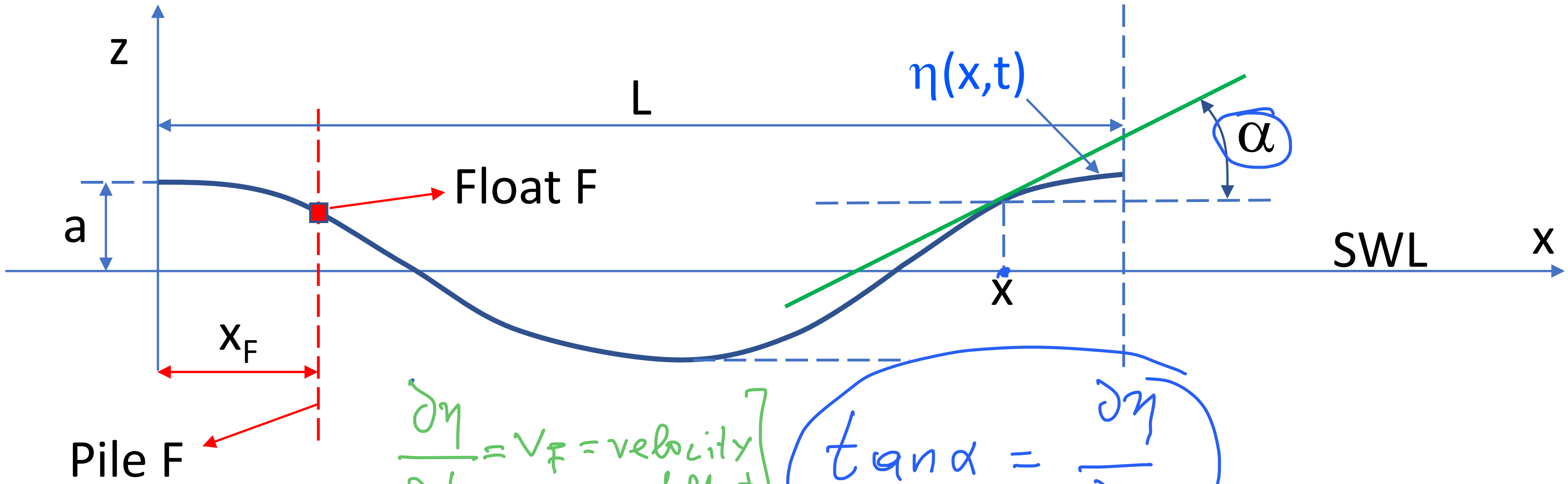
SLOPE OF WAVE



$\frac{\partial \eta}{\partial t} = V_F =$ velocity of float

$$\tan\alpha = \frac{\partial \eta}{\partial x}$$

$$\eta(x,t) = a\cos(kx - \omega t)$$

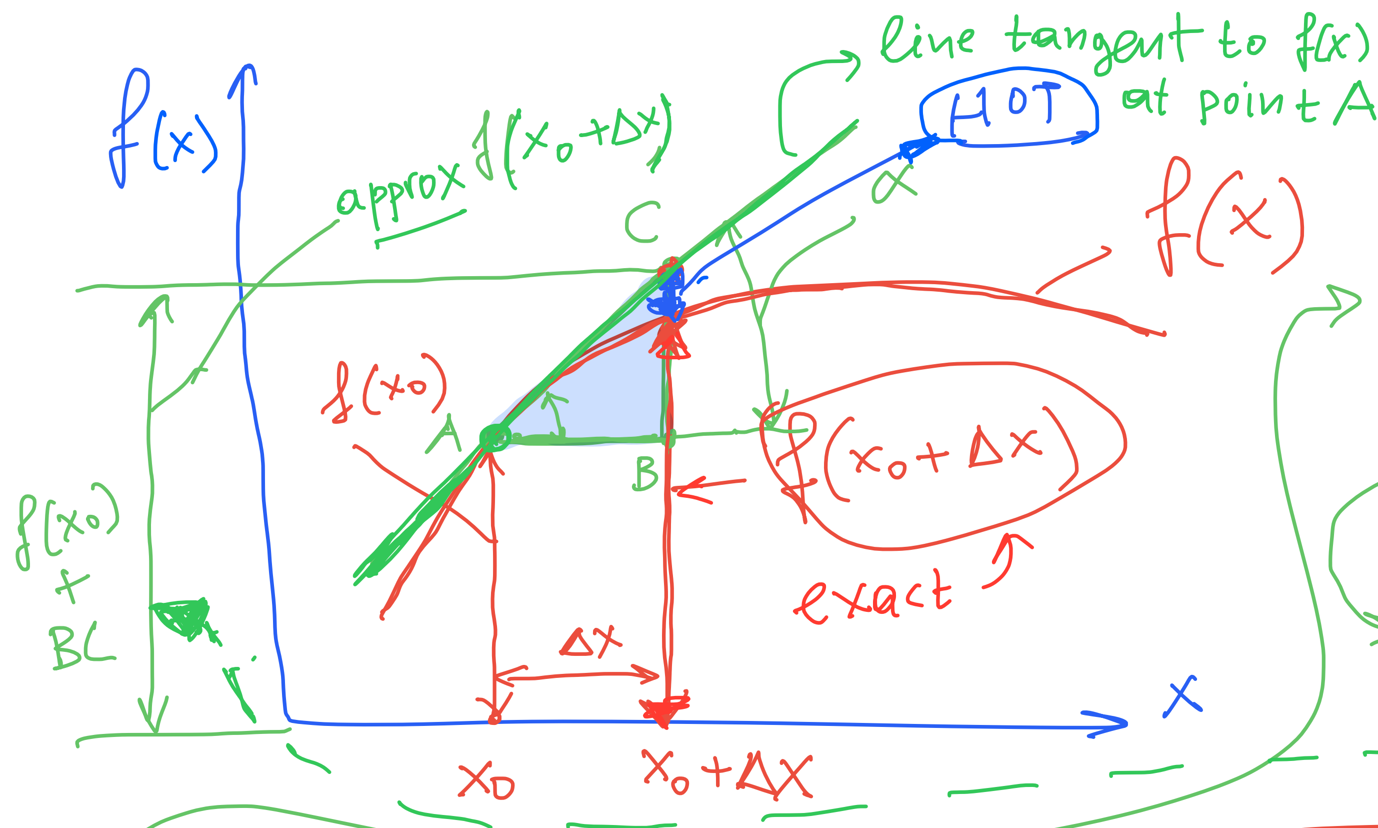
$$\frac{\partial \eta}{\partial x} = a\left[-\sin(\underbrace{kx-\omega t})\right]k = -\underbrace{ak}\sin(kx-\omega t)$$

max slope is $\underline{ak}$

In general

if: $f = A\,\underline{\cos}(kx \pm \omega t)$ or $A\,\underline{\sin}(kx \pm \omega t)$

