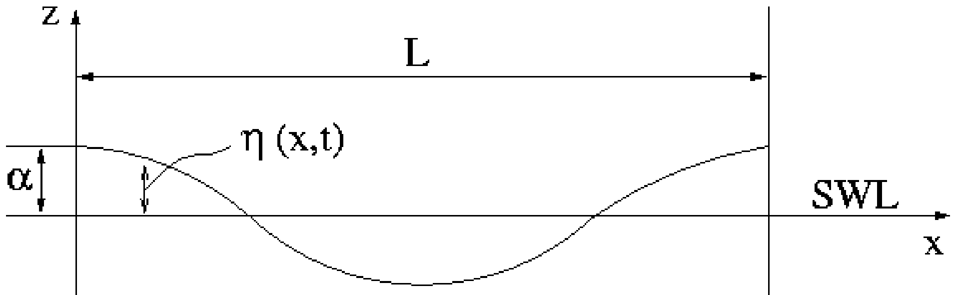
LINEAR WAVE THEORY – DEEP WATER



Finally :

$$\varphi(x,z,t) = \frac{a \cdot \omega}{k} \cdot e^{kz} \cdot \sin(kx - \omega t) \qquad (12)$$

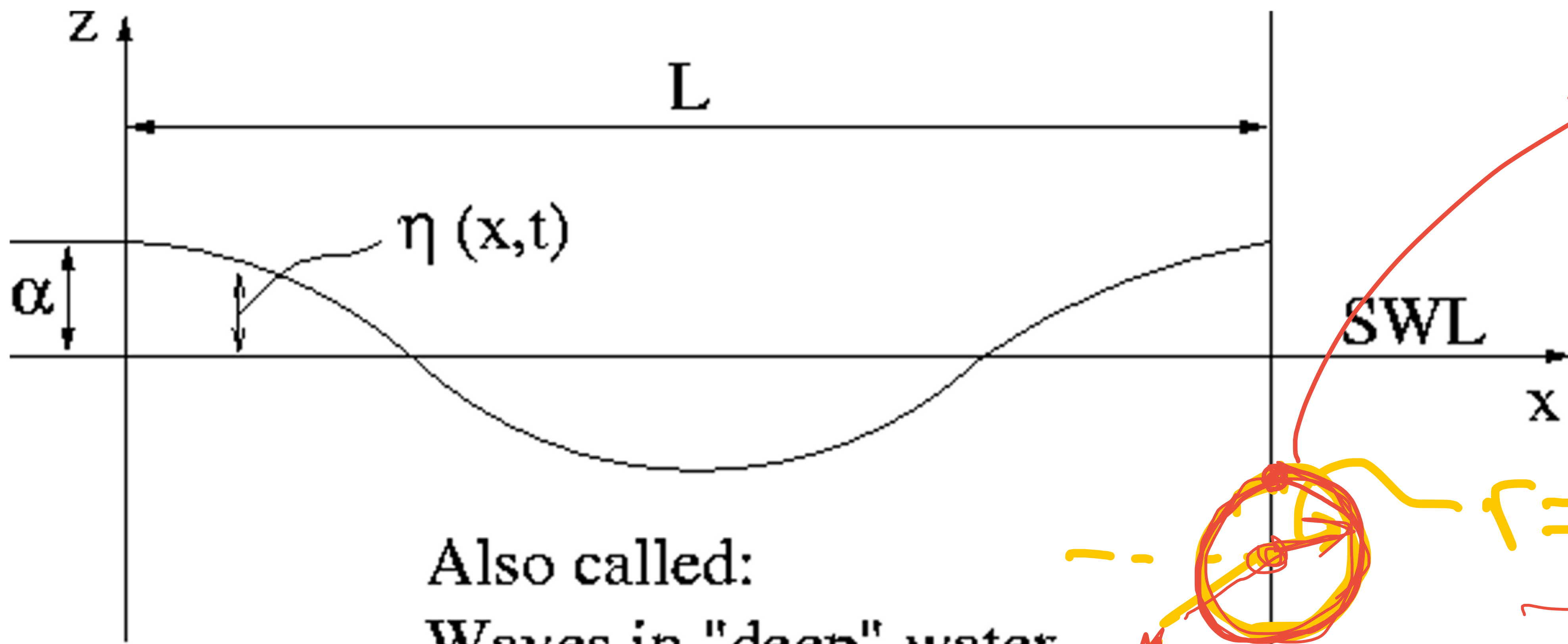
$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = a\omega e^{kz} \cos(kx - \omega t) \qquad (28)$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = a\omega e^{kz} \sin(kx - \omega t) \qquad (29)$$

