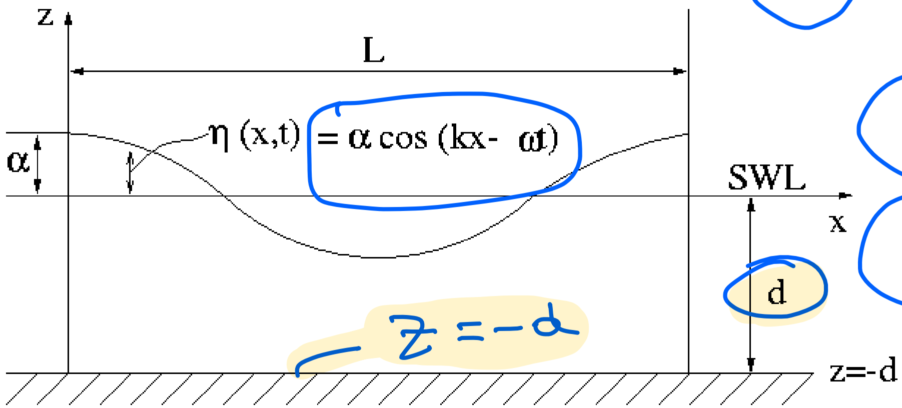
LINEAR WAVE THEORY – FINITE DEPTH WATER

In deep H_2O $\quad \phi = \frac{a\omega}{k} e^{kz} \sin(kx - \omega t) \Rightarrow \nabla^2 \phi = 0$



$k = \frac{2\pi}{L}$

$\omega = \frac{2\pi}{T}$

$\nabla^2 \phi = 0$ ✓

- kbc + dbc on $z = \eta$
- kbc at $z = -d$

$$\rightarrow \varphi(x,z,t) = (Ae^{kz} + Be^{-kz}) \sin(kx - \omega t) \tag{62}$$

$$\boxed{\varphi(x,z,t) = \frac{ga}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \sin(kx - \omega t)} \quad \text{Finite Depth} \tag{63}$$

- A, B can be determined from applying the kbc at the sea-floor ($\frac{\partial \phi}{\partial z} = 0$ at $z = -d$) and the kbc on the free-surface ($\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}$).
- The final expression for ϕ is

LINEAR WAVE THEORY – FINITE DEPTH WATER – DISPERSION RELATIONSHIP

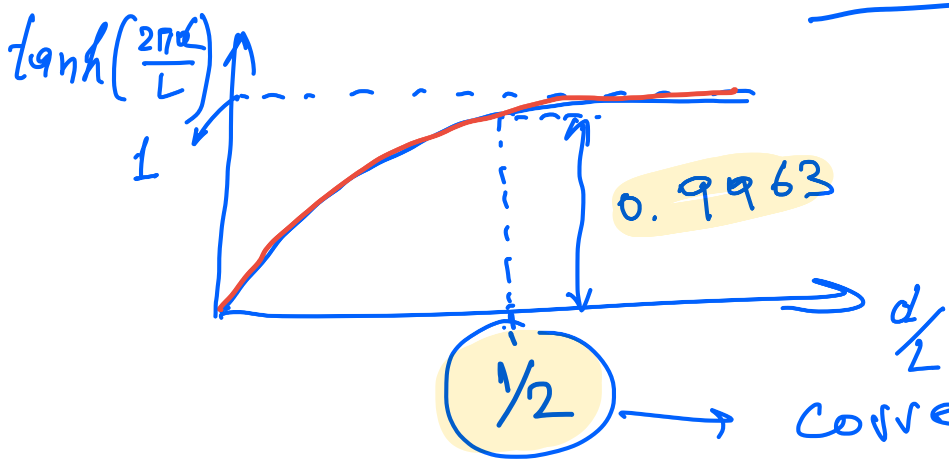
- From dbc on the free surface ($p=0$) we get:

FINITE DEPTH H_2O

$$\boxed{\frac{\omega^2}{k\tanh(kd)} = g} \; *$$

Dispersion Relationship for finite depth H_2O

Remember in deep H_2O: $\boxed{\frac{\omega^2}{k} = g}$



as $\frac{d}{L} \to \infty$ $\tanh\left(\frac{2\pi d}{L}\right) \to 1$

$\frac{1}{2}$ → corresponds to deep water limit

$\frac{d}{L} > \frac{1}{2} \Rightarrow$ deep H_2O

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\Rightarrow \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \to 1 \text{ as } x \to \infty$$

as $x \to \infty$ $e^{-x} \to 0$

LINEAR WAVE THEORY – DISPERSION RELATIONSHIP - EXAMPLE

$$\frac{\omega^2}{k\tanh(kd)} = g \Rightarrow \frac{\left(\frac{2\pi}{T}\right)^2}{\left(\frac{2\pi}{L}\right)\tanh(kd)} = g \Rightarrow$$

$$\Rightarrow \boxed{L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)} \quad (1)$$

→ If d and L are given then (1) can be solved w.r.t. $T = \sqrt{\frac{2\pi}{g}\frac{L}{\tanh\left(\frac{2\pi d}{L}\right)}}$

→ If d & T are given how can we find L?

