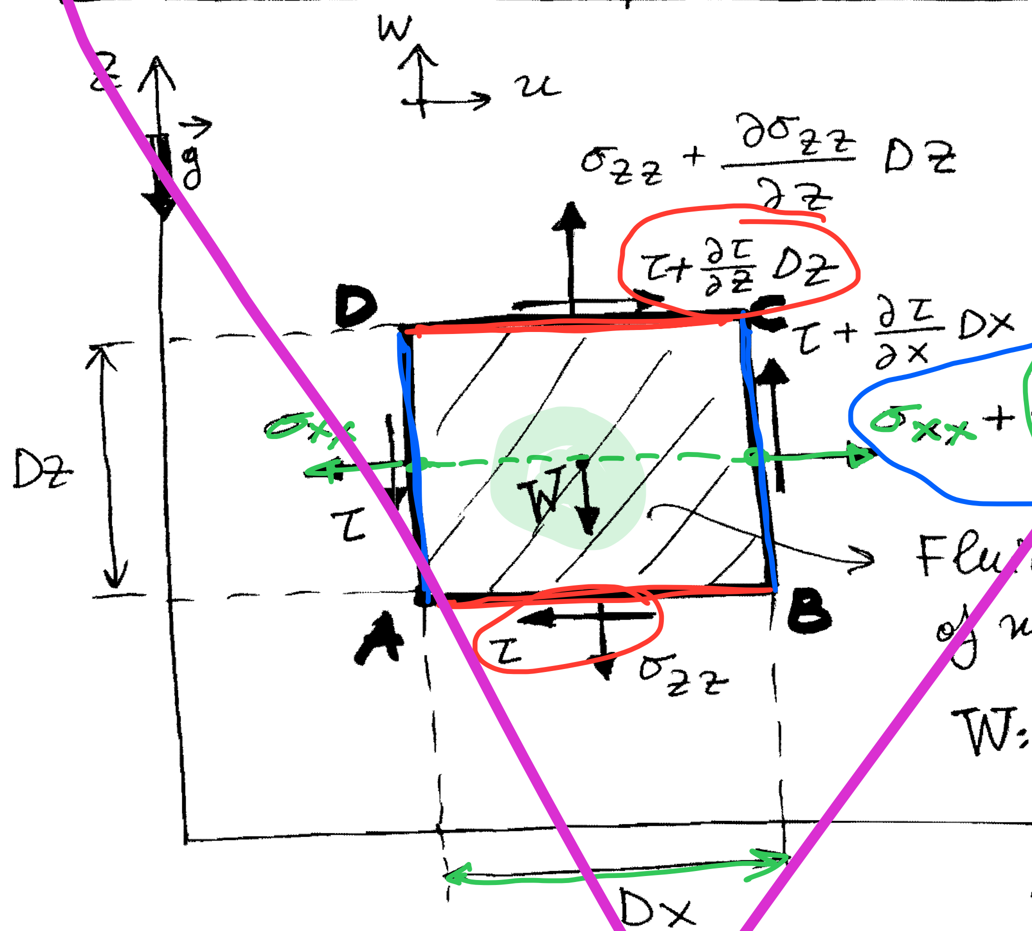
Note that $\overline{u'} = 0$, $\overline{w'} = 0$, but $\overline{(u')^2}, \overline{(w')^2}, \overline{(u'w')} \neq 0$

Turbulence modeling attempts to express the Reynolds stresses in terms of the mean quantities ($\bar{u}, \bar{w}, \bar{p}, \ldots$).

Equations (1t), (2t), (5t) with (3t) consist the Reynolds Averaged Navier Stokes (RANS) equations.

Check the next slide to see how these additional stresses showed up!

Navier-Stokes equations in 2-D



Fluid particle of mass: $\rho\, Dx\, Dz$

W: weight of particle $= \rho g\, Dx\, Dz$

$\tau = \sigma_{xz} = \sigma_{zx}$ (shear stress)

$$\sum F_x = (\rho\, Dx\, Dz) \cdot a_x$$

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \rightarrow \text{fluid particle acceleration along } x$$

$$\sum F_x = \left(\cancel{\sigma_{xx}} + \frac{\partial \sigma_{xx}}{\partial x} Dx\right) \cdot \underbrace{BC}_{Dz} - \left(\cancel{\sigma_{xx}}\right) \underbrace{AD}_{Dz}$$

$$+ \left(\cancel{\tau} + \frac{\partial \tau}{\partial z} Dz\right) \cdot \underbrace{DC}_{Dx} - \left(\cancel{\tau}\right) \underbrace{AB}_{Dx} =$$

$$= \frac{\partial \sigma_{xx}}{\partial x} Dx\, \cancel{Dz} + \frac{\partial \tau}{\partial z} Dz\, \cancel{Dx} = \rho\, \cancel{Dx\, Dz}\, a_x$$

$$\Rightarrow \boxed{\rho a_x = \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right] = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau}{\partial z}} \quad (1)$$

$$u \frac{\partial u}{\partial x} = \frac{\partial u \cdot u}{\partial x} - \frac{\partial u}{\partial x} u \Leftarrow \frac{\partial (u u)}{\partial x} = u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} u$$

$$w \frac{\partial u}{\partial z} = \frac{\partial u w}{\partial z} - \frac{\partial w}{\partial z} u \Leftarrow \frac{\partial (u w)}{\partial z} = u \frac{\partial w}{\partial z} + \frac{\partial u}{\partial z} w$$

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = \boxed{\frac{\partial u^2}{\partial x} + \frac{\partial u w}{\partial z}} - \frac{\partial u}{\partial x} u - \frac{\partial w}{\partial z} u$$

$$= -u \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0$$

continuity equ!