$$a_x = a\omega^2 e^{kz} \sin(kx - \omega t) \qquad (58)$$

$$a_z = -a\omega^2 e^{kz} \cos(kx - \omega t) \qquad (59)$$

LINEAR WAVE THEORY - DEEP WATER - PARTICLE TRAJECTORIES

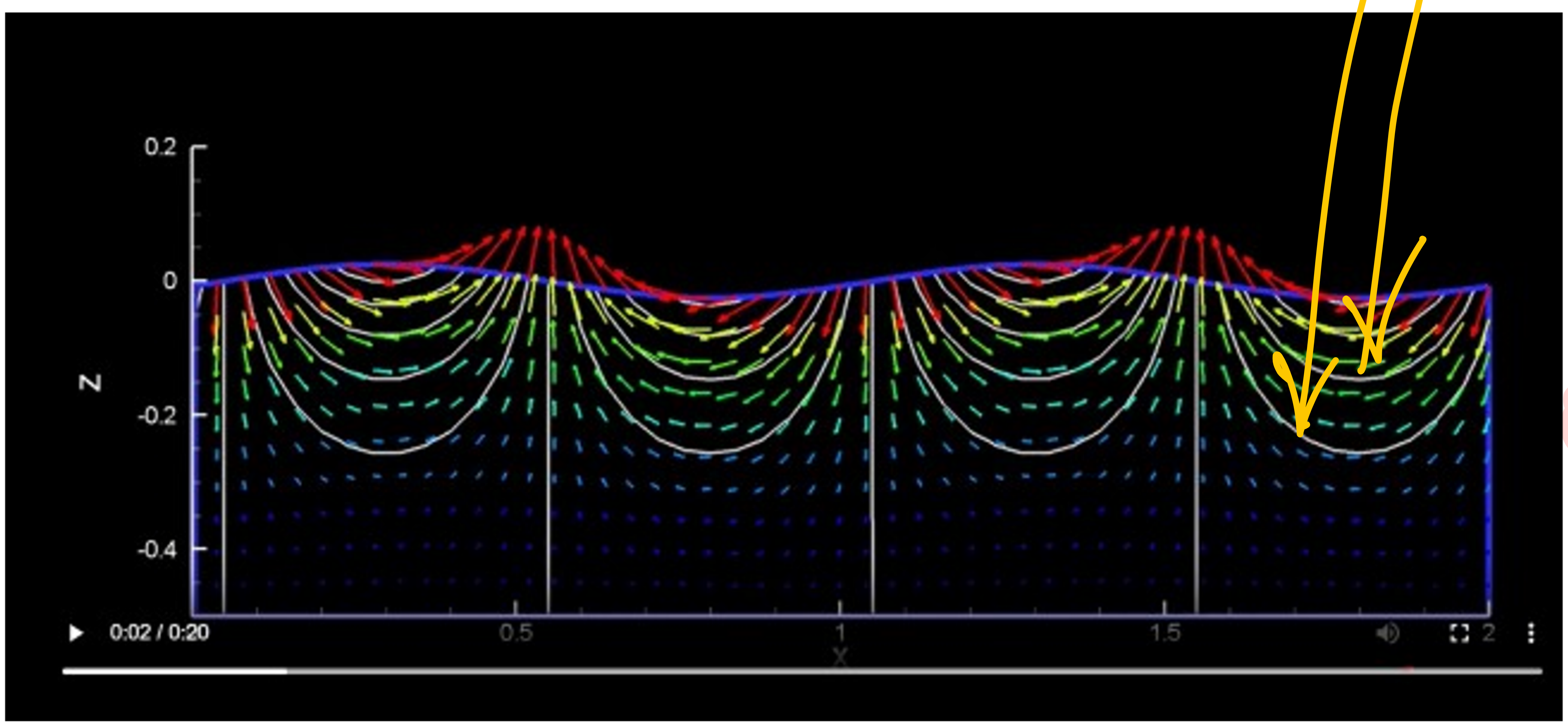


Also called:
Waves in "deep" water

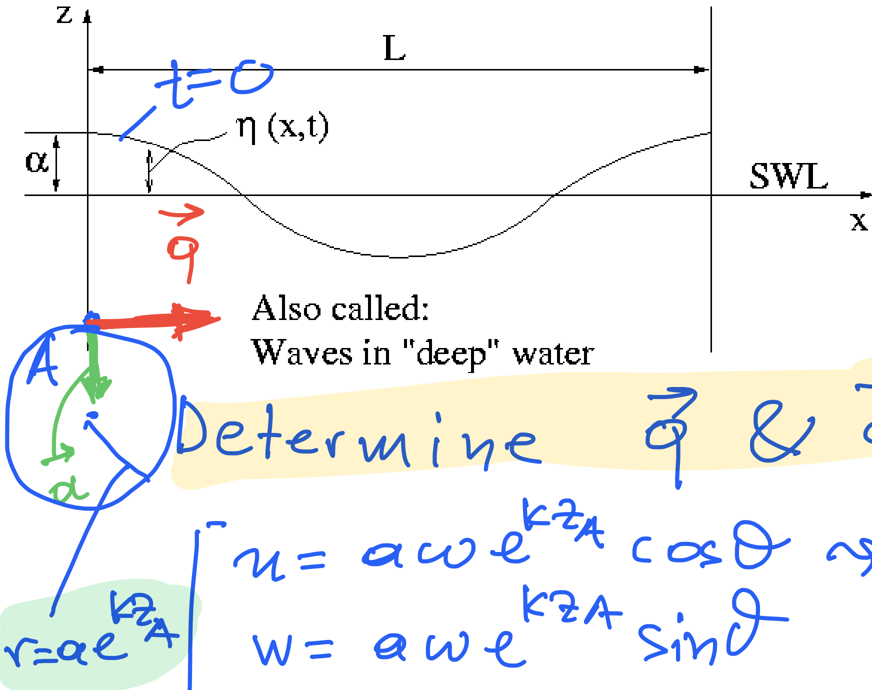
$r = a e^{kz} A$

pathlines (or particle trajectories)

streamlines ≠ pathlines since flow is unsteady



How about the streamlines of the flow -field under the wave?