$L_0 = \frac{gT^2}{2\pi}$: deep water wave length

$$\boxed{L = L_0 \tanh\left(\frac{2\pi d}{L}\right)}$$

$$\frac{d}{L_0} = \frac{d}{L} \tanh\left(\frac{2\pi d}{L}\right)$$

$\Longrightarrow$ Table C-1

Example: Find L for T = 7 sec at d = 8m

$$\boxed{L_0} = \frac{gT^2}{2\pi} = \frac{9.81 \times (7)^2}{2\pi} = \boxed{76.5m}$$

$$\frac{d}{L_0} = \frac{8}{76.5} = 0.1046 \xrightarrow[C-1]{Table} \frac{d}{L} = 0.1449$$

$$\Rightarrow \boxed{L} = \frac{d}{0.1449} = \boxed{55.2m}$$

Always $\boxed{L = L_0 \tanh\left(\frac{2\pi d}{L}\right) < L_0}$

// CE358 (Kinnas) – Fall 1998 – UT Austin

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Related to Example Problem under section: "How to find L for given T and d?"

Table C-1. Continued.

d/L_o	d/L	$2\pi d/L$	TANH $2\pi d/L$	SINH $2\pi d/L$	COSH $2\pi d/L$	H/H'_o	K	$4\pi d/L$	SINH $4\pi d/L$	COSH $4\pi d/L$	n	C_G/C_o	M
.09000	.1322	.8306	.6808	.9295	1.3653	.9422	.7324	1.661	2.538	2.728	.8273	.5632	10.65
.09100	.1331	.8363	.6838	.9372	1.3706	.9411	.7296	1.672	2.568	2.756	.8255	.5645	10.55
.09200	.1340	.8420	.6868	.9450	1.3759	.9401	.7268	1.684	2.599	2.785	.8238	.5658	10.46
.09300	.1349	.8474	.6897	.9525	1.3810	.9391	.7241	1.695	2.630	2.814	.8221	.5670	10.37
.09400	.1357	.8528	.6925	.9600	1.3862	.9381	.7214	1.706	2.662	2.843	.8204	.5682	10.29
.09500	.1366	.8583	.6953	.9677	1.3917	.9371	.7186	1.717	2.693	2.873	.8187	.5693	10.21
.09600	.1375	.8639	.6982	.9755	1.3970	.9362	.7158	1.728	2.726	2.903	.8170	.5704	10.12
.09700	.1384	.8694	.7011	.9832	1.4023	.9353	.7131	1.739	2.757	2.933	.8153	.5716	10.04
.09800	.1392	.8749	.7039	.9908	1.4077	.9344	.7104	1.750	2.790	2.963	.8136	.5727	9.962
.09900	.1401	.8803	.7066	.9985	1.4131	.9335	.7076	1.761	2.822	2.994	.8120	.5737	9.884
.1000	.1410	.8858	.7093	1.006	1.4187	.9327	.7049	1.772	2.855	3.025	.8103	.5747	9.808
.1010	.1419	.8913	.7120	1.014	1.4242	.9319	.7022	1.783	2.888	3.057	.8086	.5757	9.734
.1020	.1427	.8967	.7147	1.022	1.4297	.9311	.6994	1.793	2.922	3.088	.8069	.5766	9.661
.1030	.1436	.9023	.7173	1.030	1.4354	.9304	.6967	1.805	2.956	3.121	.8052	.5776	9.590
.1040	.1445	.9076	.7200	1.037	1.4410	.9297	.6940	1.815	2.990	3.153	.8036	.5785	9.519
.1050	.1453	.9130	.7226	1.045	1.4465	.9290	.6913	1.826	3.024	3.185	.8019	.5794	9.451
.1060	.1462	.9184	.7252	1.053	1.4523	.9282	.6886	1.837	3.059	3.218	.8003	.5803	9.384
.1070	.1470	.9239	.7277	1.061	1.4580	.9276	.6859	1.848	3.094	3.251	.7986	.5812	9.318
.1080	.1479	.9293	.7303	1.069	1.4638	.9269	.6833	1.858	3.128	3.284	.7970	.5820	9.254
.1090	.1488	.9343	.7327	1.076	1.4692	.9263	.6806	1.869	3.164	3.319	.7954	.5828	9.191
.1100	.1496	.9400	.7352	1.085	1.4752	.9257	.6779	1.880	3.201	3.353	.7937	.5836	9.129
.1110	.1505	.9456	.7377	1.093	1.4814	.9251	.6752	1.891	3.237	3.388	.7920	.5843	9.068
.1120	.1513	.9508	.7402	1.101	1.4871	.9245	.6725	1.902	3.274	3.423	.7904	.5850	9.009
.1130	.1522	.9563	.7426	1.109	1.4932	.9239	.6697	1.913	3.312	3.459	.7888	.5857	8.950
.1140	.1530	.9616	.7450	1.117	1.4990	.9234	.6671	1.923	3.348	3.494	.7872	.5864	8.891