$\underline{\text{max } f}$: $|A|$ $\quad$ $\underline{\text{min} f}$ $-|A|$



$\tan\alpha = \frac{df}{dx}$

$f(x_0+\Delta x) = f(x_0) + \underline{\tan\alpha(\Delta x)}$

$\tan\alpha(\Delta x) = BC$

$\underline{f(x_0) + BC}$

Taylor expansion:

$$f(x_0+\Delta x) = f(x_0) + \frac{df}{dx}\Big|_{x=x_0}\Delta x + \boxed{\frac{d^2f}{dx^2}\Big|_{x=x_0}\frac{\Delta x^2}{2!} + \frac{d^3f}{dx^3}\Big|_{x=x_0}\frac{\Delta x^3}{3!} + \cdots}$$

If: $\Delta x = 0.1 \to \Delta x^2 = 0.01 \to \Delta x^3 = 0.001$

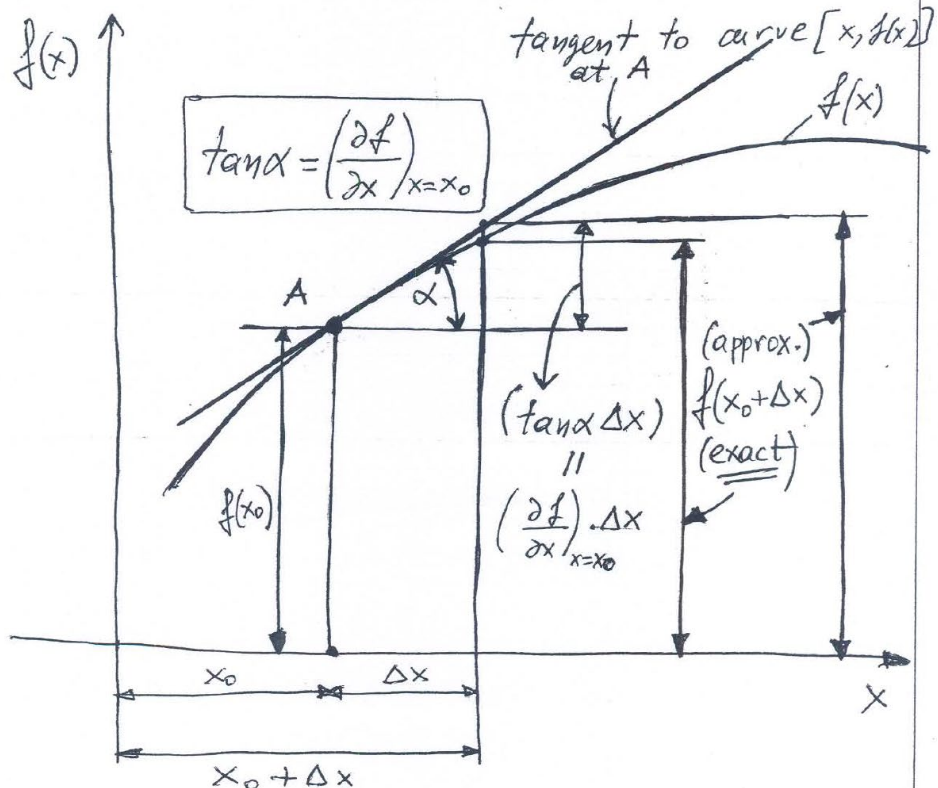
Higher Order Terms $\boxed{\text{HOT}}$ can be ignored for small Δx

Linear Taylor expansion

$$f(x_0+\Delta x) \approx f(x_0) + \frac{df}{dx}\Delta x$$

PHYSICAL MEANING OF 1st ORDER APPROXIMATION

$$f(x_0+\Delta x) \approx f(x_0) + \left(\frac{\partial f}{\partial x}\right)_{x=x_0} \cdot \Delta x$$



Taylor expansion:

$$f(x_0+\Delta x) = f(x_0) + f'(x_0)\frac{\Delta x}{1!} + f''(x_0)\frac{(\Delta x)^2}{2!} + $$
$$+ f'''(x_0)\cdot\frac{(\Delta x)^3}{3!} + \cdots$$

EXAMPLE ON 1st ORDER APPROXIMATION

Provide an *approximation* (first order expansion) of the following expression, H, for *small* values of a. Then apply the formula for $a = 0.1$ and compare the answer to the exact value. Determine the percentage error: $(H_{approx} - H_{exact})/H_{exact} = ?$

α α

$$H = \sqrt{1 + \alpha} = ? \tag{1}$$

$$f(x_0 + \Delta x) = f(x_0) + \frac{df}{dx} \Delta x$$

$x_0 \to 1$, $\Delta x \to \alpha$

$$\frac{d}{dx} x^b = b\, x^{b-1}$$

$$f(x) = \sqrt{x}$$

$$\boxed{\sqrt{1+\alpha}} = \sqrt{1} + \frac{df}{dx} \cdot \alpha = \boxed{1 + \frac{\alpha}{2}}$$

$$\frac{df}{dx} = \frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\left.\frac{df}{dx}\right|_{x=1} = \frac{1}{2}$$

EXAMPLE ON 1st ORDER APPROXIMATION

Provide an *approximation* (first order expansion) of the following expression, H, for *small* values of a. Then apply the formula for $a = 0.1$ and compare the answer to the exact value. Determine the percentage error: $(H_{approx} - H_{exact})/H_{exact} =?$

$$H = \sqrt{1+\alpha} =? \tag{1}$$

$$\sqrt{1+\alpha} = 1 + \frac{\alpha}{2}$$

Try $\alpha = 0.1$ $\quad \sqrt{1+\alpha} = 1 + \frac{0.1}{2} = 1.05 = H_{approx.}$

H_{exact} $\quad \sqrt{1+\alpha} = 1.0488$

$$\% \text{ error} = \frac{H_{approx.} - H_{exact}}{H_{exact}} = \frac{1.05 - 1.0488}{1.0488} = 0.00114 = 0.11\%$$