$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = a\omega e^{kz}\cos(kx - \omega t)$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = a\omega e^{kz}\sin(kx - \omega t)$$

$$\theta = kx - \omega t$$

$$a_x = a\omega^2 e^{kz}\sin(kx - \omega t)$$

$$a_z = -a\omega^2 e^{kz}\cos(kx - \omega t)$$

Determine $\vec{q}$ & $\vec{a}$ at A below crest at t=0

$u = a\omega e^{kz_A}\cos\theta \rightsquigarrow u = a\omega e^{kz_A}$

$w = a\omega e^{kz_A}\sin\theta \qquad w = 0$

at $x=0, t=0, \theta = 0$

$|\vec{q}| = a\omega e^{kz_A}$

$\theta = 0$

$a_x = a\omega^2 e^{kz_A}\,\cancel{\sin(\theta)}^{=0} = 0$

$a_z = -a\omega^2 e^{kz_A}\,\cancel{\cos(\theta)}^{=1} = -a\omega^2 e^{kz_A}$

$|\vec{a}| = \boxed{a\omega^2 e^{kz_A}}$

The same!

Based on circular trajectory of particle

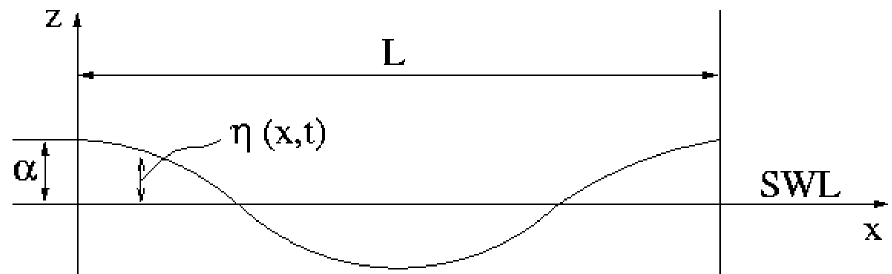
$\vec{a} \perp \vec{q}$

centripetal accel.

$|\vec{a}| = \frac{q^2}{r} = \frac{a^2\omega^2 e^{2kz_A}}{\cancel{a e^{kz_A}}} = \boxed{a\omega^2 e^{kz_A}}$

LINEAR WAVE THEORY – DEEP WATER - PRESSURES

$$\varphi(x,z,t)=\frac{a\cdot\omega}{k}\cdot e^{kz}\cdot\sin(kx-\omega t) \qquad (12)$$



Also called:
Waves in "deep" water

Bernoulli equ.

$$p+\rho\frac{\partial\phi}{\partial t}+\rho\frac{q^2}{2}+\rho g z=0$$

$$p=-\rho\frac{\partial\phi}{\partial t}-\rho\cancel{\frac{q^2}{2}}-\rho g z \qquad \frac{q^2}{2}\sim a^2 \text{ (HOT)}$$

$$\frac{\partial\phi}{\partial t}=\frac{a\omega}{k}e^{kz}\cos(kx-\omega t)(-\omega)=-\frac{a\omega^2}{k}e^{kz}\cos(kx-\omega t)$$

$$q=a\omega e^{kz}$$

$$p=+\rho\frac{a\omega^2}{k}e^{kz}\cos(kx-\omega t)-\rho g z=$$

$$=\rho g\eta e^{kz}-\rho g z$$

$\frac{\omega^2}{k}=g$; $a\cos(kx-\omega t)=\eta$

$$\boxed{\frac{\omega^2}{k}=g}$$

Dispersion relationship

LINEAR WAVE THEORY – DEEP WATER - PRESSURES

$$\varphi(x,z,t) = \frac{a \cdot \omega}{k} \cdot e^{kz} \cdot \sin(kx - \omega t) \qquad (12)$$

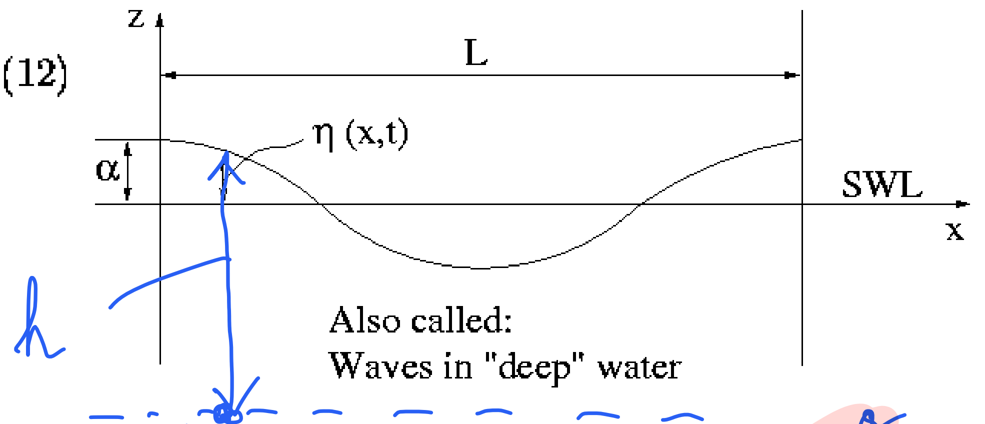
$$p = \underbrace{\rho g \eta e^{kz}}_{P_{wave}} - \underbrace{\rho g z}_{P_{hydro}}$$