$d/L_o = 0.1046$

LINEAR WAVE THEORY – PARTICLE VELOCITIES

$$\varphi(x,z,t) = \frac{ga}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \sin(kx - \omega t) \qquad \text{Finite Depth} \qquad (63)$$

$$u = \frac{\partial \phi}{\partial x} = \frac{ga}{\omega} \frac{\cosh[k(z+d)]}{\cosh[kd]} \cos(kx - \omega t) \cdot k$$

$$w = \frac{\partial \phi}{\partial z} = \ldots$$

$$\frac{g \frac{H}{2} \cdot \frac{\cancel{2\pi}}{L}}{\frac{\cancel{2\pi}}{T}} = \frac{gHT}{2L}$$

$$u(x,z,t) = \frac{\partial\varphi}{\partial x} = \frac{gak}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \qquad (79)$$

$$w(x,z,t) = \frac{\partial\varphi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \qquad (80)$$

LINEAR WAVE THEORY – ACCELERATIONS

$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = \frac{gak}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \tag{79}$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \tag{80}$$

$$a_x = \frac{\partial u}{\partial t} + \cancel{\underbrace{u \frac{\partial u}{\partial x}}_{\sim a^2}} + \cancel{\underbrace{w \frac{\partial u}{\partial z}}_{\sim a^2}}$$

$u \sim a$, $\frac{\partial u}{\partial x} \sim a$, $w \sim a$, $\frac{\partial u}{\partial z} \sim a$

H.O.T.

$$a_z = \frac{\partial w}{\partial t}$$

$$g \frac{H}{2} \frac{2\pi}{L} = \frac{gH\pi}{L}$$

$$\boxed{a_x = \frac{\partial u}{\partial t} = gak \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t)} \tag{81}$$

$$\boxed{a_z = \frac{\partial w}{\partial t} = -gak \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t)} \tag{82}$$

Intermediate H_2O
or

$\theta = kx - \omega t$

RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25}$ or $d < 0.04$	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	← Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_o = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_o = \frac{gT^2}{2\pi} = C_oT$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]\cdot C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal, u	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\cos\theta$
(b) Vertical, w	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\sin\theta$
6. Water Particle Accelerations (a) Horizontal, a_x	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical, a_z	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\cos\theta$
7. Water Particle Displacements (a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2}e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right)\cos\theta$	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2}e^{\frac{2\pi z}{L}}\cos\theta$
8. wave Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho gz$	$p = \rho g\eta e^{\frac{2\pi z}{L}} - \rho gz$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

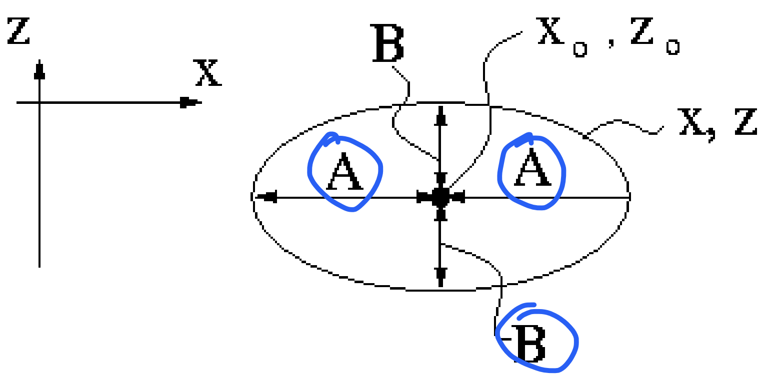
LINEAR WAVE THEORY – PARTICLE TRAJECTORIES

$$u(x,z,t) = \frac{\partial \varphi}{\partial x} = \frac{gak}{\omega} \cdot \frac{\cosh[k(z+d)]}{\cosh(kd)} \cdot \cos(kx - \omega t) \tag{79}$$

$$w(x,z,t) = \frac{\partial \varphi}{\partial z} = \frac{gak}{\omega} \cdot \frac{\sinh[k(z+d)]}{\cosh(kd)} \cdot \sin(kx - \omega t) \tag{80}$$

$$u(x,z,t) \approx u(x_0, z_0, t) = \frac{dx}{dt} \tag{83}$$

$$w(x,z,t) \approx w(x_0, z_0, t) = \frac{dz}{dt} \tag{84}$$



$$\boxed{A = a \cdot \frac{\cosh[k(z_0+d)]}{\sinh(kd)}}$$

$$\boxed{B = a \cdot \frac{\sinh[k(z_0+d)]}{\sinh(kd)}}$$

1) what should $\underline{B}$ be at $z = -d$ (sea-floor)

$$\boxed{B} = a \frac{\sinh[k(-d+d)]}{\sin[kd]} = a \frac{\sinh(0)}{\sinh(kd)} = \boxed{0}$$

2) what should B be at the free-surface.