The above result can be generalized (using Taylor exp.) to the binomial approximation;

$$(1+\alpha)^r \simeq 1 + r\alpha$$

$|\alpha| << 1$; r can be any real number

special case $(r=-1)$

$$\boxed{\frac{1}{1+\alpha}} = (1+\alpha)^{-1} = \boxed{1-\alpha}$$

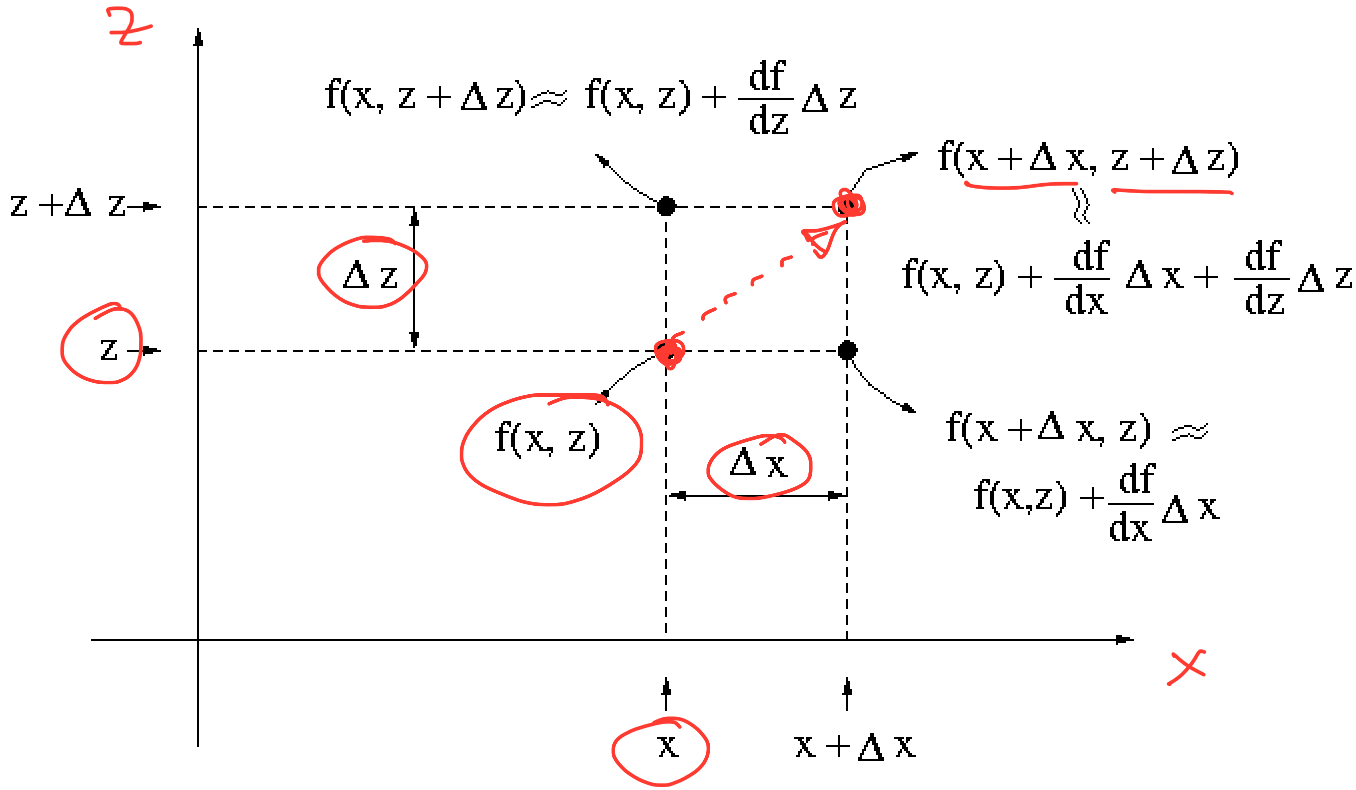
$e^x \approx 1 + x$ $\quad$ $\tan\theta \approx \theta$

Based on Taylor expansion it can be shown

θ in RADIANS

1st ORDER APPROXIMATION IN 2-D

f can be any function of x, z
e.g. pressure, velocity of fluid, etc.



$$\boxed{f(x+\Delta x, z+\Delta z) \approx f(x,z) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial z}\Delta z} + \text{HOT} \qquad (19)$$

those partial derivaties must be evaluated at the location (x, z)

EXAMPLE ON 1st ORDER APPROXIMATION IN 2-D

$$f(x,z) = x^3 + xz^2$$

$$\left.\begin{array}{l} x = 1 \\ z = 1 \end{array}\right\} \quad f = 1^3 + 1.1^2 = 2 = f(1,1)$$

$$f(\underbrace{1.1}_{1+0.1}, \underbrace{1.1}_{1+0.1}) = f(1,1) + \frac{\partial f}{\partial x} \underbrace{\Delta x}_{=0.1} + \frac{\partial f}{\partial z} \underbrace{\Delta z}_{=0.1}$$

$$\frac{\partial f}{\partial x} = 3x^2 + z^2 \sim \left.\frac{\partial f}{\partial x}\right|_{1,1} = 3.1^2 + 1^2 = 4$$

$$\frac{\partial f}{\partial z} = 2xz \sim \left.\frac{\partial f}{\partial z}\right|_{1,1} = 2.1.1 = 2$$

$$f(1.1, 1.1) = 2 + 4.\Delta x + 2\Delta z =$$

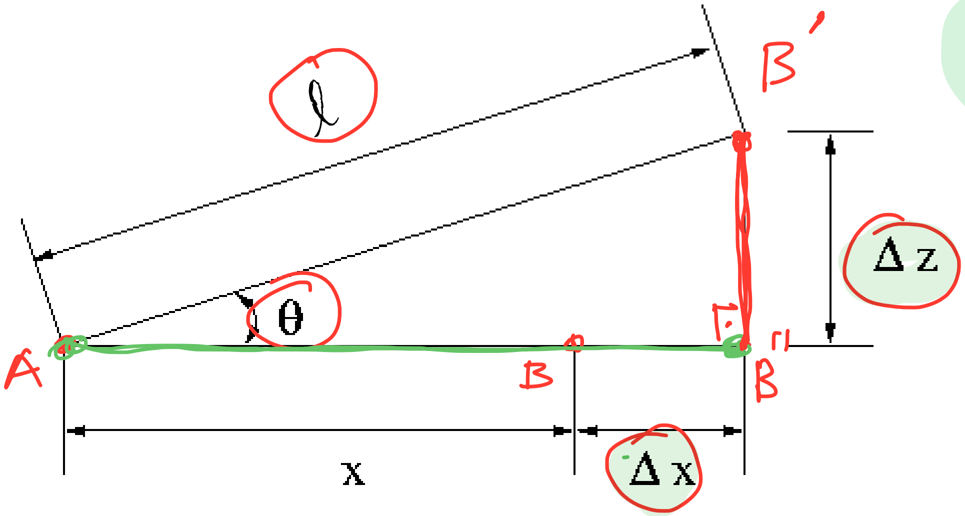
$$= 2 + 4 * (0.1) + 2 * (0.1) = \boxed{2.6} \leftarrow \text{APROX.}$$

$$f(1.1, 1.1) = \boxed{2.662} \text{ (exact)}$$

$$\% \text{ error} = -2.3\%$$

SOME USEFUL APPROXIMATIONS

Q: How can ℓ and θ be approximated for ``small'' values of $\Delta x, \Delta z$ compared to x?