P_{wave}

Wave pressure

P_{hydro}

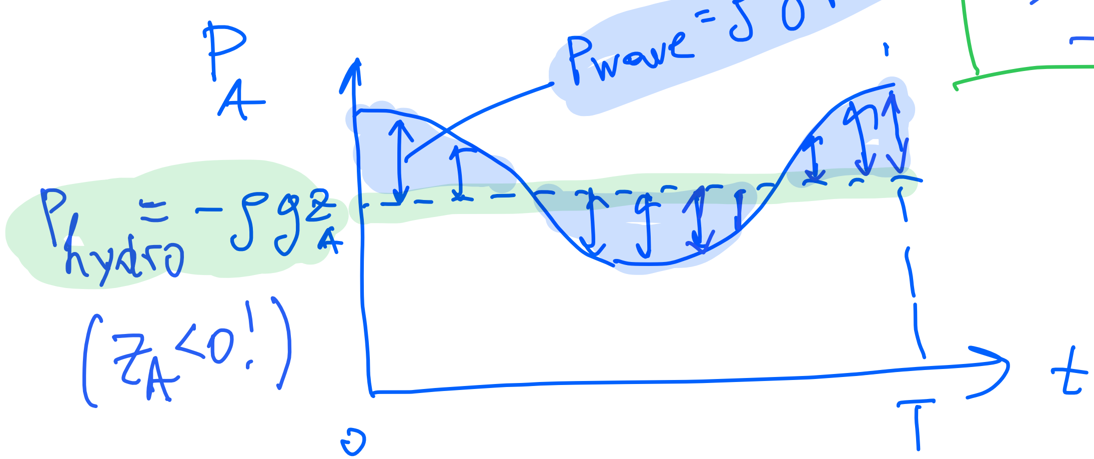
hydrostatic pressure in the absence of waves



$p = \rho g h \qquad h = \eta - z$

$p = \rho g(\eta - z) = \cancel{\rho g \eta} - \rho g z$ wrong wave pressure!

→ quasi-steady is wrong!

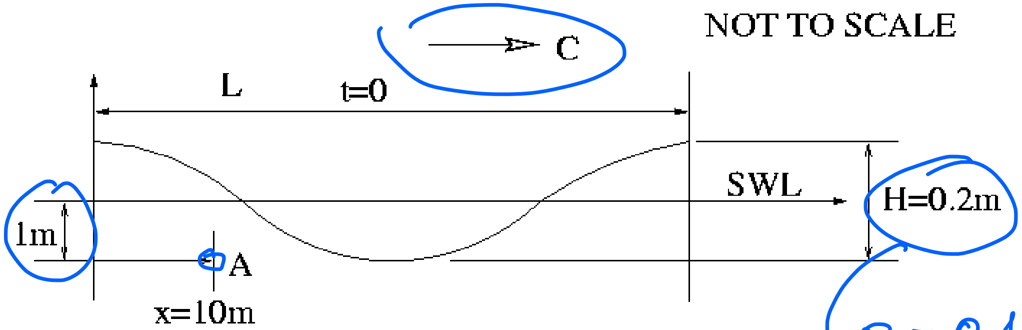


P_A: pressure at point A at location z_A under free-surface

LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 1

$T = 5\,sec$

5sec sinusoidal wave is propagating in deep water. Find the velocity vector and wave pressure at a distance = 10m from the crest, depth of 1m and the time t=3 sec.



$a = 0.1\,m$

$$u = a\omega e^{kz} \cos(\underbrace{kx-\omega t}_{=\theta})$$

$$w = a\omega e^{kz} \sin(\underbrace{kx-\omega t}_{\theta})$$

$$\theta = kx - \omega t = 0.162 \times 10 - 1.257 \times 3 = -2.161\ rad$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{5} = 1.257\ rad/sec$$

$$L = \frac{gT^2}{2\pi} = \frac{9.81 \times 5^2}{2 \times \pi} = 39m$$

$$k = \frac{2\pi}{L} = 0.161\ m^{-1}$$

$z = -1m$

$x = 10m$

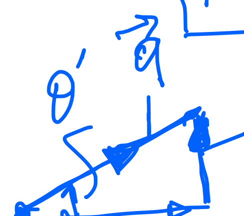
$$e^{kz} = e^{0.161(-1)} = 0.85$$

Deep Water ← $\boxed{\theta = kx - \omega t}$

$$u = -0.06\ m/s$$

$$w = -0.089\ m/s$$

$$q = \sqrt{u^2 + w^2} = a\omega e^{kz} = 0.1073\ \frac{m}{s}$$



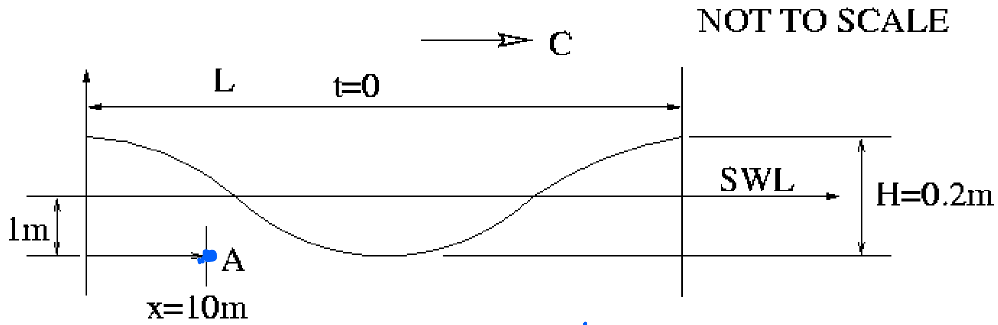
$$w = a\omega e^{kz} \sin(kx-\omega t)$$

$$u = a\omega e^{kz} \cos(kx-\omega t)$$

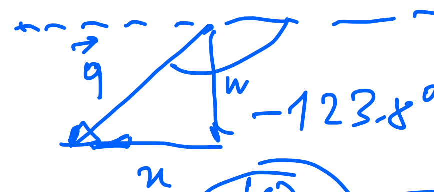
$$\tan\theta' = \frac{w}{u} = \tan(kx-\omega t)$$

LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 1

5sec sinusoidal wave is propagating in deep water. Find the velocity vector and wave pressure at a distance = 10m from the crest, depth of 1m and the time t=3 sec.