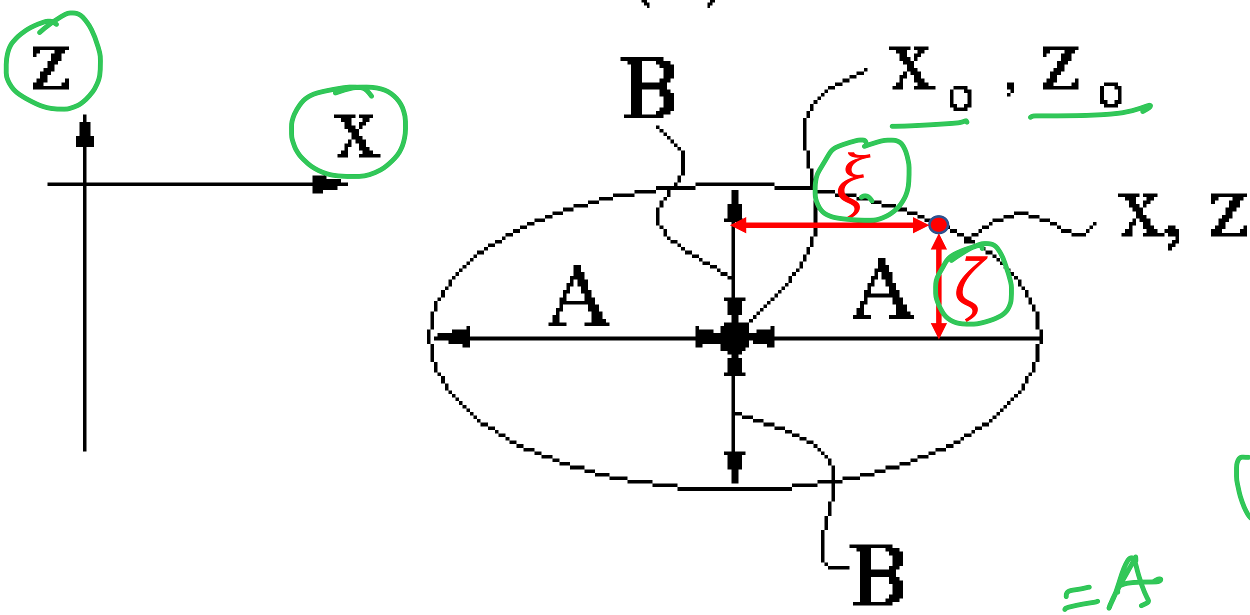
$z_0 = 0$ $\boxed{B} = a \sinh[kd] / \sinh(kd) = \boxed{a}$

a

LINEAR WAVE THEORY – PARTICLE TRAJECTORIES

$$A = a \cdot \frac{\cosh[k(z_0 + d)]}{\sinh(kd)} \tag{85}$$

$$B = a \cdot \frac{\sinh[k(z_0 + d)]}{\sinh(kd)} \tag{86}$$



$$\left(\frac{\xi}{A}\right)^2 + \left(\frac{\zeta}{B}\right)^2 = 1$$

$$\xi = -\underbrace{\frac{H}{2}\,\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}}_{=A}\,\sin\theta \;\rightsquigarrow\; \frac{\xi}{A} = -\sin\theta$$

$$\zeta = \underbrace{\frac{H}{2}\,\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}}_{=B}\,\cos\theta \;\rightsquigarrow\; \frac{\zeta}{B} = \cos\theta$$

LINEAR WAVE THEORY – PRESSURES

$$\varphi(x,z,t) = \frac{ga}{\omega}\cdot\frac{\cosh[k(z+d)]}{\cosh(kd)}\sin(kx-\omega t) \qquad \text{Finite Depth} \tag{63}$$

Bernoulli equ: $\rho\frac{\partial\phi}{\partial t} + \rho\frac{q^2}{2} + p + \rho g z = 0$

HOT

$\rightsquigarrow p = -\rho\frac{\partial\phi}{\partial t} - \cancel{\rho\frac{q^2}{2}} - \rho g z$

$q = \sqrt{u^2 + w^2}$ $(\sim a)^2$ $\sim(a)^2$ $\rightsquigarrow q^2 = u^2 + w^2$ $\sim a^2$ $\sim a^2$ $\Rightarrow$