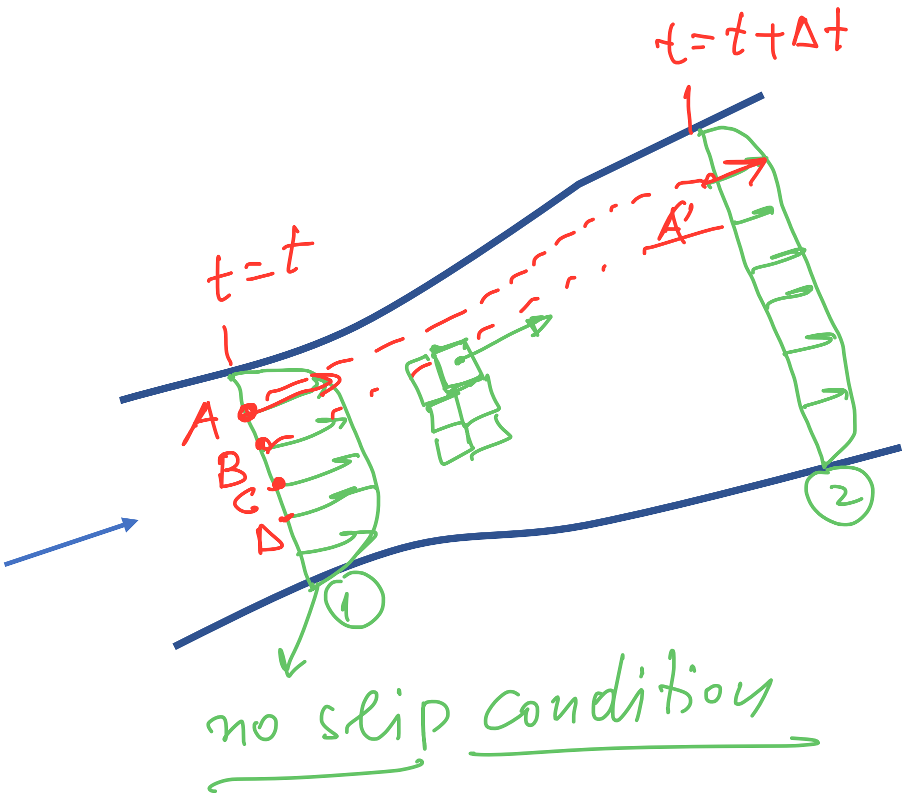
$$\ell \approx 1 + \Delta x = AB''$$

θ in RADIANS

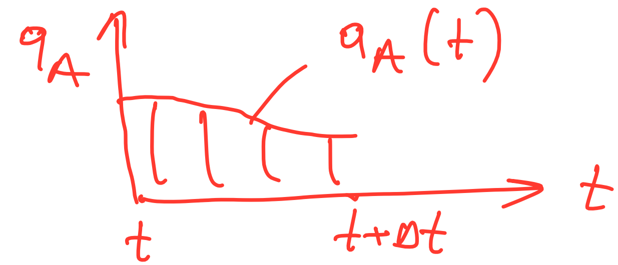
$$\tan\theta \approx \theta = \frac{\Delta z}{x + \Delta x} = \frac{\Delta z}{x\left(1 + \frac{\Delta x}{x}\right)} = \frac{\Delta z}{x}\,\frac{1}{\left(1 + \frac{\Delta x}{x}\right)} \approx$$

$$\frac{\Delta z}{x}\left[1 - \frac{\Delta x}{x}\right] = \frac{\Delta z}{x} - \cancel{\frac{\Delta z\,\Delta x}{x^2}}\ \text{HOT} = \frac{\Delta z}{x}$$

TWO APPROACHES TO DEFINE A FLOW-FIELD



LAGRANGIAN APPROACH



Follows the particle and records ~~its~~ velocity

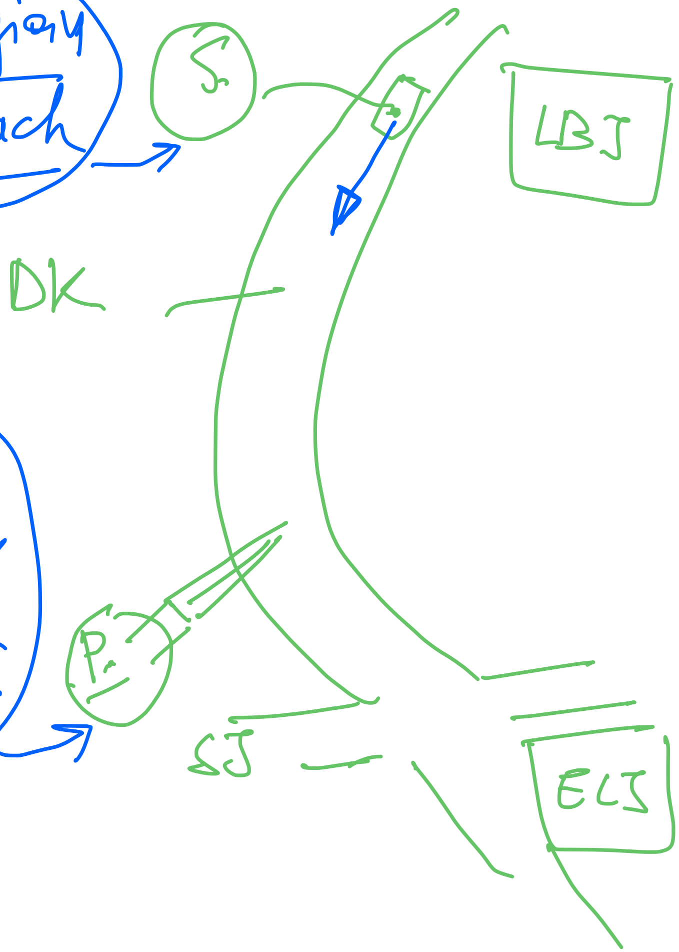
EULERIAN APPROACH

It does not follow the particles.
Record velocities of particles that pass through a fixed point.