$\theta = -2.161 \text{ rad} = -123.8^0$



$\rho_{\text{fresh water}} = 1{,}000 \frac{kg}{m^3}$

$\rho_{\text{sea water}} = 1{,}025 \frac{kg}{m^3}$

$P_{wave} = \rho g \eta e^{kz}$ — 0.85

$\eta = a \cos(kx - \omega t) =$

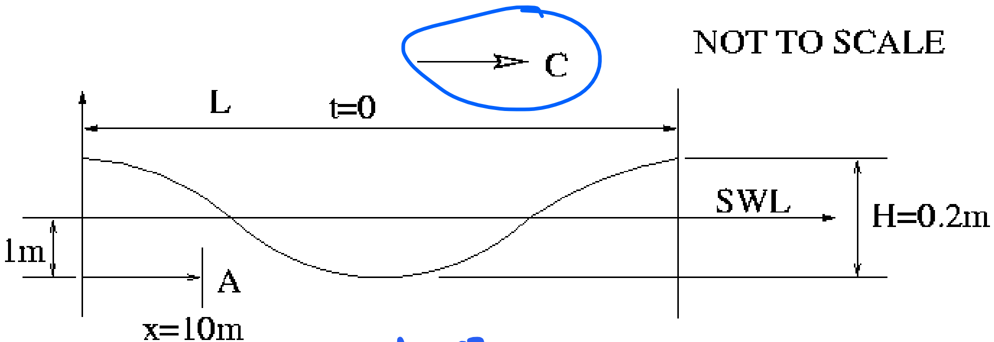
$= 0.1 \cos(-2.161 \text{ rad}) =$

$= -0.05565 \text{ m}$

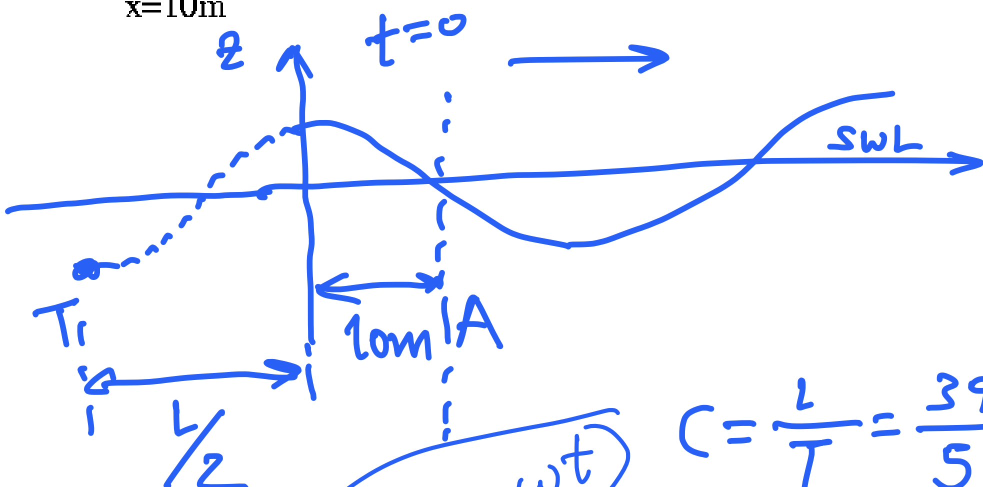
$= 1{,}025 \times 9.81 \times (-0.05565) \times (0.85) = \boxed{-457.6 \text{ Pa}}$

LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 1

5sec sinusoidal wave is propagating in deep water. Find the velocity vector and wave pressure at a distance   = 10m from the crest, depth of 1m and the time t=3 sec.



What would your answers be if we ask for the moment a trough is above A



u, w, P_{wave}

$$t = \frac{\frac{L}{2}+10}{C} = \frac{\frac{39}{2}+10}{7.8} = 3.782\,\text{sec}$$

$$C = \frac{L}{T} = \frac{39}{5} = 7.8\ m/s$$

$$L = 39\ m$$

$$\theta = kx - \omega t$$

$$u = a\omega e^{kz_A}\cos\theta$$

$$w = a\omega e^{kz_A}\sin\theta$$

$$P_{wave} = \rho g \eta e^{kz_A}, \quad \eta = a\cos(\theta)$$

$$\theta = kx_A - \omega t$$

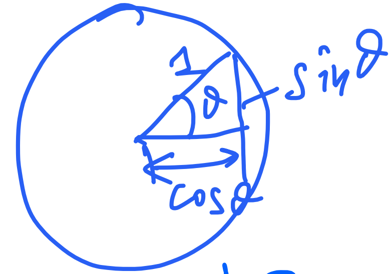
$$k = 0.161\ m^{-1}$$

$$x_A = 10\ m$$

$$\omega = 1.257\ rad/sec$$

$$\Rightarrow \theta = -3.143 = -\pi$$

$\vartheta = \pi \quad (\cos -\pi)$



$u = a\omega e^{kz_A}(-1) = -a\omega e^{kz_A}$

$w = 0$

$$\left.\begin{array}{l} P_{wave} = \rho g \eta e^{kz_A} \\ \eta = a\cos(\pi) = -a \end{array}\right\} P_{wave} = -\rho g a e^{kz_A}$$