$p = -\rho\frac{\partial\phi}{\partial t} - \rho g z$

$\frac{\partial\phi}{\partial t} = \frac{ga}{\cancel{\omega}}\frac{\cosh[k(z+d)]}{\cosh[kd]}\cos(kx-\omega t)(-\cancel{\omega}) =$

$= -g\,a\,\frac{\cosh[k(z+d)]}{\cosh[kd]}\cos(kx-\omega t) = \eta$

$$p = \rho g \eta\,\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$$

$p = \ominus\rho\left[\ominus g\eta\,\frac{\cosh[k(z+d)]}{\cosh[kd]}\right] - \rho g z$

LINEAR WAVE THEORY – PRESSURES

$$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$$

gage pressure under the wave

wave pressure

hydrostatic pressure

LINEAR WAVE THEORY – PRESSURES - EXAMPLE

$$p = \rho g \eta \underbrace{\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}}_{K_z} - \rho g z$$

K_z : pressure response factor

Example Problem 1 on p. 2-11 of SPM

T = 10 sec

1) what the L and C of the wave at d = 200 m

$L_0 = \frac{gT^2}{2\pi} = 156\ m$

$\Rightarrow$ deep $\boxed{C = \frac{gT}{2\pi}} = \boxed{15.6\ \frac{m}{s}}$

2) at d = 3m

T of wave stays the same as the wave approaches the shoreline (beach)

T = 10 sec $\quad \frac{d}{L_0} = \frac{3m}{156m} = 0.0192 \Longrightarrow \frac{d}{L} = 0.05641 \rightsquigarrow \boxed{L = 53.2m}$

C-1

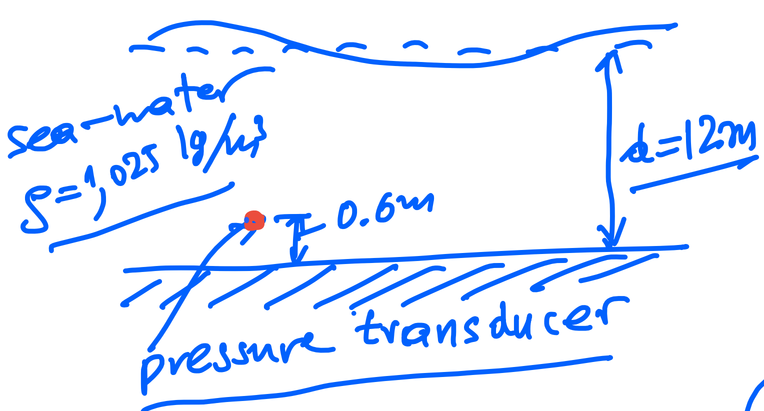
$\boxed{C} = \frac{L}{T} = \frac{53.2}{10} = \boxed{5.32\ \frac{m}{s}}$

LINEAR WAVE THEORY – PRESSURES - EXAMPLE

$$* \quad p = \rho g \eta \underbrace{\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}}_{=K_z} - \rho g z$$

$1\,Pa = 1\,\frac{N}{m^2}$

Example Prob. 5 on p. 2-22 of SPM



Maximum gage pressure measured by the transducer: $124\,\frac{kN}{m^2}$ = 124 kPa

Frequency of the wave is

$f = 0.666$ cycles/sec $\quad (\omega = 2\pi f)^*$ $\quad (f = \frac{1}{T})^*$

(temporal frequency

Find H

$P_{max} = \rho g a K_z - \rho g z$ $\qquad z = -(12-0.6) = -11.4\,m$

Need to find L

$f = 0.666$ cycles/sec $\rightarrow \boxed{T} = \frac{1}{f} = \frac{1}{0.666} = \boxed{15\ sec}$

$L_0 = \frac{gT^2}{2\pi} = 351\,m \rightarrow \frac{d}{L_0} = \frac{12}{351} = 0.0342 \xrightarrow{C-1} \frac{d}{L} = 0.07651 \Rightarrow$

