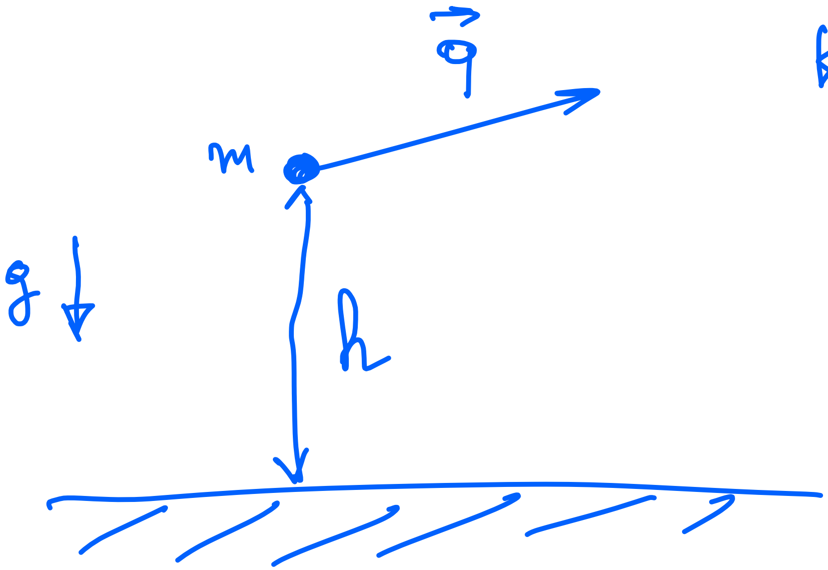
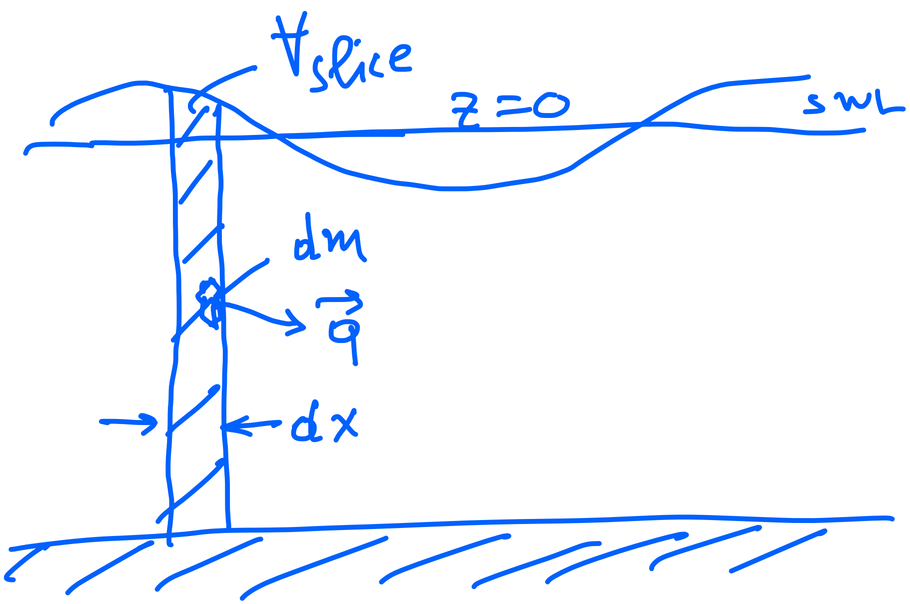
WAVE ENERGY