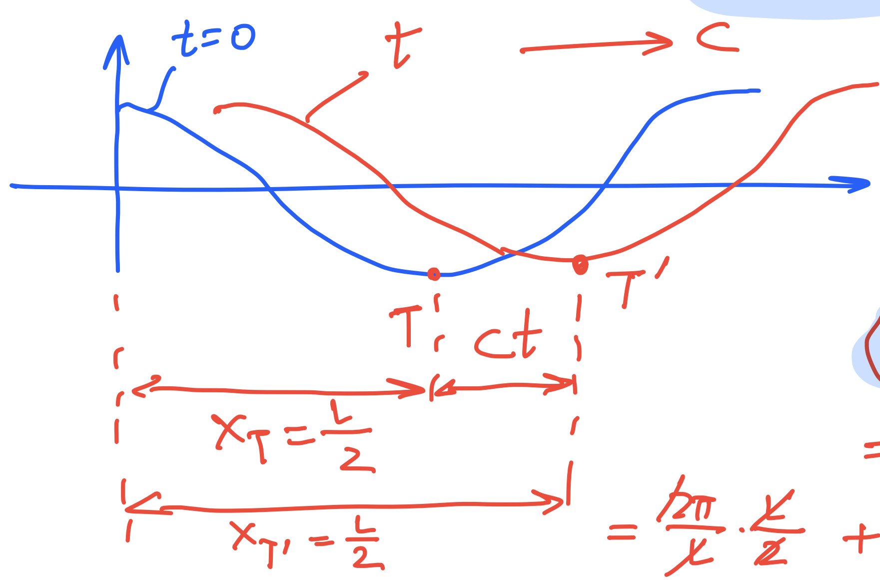
$$\Rightarrow \quad \boxed{u} = -(0.1) \times 1.257 \times 0.85 = \boxed{0.107 \frac{m}{s}}$$

$$\boxed{w = 0}$$

$$\boxed{P_{wave}} = -1{,}025 \times 9.81 \times (0.1)(0.85) =$$

$$= \boxed{-854.7 \ Pa}$$

VALUES OF θ=kx-ωt AT SPECIAL POINTS



$$\vartheta_T = kx_T - \omega t$$

$$t=0: \quad \vartheta_T = \frac{2\pi}{\cancel{L}}\frac{\cancel{L}}{\cancel{2}} - 0$$

$$\vartheta_T = \pi$$

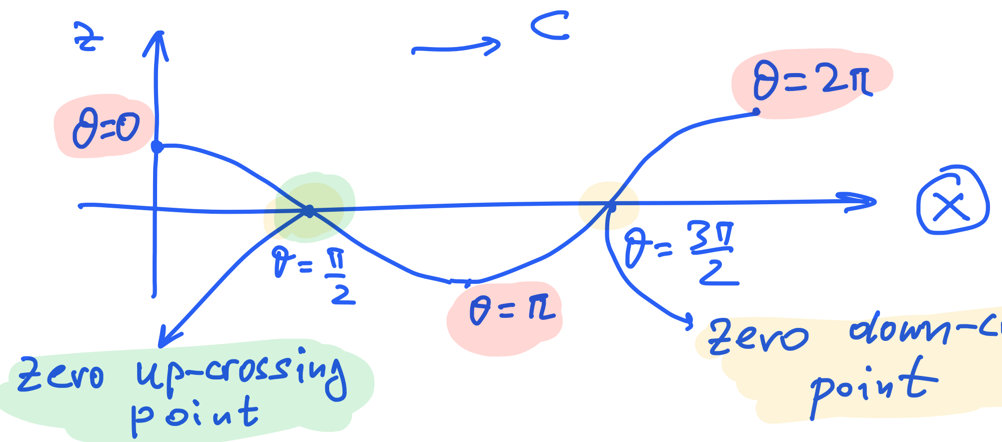
$$\vartheta_{T'} = kx_{T'} - \omega t =$$

$$= k\left(\frac{L}{2} + ct\right) - \omega t =$$

$$= \frac{\cancel{2}\pi}{\cancel{L}} \cdot \frac{\cancel{L}}{\cancel{2}} + \underbrace{kct}_{\omega t} - \omega t = \pi$$