$\Rightarrow \boxed{L = 156.8\,m}$

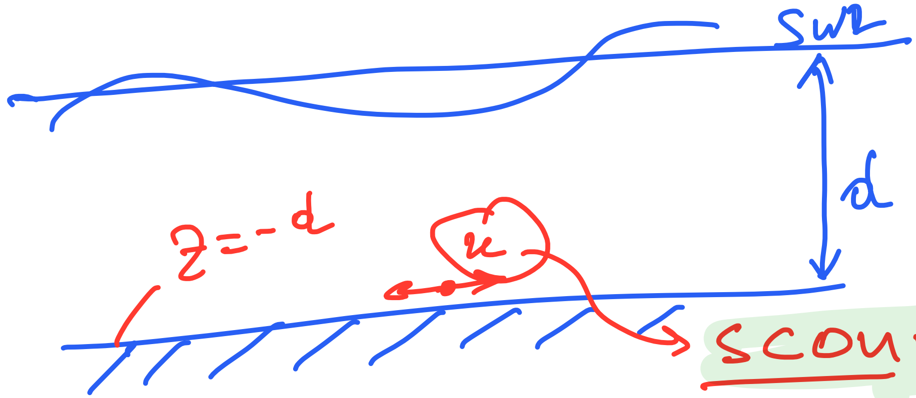
LINEAR WAVE THEORY – SHALLOW WATER - APPROXIMATIONS as d/L -->0/PRACTICAL LIMIT

$$\underline{K_z} = \frac{\cosh\left[2\pi(-11.4 + 12)/156.8\right]}{\cosh\left[2\pi \times \frac{12}{156.8}\right]} = \frac{1.0003}{1.1178} = \underline{0.8949}$$

$$\Longrightarrow \boxed{H} = 2a = \boxed{1.04\,m}$$

Remember that $z < 0$ but $\underline{d > 0}$.

However $z = -d$ at the sea-floor



scour velocity is

u at $z = -d$

For small x ($|x|<1$) can be shown that $e^x \simeq 1+x$ when $|x|<1$, as $x \to 0$

$$\boxed{\sinh x} = \frac{e^x - e^{-x}}{2} = \frac{(\cancel{1}+x) - (\cancel{1}-x)}{2} = \boxed{\frac{2x}{2} = x}$$

$$\boxed{\cosh x} = \frac{e^x + e^{-x}}{2} = \frac{(1+\cancel{x})+(1-\cancel{x})}{2} = \boxed{1}$$

$$\boxed{\tanh x} = \frac{\sinh x}{\cosh x} \rightarrow \boxed{x} \quad \text{as } x \to 0$$

LINEAR WAVE THEORY – SHALLOW WATER - WAVE SPEED

$$\boxed{C = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)} \quad \text{(wave speed for finite depth water)} \tag{90}$$

as $\underline{\frac{d}{L} \to 0}$ $\quad \tanh\left(\frac{2\pi d}{L}\right) \approx \frac{2\pi d}{L}$

$$C = \frac{gT}{2\pi} \cdot \frac{2\pi d}{L} = \frac{gTd}{L} = \frac{gd}{C} \Rightarrow$$

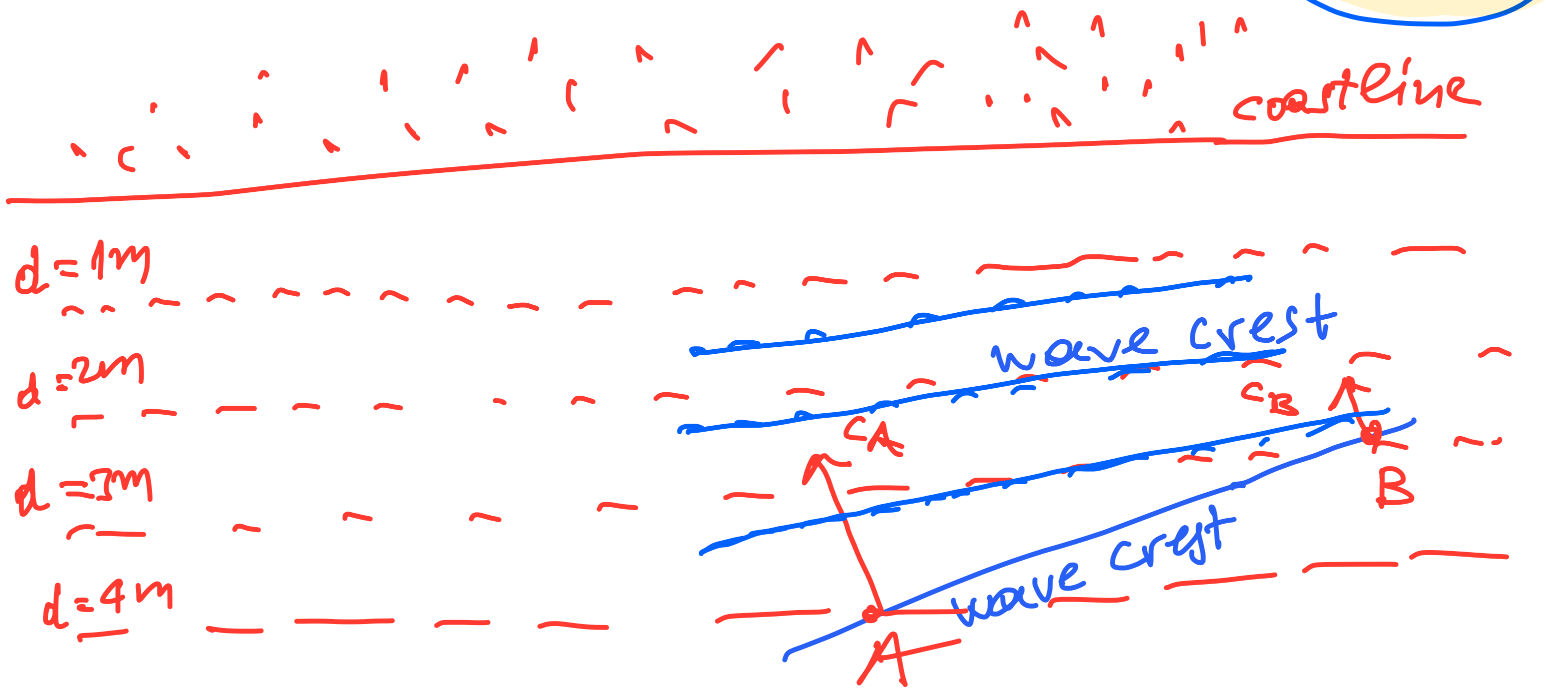
$$\Rightarrow \quad C^2 = gd \Rightarrow \boxed{C = \sqrt{gd}}$$