EXAMPLE OF LAGRANGIAN VS. EULERIAN APPROACH

Lagrangian Approach

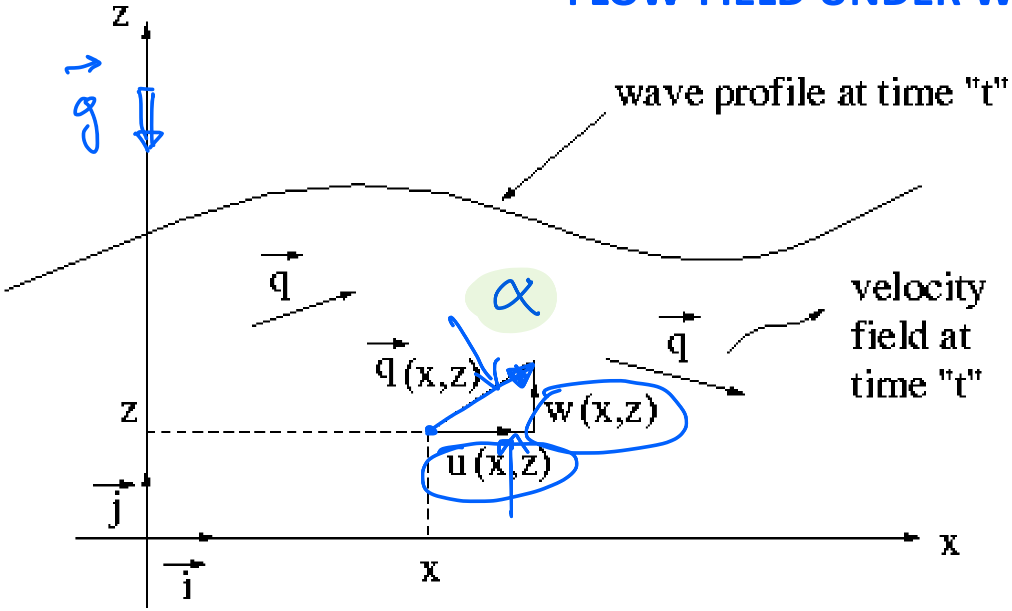
Eulerian approach



S: Spyros in his car, checks his speed on the speedometer

P: Police measure speeds of cars passing through fixed location

FLOW-FIELD UNDER WAVE SURFACE



Q: What laws should the fluid field follow?

$\begin{bmatrix} u(x,z) \\ w(x,z) \end{bmatrix}$ components of velocity vector $\vec{q}$

Laws of Physics!

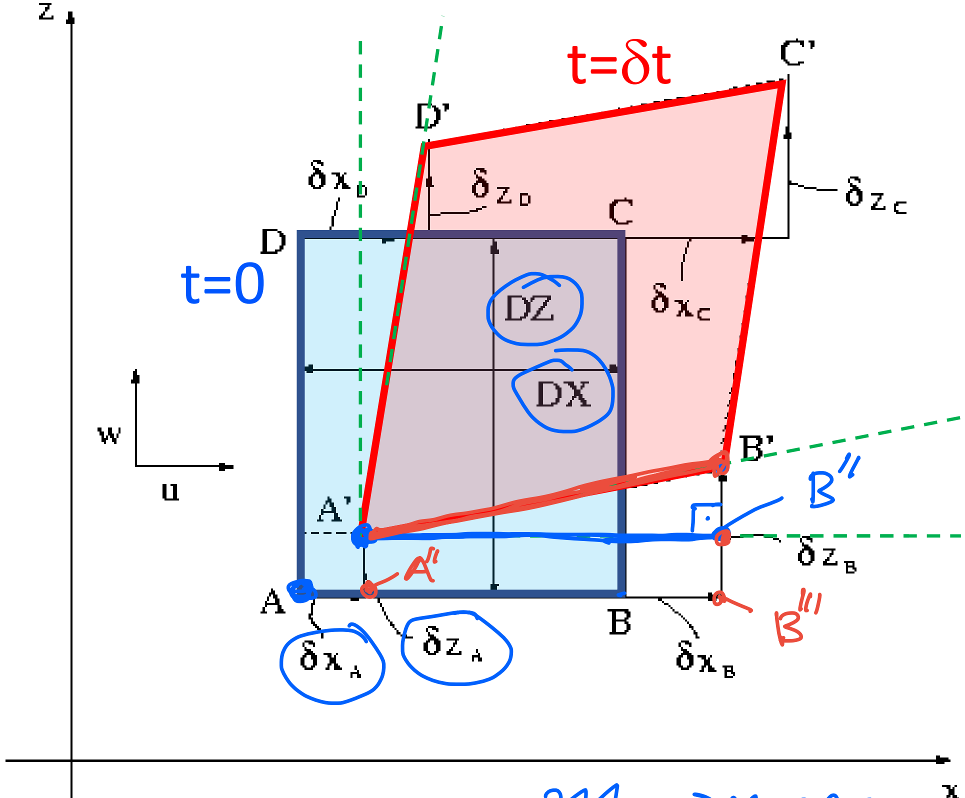
$$q = |\vec{q}| = \sqrt{u^2 + w^2}$$

magnitude of velocity vector

$$\tan\alpha = \frac{w}{u}$$

$\rightsquigarrow$ angle α of velocity vector with horizontal axis x

DEFORMATION OF FLUID PARTICLE & CONSERVATION OF MASS



$$m_{ABCD} = m_{A'B'C'D'}$$

$$\rho\, \forall_{ABCD} = \rho' \forall_{A'B'C'D'}$$

Incompressible flow

$$\rho = \rho'$$

$$\forall_{ABCD} = \forall_{A'B'C'D'}$$

$$DX \cdot DZ = (A'B'C'D')$$

density $\rightarrow \rho = \dfrac{m}{\forall}$, $m \rightarrow$ mass, $\forall \rightarrow$ volume