$$\cancel{\omega t - \omega t} = 0$$

$$c = \frac{L}{T} = \frac{\omega}{k} \rightsquigarrow kc = \omega$$

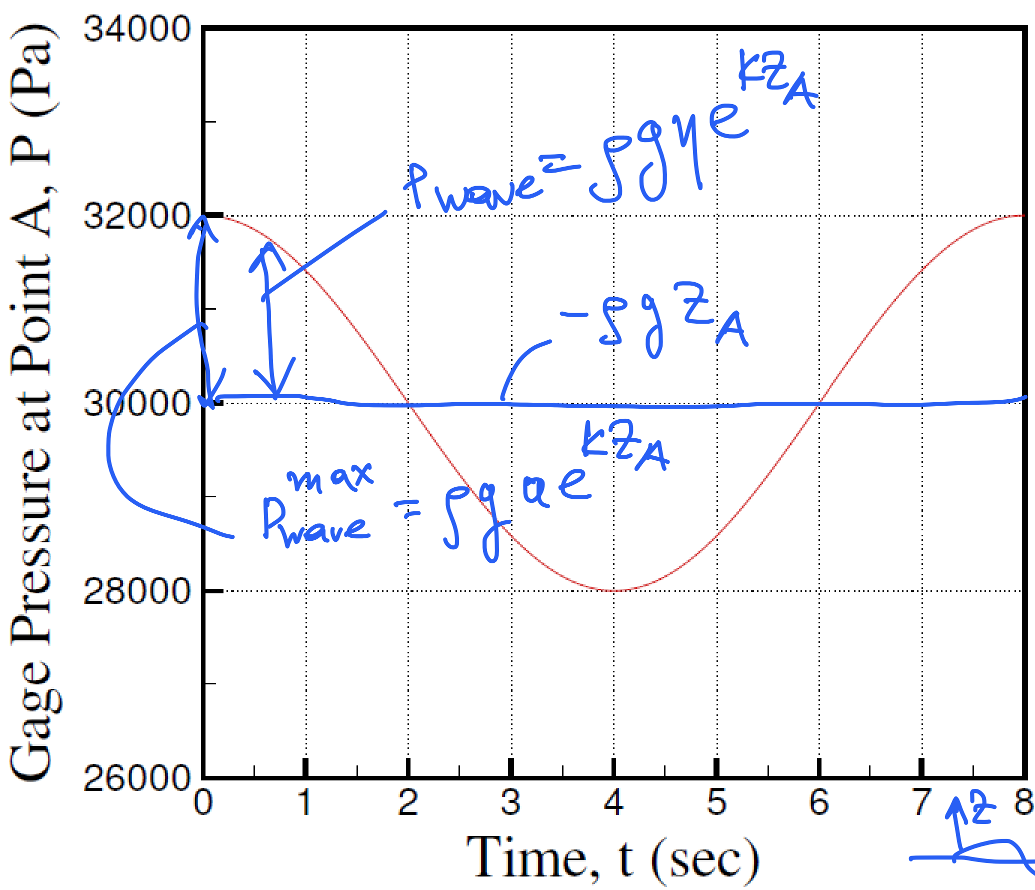


θ of trough is π independent of x, t

LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 2

$\rho = 1,025 \ kg/m^3$

A sinusoidal wave is propagating in infinite depth sea-water (in the .+ x. direction). A pressure gage is mounted at point A under the free surface. The time history of the gage pressure at point A over one wave period is shown in the figure below. Using the information on the given graph, apply linear wave theory and determine the following:



a) The depth at point A. (5 points)

b) The wave height. (10 points)

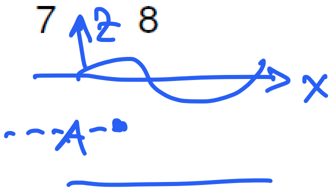
c) The wave elevation above point A at t = 6.8 sec. (10 points)

d) The values of the *horizontal* particle velocity and acceleration at point A and at t = 6.8 sec. (10 points)

$P = \boxed{\rho g \eta e^{kz} - \rho g z}$

gage pressure

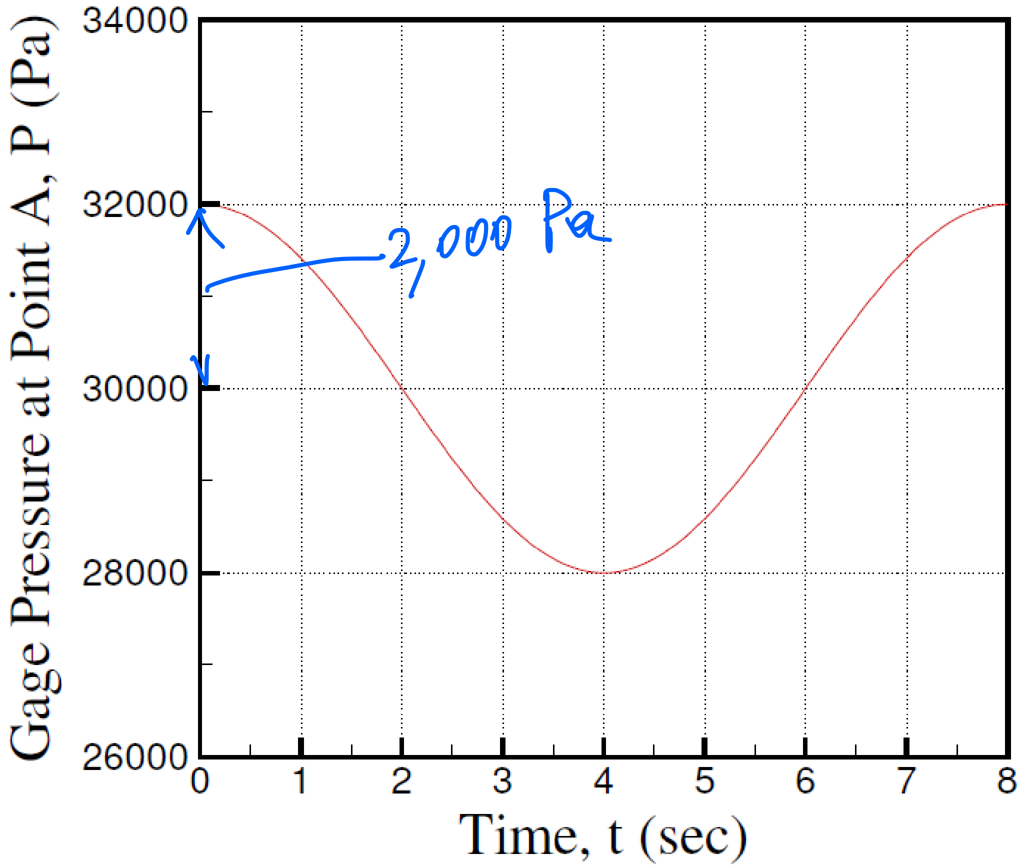
a) $z_A = ?$



$-\rho g z_A = 30,000$

$\boxed{z_A} = -\dfrac{30,000}{1,025 \times 9.81} = \boxed{-2.98 \ m}$

LINEAR WAVE THEORY – DEEP WATER – EXAMPLE 2



a) The depth at point A. (5 points)