Note C for shallow H_2O is only a function of $\boxed{d}$

$C = \frac{L}{T}$

$\rightsquigarrow \frac{T}{L} = \frac{1}{C}$

LINEAR WAVE THEORY – SHALLOW WATER - WAVE SPEED – APPLICATION TO REFRACTION



$d_A > d_B \Rightarrow c_A > c_B$

Thus A will propagate faster than B, and that leads to the bending of the waves crests, as can be seen on the video shown on the 1st day of class!

ALWAYS VALID

RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25}$ =0.04	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	← Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]\cdot C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T}e^{\frac{2\pi z}{L}}\sin\theta$
6. Water Particle Accelerations (a) Horizontal	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}}\cos\theta$
7. Water Particle Displacements (a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$ A	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2}e^{\frac{2\pi z}{L}}\sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right)\cos\theta$ B	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2}e^{\frac{2\pi z}{L}}\cos\theta$
8. Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$	$p = \rho g\eta e^{\frac{2\pi z}{L}} - \rho g z$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

LINEAR WAVE THEORY – SHALLOW WATER - VELOCITIES

Transitional

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$$

~ 1 ~ 1

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$$

$\sim 2\pi(z+d)/L$

Shallow

$$\boxed{u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta}$$

$$w = \frac{H\pi}{T}\left(1+\frac{z}{d}\right)\sin\theta$$

as $x \to 0$ $\cosh x \sim 1$

$\sinh x \sim x$

$\tanh x \sim x$

$$u = \frac{H}{2}\frac{gT}{L}\cos\theta = \frac{H}{2}\frac{g}{c}\cos\theta = \frac{H}{2}\frac{g}{\sqrt{gd}}\cos\theta =$$

$$c = \frac{L}{T} = \sqrt{gd} \leftarrow \text{(shallow)}$$

Note $\underline{u}$ does not depend on $\underline{z}$

LINEAR WAVE THEORY – SHALLOW WATER - ACCELERATIONS

Transitional

$$a_x = \frac{g\pi H}{L}\,\frac{\overbrace{\cosh\left[2\pi(z+d)/L\right]}^{\sim 1}}{\underbrace{\cosh(2\pi d/L)}_{\sim 1}}\sin\theta$$

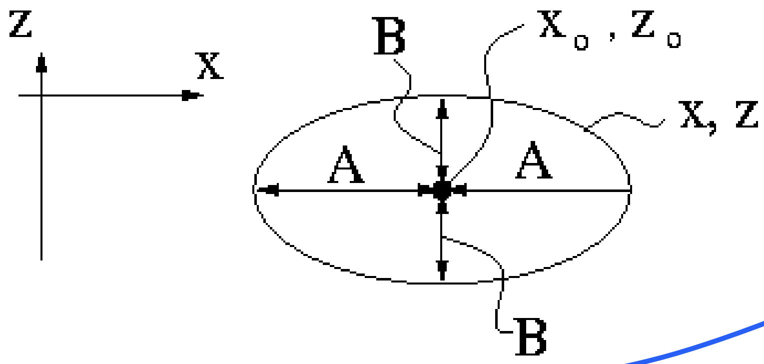
$$a_z = -\frac{g\pi H}{L}\,\frac{\sinh\left[2\pi(z+d)/L\right]}{\cosh(2\pi d/L)}\cos\theta$$

Shallow

$$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$$

$$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1+\frac{z}{d}\right)\cos\theta$$