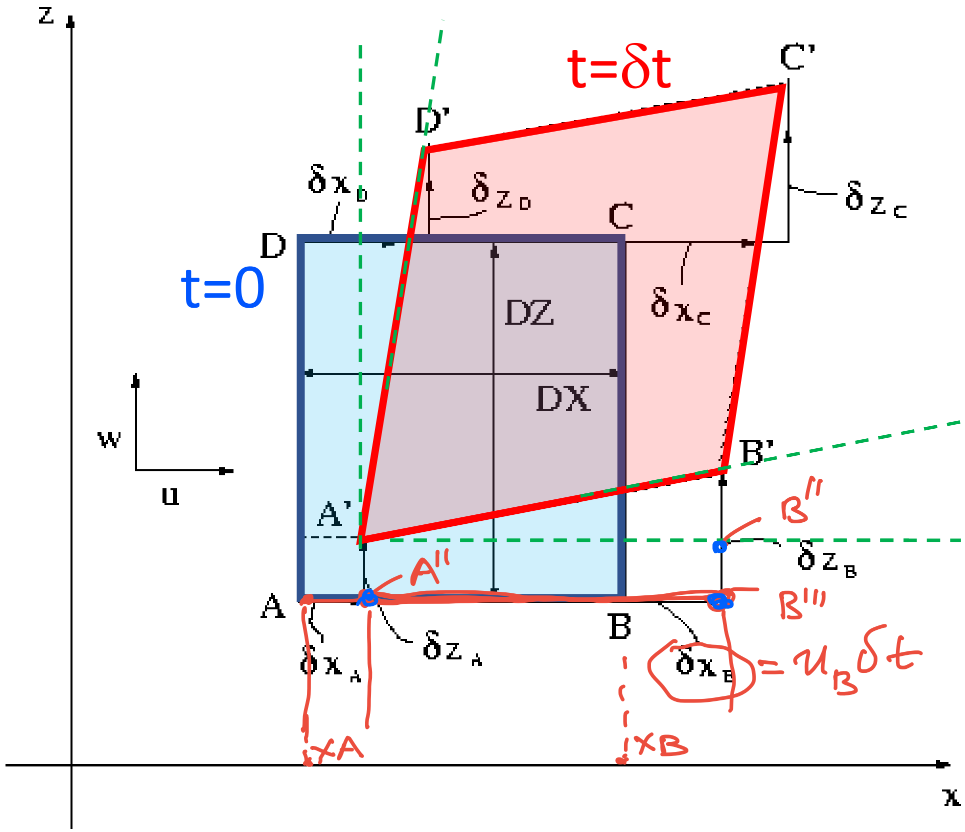
$$(A'B'C'D') \approx (A'B')(A'D')$$

$$A'B' \approx A'B'' = A''B'''$$

If u_A, w_A are the velocities at A

then $\begin{cases} \delta x_A = u_A\, \delta t \\ \delta z_A = w_A\, \delta t \end{cases}$

DEFORMATION OF FLUID PARTICLE & CONSERVATION OF MASS



$\delta x_B = u_B \delta t$

$A''B''' = AB''' - AA'' = (AB + BB''') - AA''$

$= AB + BB''' - AA'' =$

$= Dx + u_B \delta t - u_A \delta t =$

$= Dx + (u_B - u_A)\delta t$

Taylor in 2-D: $u(x_B, z_B) = u(x_A, z_A) + \frac{\partial u}{\partial x}\underbrace{\Delta x}_{Dx} + \frac{\partial u}{\partial z}\underbrace{\Delta z}_{=0\ (z_B = z_A)}$

$\sim\!\!> \quad u_B = u_A + \frac{\partial u}{\partial x} Dx$

$\sim\!\!> \quad (u_B - u_A) = \frac{\partial u}{\partial x} Dx$

$A'B' = A'B'' = A''B''' = Dx + \frac{\partial u}{\partial x} Dx\,\delta t$

$= Dx\left[1 + \frac{\partial u}{\partial x}\delta t\right]$

CONTINUITY EQUATION FOR INCOMPRESSIBLE FLUIDS

$$A'B' = Dx\left[1+\frac{\partial u}{\partial x}\delta t\right]$$

Similarly: $$A'D' = Dz\left[1+\frac{\partial w}{\partial z}\delta t\right]$$

$$(A'B'C'D') = (A'B')(A'D') = Dx\,Dz\left[1+\frac{\partial u}{\partial x}\delta t\right]\left[1+\frac{\partial w}{\partial z}\delta t\right]$$

Due to conservation of mass

$$= (ABCD) = Dx\,Dz$$

$$\cancel{Dx\,Dz} = \cancel{Dx\,Dz}\left[1+\frac{\partial u}{\partial x}\delta t + \frac{\partial w}{\partial z}\delta t + \frac{\partial u}{\partial x}\frac{\partial w}{\partial z}(\delta t)^2\right]$$

$$\cancel{1} = \cancel{1} + \frac{\partial u}{\partial x}\delta t + \frac{\partial w}{\partial z}\delta t + \cancel{\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}(\delta t)^2}$$

HOT

$$\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)\delta t = 0$$

$$\Longrightarrow \boxed{\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0}$$

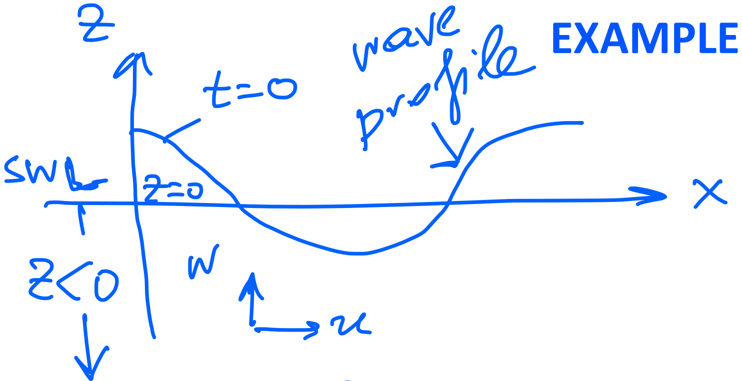
Continuity equation

must be valid at ALL points and at ALL times

- Valid for steady and unsteady flow
- valid for inviscid or viscous flow

In 3-D: $$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0}$$

EXAMPLE ON CONTINUITY EQUATION



$$\frac{de^x}{dx} = e^x$$

Flow-field under the wave can be expressed as follows: (we will prove that later)

- $u = e^{\lambda z}\cos(kx)k$
- $w = \lambda e^{\lambda z}\sin(kx)$

$\underline{\lambda = ?}$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = e^{\lambda z}\left[-\sin(kx)k\right]k = -e^{\lambda z}k^2\sin(kx)$$

$$\frac{\partial w}{\partial z} = \lambda e^{\lambda z}\lambda\sin(kx) = e^{\lambda z}\lambda^2\sin(kx)$$