Take average of equ. (A)

$$\rho\frac{\partial u}{\partial t} + \rho\frac{\partial u^2}{\partial x} + \rho\frac{\partial (uw)}{\partial z} = \frac{\partial \bar\sigma_{xx}}{\partial x} + \frac{\partial \tau}{\partial z} \qquad (A)$$

$u = \bar u + u'$

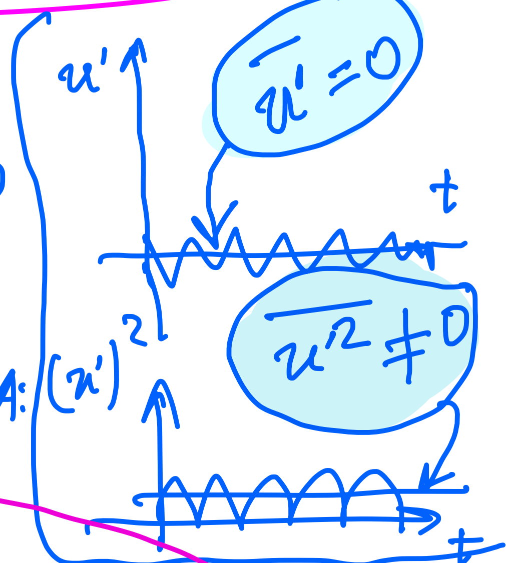
$$\overline{\frac{\partial u}{\partial t}} = \overline{\frac{\partial \bar u}{\partial t}} + \overline{\frac{\partial u'}{\partial t}} = \overline{\frac{\partial \bar u}{\partial t}} + \cancel{\frac{\partial \overline{u'}}{\partial t}}^{=0} = \frac{\partial \bar u}{\partial t}$$

(since $\bar{\bar u} = \bar u$ & $\overline{u'} = 0$)

$$\overline{\rho\frac{\partial u^2}{\partial x}} = \rho\,\overline{\frac{\partial (\bar u + u')^2}{\partial x}} = \rho\frac{\partial}{\partial x}\overline{\left[\bar u^2 + u'^2 + 2\bar u u'\right]} =$$

$$= \rho\frac{\partial \overline{\bar u^2}}{\partial x} + \rho\frac{\partial \overline{u'^2}}{\partial x} + 2\rho\frac{\partial \overline{\bar u u'}}{\partial x}$$

$$\rho\frac{\partial \bar u^2}{\partial x} + \frac{\partial \rho\overline{u'^2}}{\partial x} + 2\rho\frac{\partial \bar u \cancel{\overline{u'}}^{=0}}{\partial x}$$



we will put this term on the RHS of A:

$$\frac{\partial}{\partial x}\left(\bar\sigma_{xx} - \rho\overline{u'^2}\right)$$

similarly, we take the average of $\rho\frac{\partial (uw)}{\partial z}$:

$$\overline{\rho\frac{\partial (uw)}{\partial z}} = \rho\frac{\partial}{\partial z}\overline{\left[(\bar u + u')(\bar w + w')\right]} =$$

$$= \rho\frac{\partial}{\partial z}\left(\overline{\bar u\bar w + u'\bar w + \bar u w' + u'w'}\right) = \rho\frac{\partial}{\partial z}\left(\bar u\bar w + \underbrace{\cancel{\overline{u'}}\bar w}_{=0} + \underbrace{\bar u\cancel{\overline{w'}}}_{=0} + \rho\overline{u'w'}\right)$$

This term does NOT vanish and can be grouped in the RHS of A:

$$\frac{\partial}{\partial z}\left(\bar\tau - \rho\overline{u'w'}\right)$$

$\boxed{\text{REYNOLDS TURBULENT STRESSES}}$

* Note the Reynolds turbulent stresses: $-\rho\overline{u'^2}$, $-\rho\overline{w'^2}$, and $-\rho\overline{u'w'}$ were a result of the averaging of the Navier-Stokes equations!

There are many turbulence models, some simplified, some more elaborate.

A very popular model is the k-ε model. k is the turbulence kinetic energy (per unit mass): TKE

$$k = \frac{1}{2}\left[\overline{(u')^2} + \overline{(w')^2}\right]$$

and ε is the turbulent dissipation (per unit mass):

$$\varepsilon = \underbrace{\frac{\mu}{\rho}}_{=\nu}\left[\overline{\frac{\partial u'}{\partial x}\cdot\frac{\partial u'}{\partial x}} + \overline{\frac{\partial w'}{\partial z}\cdot\frac{\partial w'}{\partial z}} + \overline{\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}} + \overline{\frac{\partial w'}{\partial x}\cdot\frac{\partial w'}{\partial x}}\right]$$

We then replace μ with $\mu + \mu_t$ (*) in Equ.(3) (in addition to some other modifications)

where $\boxed{\mu_t}$ = turbulent (or eddy) viscosity $= \boxed{\rho C_\mu \frac{k^2}{\varepsilon}}$

$C_\mu = 0.09$

(*) For a detailed description you may look in: "Turbulence Modeling for CFD" by David C. Wilcox (DCW 1994)