LINEAR WAVE THEORY – SHALLOW WATER – PARTICLE TRAJECTORIES



$$A = a \cdot \frac{\cosh[k(z_0+d)]}{\sinh(kd)}$$

~ 1

$\sim kd$

$$B = a \cdot \frac{\sinh[k(z_0+d)]}{\sinh(kd)}$$

$A = \frac{a}{kd}$ Independent of depth! (as expected)

$$B = a\,\frac{k(z_0+d)}{kd} = a\left(1+\frac{z_0}{d}\right)$$

it can be shown to be $\boxed{\frac{HT}{4\pi}\sqrt{\frac{g}{d}}}$

$a = \frac{H}{2}$

$k = \frac{2\pi}{L}$

$L = T\sqrt{gd}$

LINEAR WAVE THEORY – SHALLOW WATER – PRESSURES

$$p = \rho g \eta \frac{\overbrace{\cosh[2\pi(z+d)/L]}^{\sim 1}}{\underbrace{\cosh(2\pi d/L)}_{\sim 1}} - \rho g z$$

gage pressure $= p = \underbrace{\rho g \eta}_{\text{wave pressure}} \underbrace{- \rho g z}_{\text{hydrostatic pressure}}$

$$p = \rho g \underbrace{(\eta - z)}$$

RELATIVE DEPTH	SHALLOW WATER $\frac{d}{L} < \frac{1}{25}$	TRANSITIONAL WATER $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	DEEP WATER $\frac{d}{L} > \frac{1}{2}$
1. Wave profile	Same As →	$\eta = \frac{H}{2} \cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2} \cos\theta$	← Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right] \cdot C$	$C_g = \frac{1}{2} C = \frac{gT}{4\pi}$
5. Water Particle Velocity (a) Horizontal	$u = \frac{H}{2} \sqrt{\frac{g}{d}} \cos\theta$	$u = \frac{H}{2} \frac{gT}{L} \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} \cos\theta$	$u = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1 + \frac{z}{d}\right) \sin\theta$	$w = \frac{H}{2} \frac{gT}{L} \frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} \sin\theta$	$w = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \sin\theta$
6. Water Particle Accelerations (a) Horizontal	$a_x = \frac{H\pi}{T} \sqrt{\frac{g}{d}} \sin\theta$	$a_x = \frac{g\pi H}{L} \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} \sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}} \sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1 + \frac{z}{d}\right) \cos\theta$	$a_z = -\frac{g\pi H}{L} \frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} \cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\frac{2\pi z}{L}} \cos\theta$
7. Water Particle Displacements (a) Horizontal	$\xi = -\frac{HT}{4\pi} \sqrt{\frac{g}{d}} \sin\theta$	$\xi = -\frac{H}{2} \frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)} \sin\theta$	$\xi = -\frac{H}{2} e^{\frac{2\pi z}{L}} \sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1 + \frac{z}{d}\right) \cos\theta$	$\zeta = \frac{H}{2} \frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)} \cos\theta$	$\zeta = \frac{H}{2} e^{\frac{2\pi z}{L}} \cos\theta$
8. Subsurface Pressure	$p = \rho g(\eta - z)$	$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$	$p = \rho g \eta e^{\frac{2\pi z}{L}} - \rho g z$

Figure 2-6. Summary of linear (Airy) wave theory--wave characteristics.

LINEAR WAVE THEORY – FINITE DEPTH WATER FORMULAS AS d/L →∞ (DEEP WATER)

$$C = \frac{L}{T} = \frac{gT}{2\pi} \underbrace{\tanh\left(\frac{2\pi d}{L}\right)}_{\sim 1}$$

$$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$$

$$\text{as } x \to \infty \quad \sinh x = \frac{e^x - \cancel{e^{-x}}^{=0}}{2} \sim \frac{e^x}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \sim \frac{e^x}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \sim 1$$

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$$

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$$

$$u = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \cos\theta \qquad e^{\frac{2\pi z}{L}} = e^{kz}$$

$$w = \frac{\pi H}{T} e^{\frac{2\pi z}{L}} \sin\theta$$

$$\frac{\dfrac{e^{k(z+\cancel{d})}}{\cancel{2}}}{\dfrac{\cancel{e^{kd}}}{\cancel{2}}} = e^{kz}$$

$$k = \frac{2\pi}{L}$$

$$\frac{H}{2}\frac{gT}{\left(\frac{gT^2}{2\pi}\right)} = \frac{H}{2}\frac{2\pi}{T} = \frac{\pi H}{T}$$

LINEAR WAVE THEORY – FINITE DEPTH WATER FORMULAS AS d/L → ∞ (DEEP WATER)

$$p = \rho g \eta \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$$

$= kz$

e^{kz} as $kd \to \infty$

$$p = \rho g \eta e^{\frac{2\pi z}{L}} - \rho g z$$

$= kz$

e^{kz}