$$-e^{\lambda z}k^2\sin(kx) + e^{\lambda z}\lambda^2\sin(kx) = 0$$

$$e^{\lambda z}\sin(kx)\left[-k^2+\lambda^2\right] = 0$$

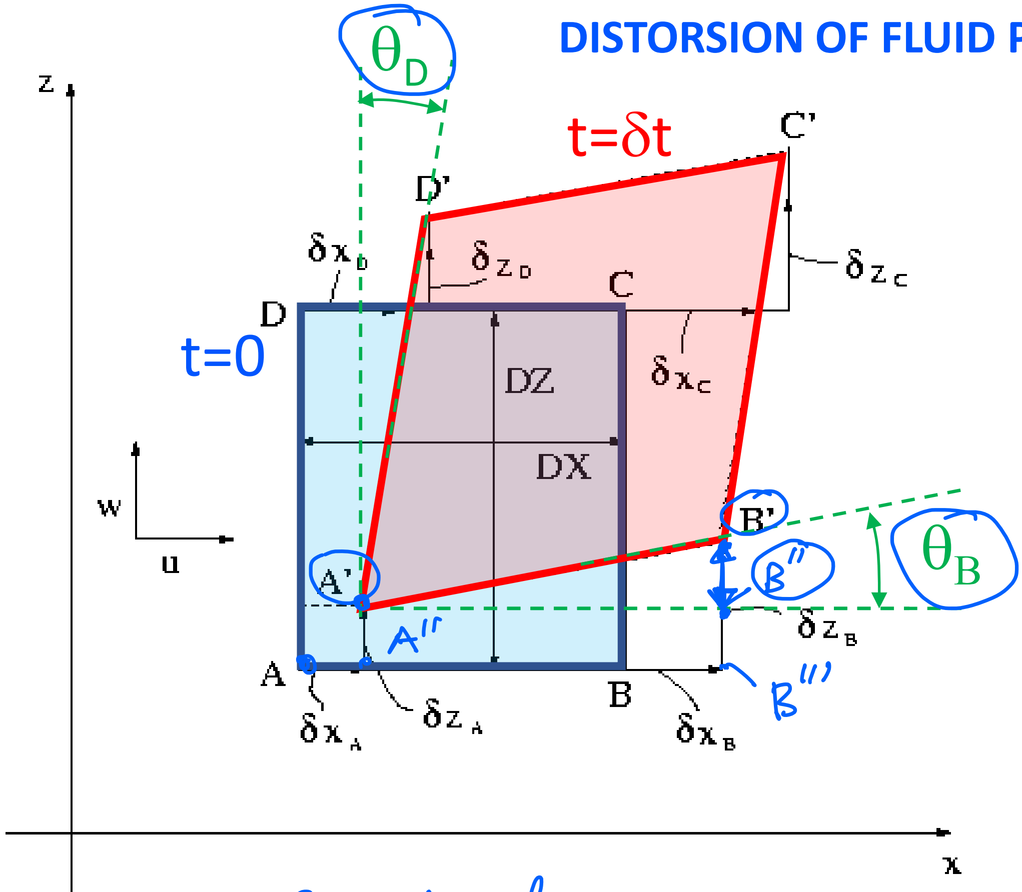
must be valid for ALL x, z

$$-k^2+\lambda^2=0 \rightsquigarrow \underline{\lambda = \pm k}$$

if $\lambda = -k$ then $e^{\lambda z} = e^{-kz}$ would blowup as $\boxed{z \to -\infty}$ → NOT Physical

Thus $\boxed{\lambda = +k}$ provides a physical solution

DISTORSION OF FLUID PARTICLE



$$\vartheta_B \approx \tan\vartheta_B = \frac{(B'B'')}{(A'B'')}$$

$$B'B'' = \underbrace{B'B'''}_{w_B\,\delta t} - \underbrace{B''B'''}_{= \underbrace{A'A''}_{w_A\,\delta t}}$$

$$= \underbrace{(w_B - w_A)}\,\delta t = \frac{\partial w}{\partial x} Dx\,\delta t$$

We already have from above: $A'B'' = Dx\left[1+\frac{\partial u}{\partial x}\delta t\right]$

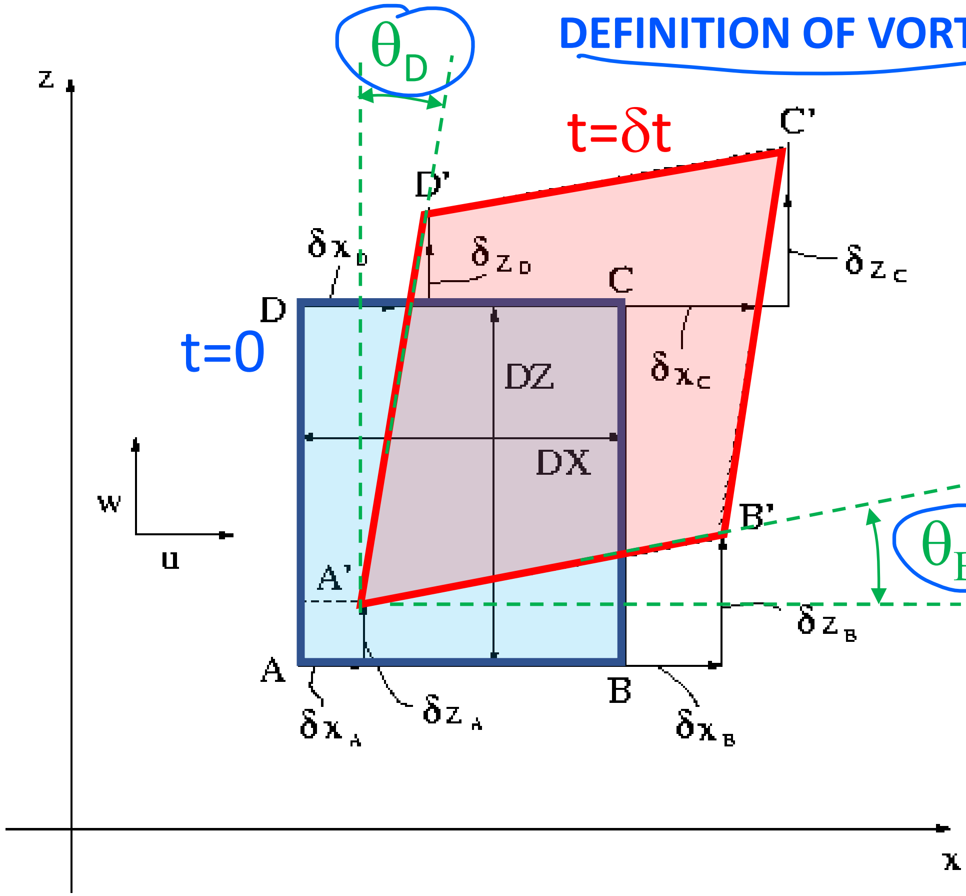
$$\boxed{\vartheta_B} = \frac{\frac{\partial w}{\partial x}\cancel{Dx}\,\delta t}{\cancel{Dx}\left[1+\frac{\partial u}{\partial x}\delta t\right]} = \frac{\frac{\partial w}{\partial x}\delta t}{\left[1+\frac{\partial u}{\partial x}\delta t\right]} = \frac{\partial w}{\partial x}\delta t\left[1-\frac{\partial u}{\partial x}\delta t\right] =$$

$$= \frac{\partial w}{\partial x}\delta t - \cancel{\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\delta t^2} = \boxed{\left(\frac{\partial w}{\partial x}\right)\delta t}$$