b) The wave height. (10 points)

b) From graph

$P_{wave}^{max} = 32,000 - 30,000 = 2,000\ Pa$

$= \rho g a e^{kz_A}$

$T = 8 sec \Rightarrow L = \frac{gT^2}{2\pi} = 100 m$

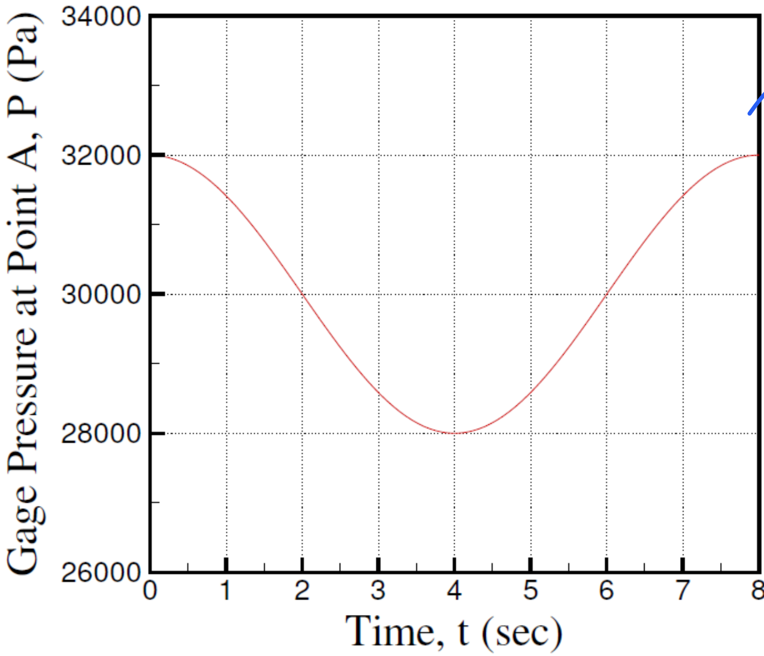
$k = \frac{2\pi}{L} = \frac{2\pi}{100} = 0.0628\ m^{-1}$

$kz_A = 0.0628 \times (-2.98) = -0.187$

$\Rightarrow e^{kz_A} = e^{-0.187} = 0.829$

$2,000 = 1,025 \times 9.81 \times a \times 0.829 \sim a = 0.24 m$

$\sim \boxed{H} = 2a = \boxed{0.48 m}$



c) The wave elevation above point A at t = 6.8 sec. (10 pts)

d) The values of the *horizontal* particle velocity and acceleration at point A and at t = 6.8 sec. (10 pts)

c) $\eta = ?$

$\eta = a\cos(kx - \omega t)$

$x = 0$

$\eta = a\cos(-0.785 \times 6.8)$ -5.43 rad

$= \boxed{0.141\,m}$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{8} = 0.785\ \frac{rad}{sec}$

d) u, a_x

$u = a\omega e^{kz_A}\cos(\theta) = 0.24 \times 0.785 \times 0.829 \times \cos(-5.43\ rad)$

$a_x = a\omega^2 e^{kz_A}\sin(\theta) = 0.24 \times (0.785)^2 \times 0.829 \times \sin(-5.43\ rad)$

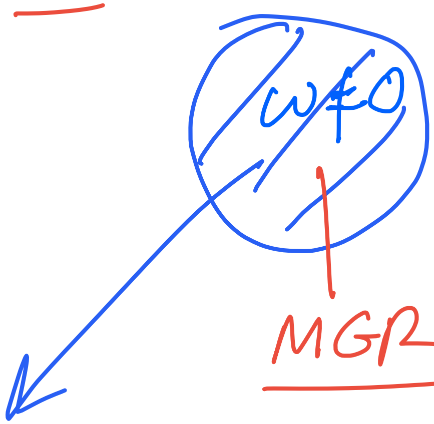
$\theta = kx - \omega t = -5.43\ rad$

$\Rightarrow \boxed{u = 0.092\ m/s \quad a_x = 0.099\ m/s^2}$

Spin of particles inside & outside tornado

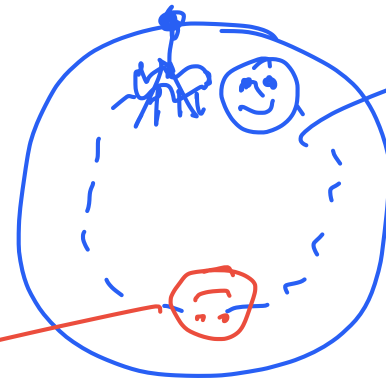
$\omega = 0$

FW.



MGR

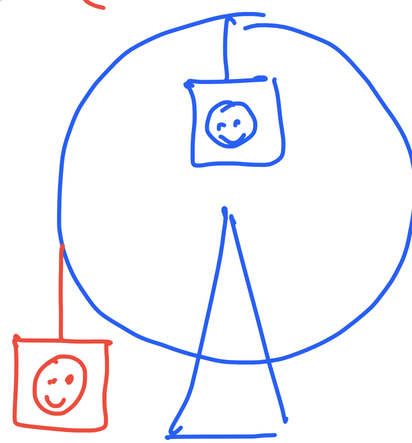
tornado core



you have spin ($\omega \neq 0$)

Merry go round (MGA)

Ferris wheel (FW)



you do not have spin

$\omega = 0$