HOT

We used binomial approx $\frac{1}{1+\alpha} = 1-\alpha$

DEFINITION OF VORTICITY



$$\boxed{\vartheta_D = \left(\frac{\partial u}{\partial z}\right)\delta t}$$

similarly we get

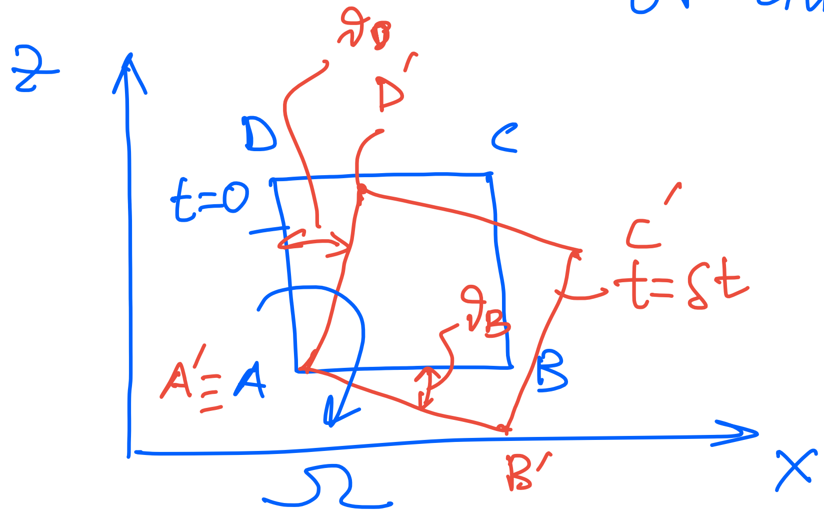
$$\boxed{\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}}$$

$$\boxed{\omega} = \frac{\vartheta_D}{\delta t} - \frac{\vartheta_B}{\delta t} = \boxed{\frac{\vartheta_D - \vartheta_B}{\delta t}}$$

mathematical definition of vorticity, ω

PHYSICAL MEANING OF VORTICITY

Special case: Particle subject to "spin" or angular speed Ω



$$\omega = \frac{\vartheta_D - \vartheta_B}{\delta t}$$

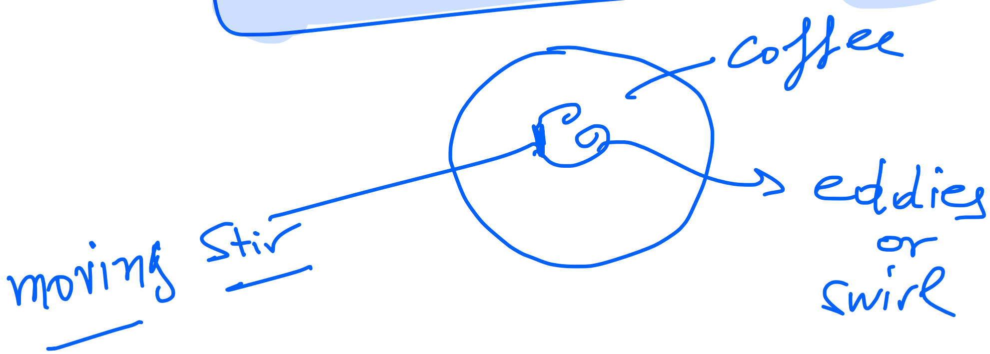
$$\vartheta_D = \Omega\,\delta t$$

$$\vartheta_B = -\Omega\,\delta t$$

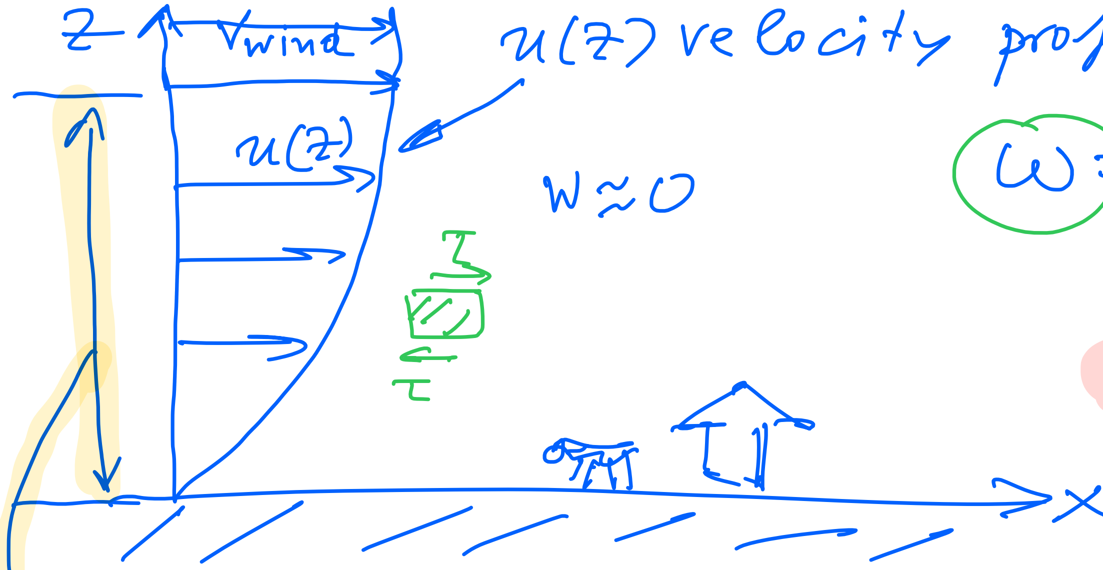
$$\Rightarrow \omega = \frac{\Omega \cancel{\delta t} - (-\Omega \cancel{\delta t})}{\cancel{\delta t}}$$

$$\sim \boxed{\omega = 2\Omega}$$

Vorticity is directly related to spin of particle



} other "words" for vorticity



$u(z)$ velocity profile

$$\omega = \frac{\partial u}{\partial z} - \underbrace{\cancel{\frac{\partial w}{\partial x}}}_{=0} = \frac{\partial u}{\partial z}$$

$\omega = 0$ outside the bl

$\omega \neq 0$ inside the bl

ω_{max} at the ground

boundary layer (bl)

τ: shear stress of fluid particles

$$\tau = \mu \frac{\partial u}{\partial z} \rightsquigarrow \boxed{\tau = \mu \cdot \omega}$$

μ: dynamic (or absolute) viscosity

ω is propotional to τ

ω is significant where τ is significant

DEFINITION OF IRROTATIONAL FLOW

$$\boxed{\omega = 0}$$