Kinetic energy: $K.E. = \frac{1}{2} m q^2$

Potential " : $P.E. = mgh$

Total Energy: $T.E. = K.E + P.E. = \frac{1}{2} m q^2 + mgh$

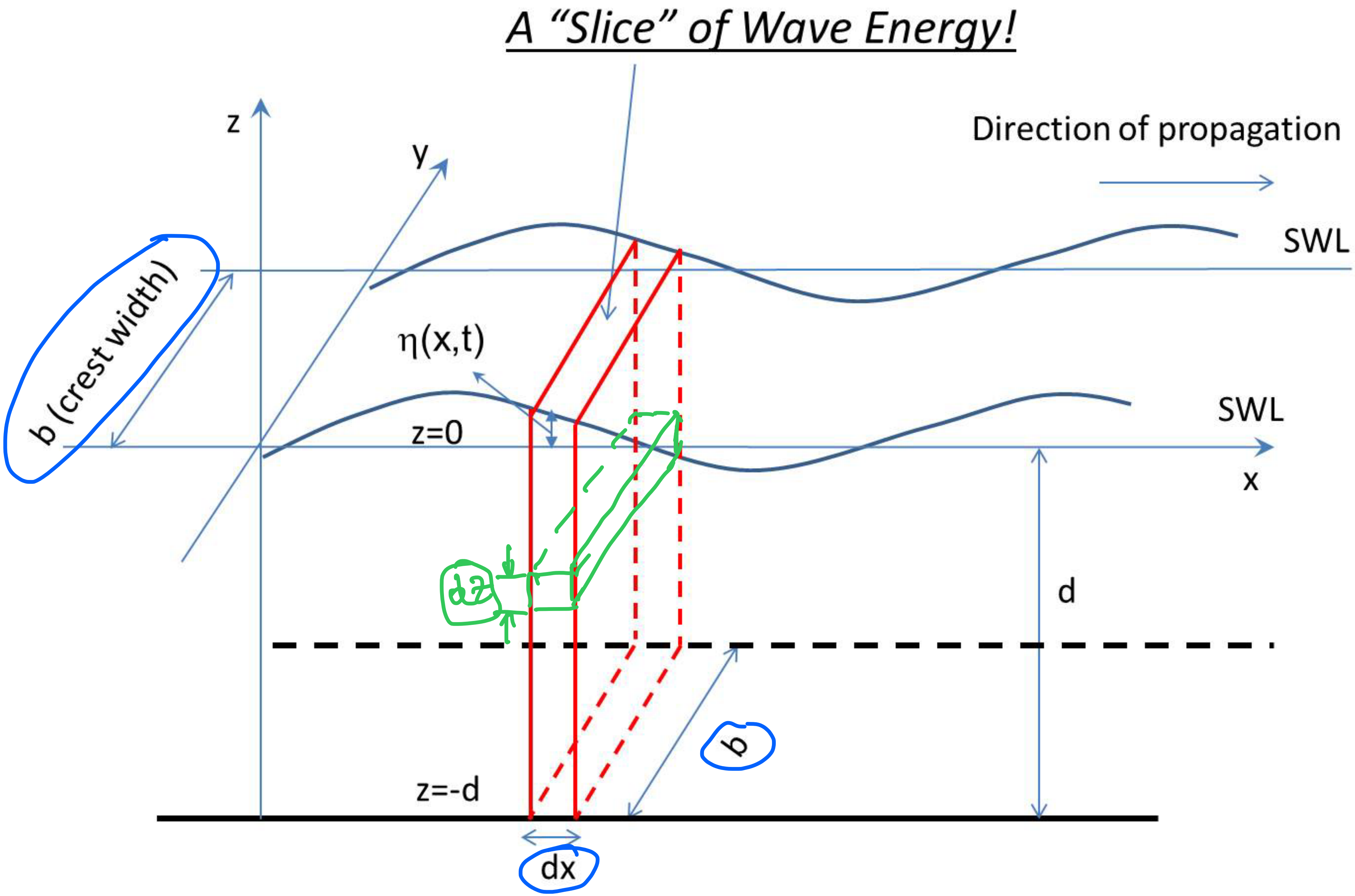


$$\int_{\forall_{slice}} dm \frac{q^2}{2} + dm\, g\, z$$

$$dm = \rho\, d\forall$$

$$\boxed{\underline{T.E._{slice}} = \int_{\forall_{slice}} \left(\underline{\rho g} z + \underline{\tfrac{\rho}{2}} q^2\right) d\forall}$$

Perspective view of a plane progressive wave



WAVE ENERGY

$$T.E._{slab} = \int_{\forall_{slab}} \left(\rho g z + \rho \frac{q^2}{2}\right) \underbrace{d\forall}_{b.dx.dz} = \underline{b.dx.dz} \left[\rho g z + \rho \frac{q^2}{2}\right]$$

$$T.E._{slice} = \int_{-d}^{\eta} \rho \left[g z + \frac{q^2}{2}\right] \underline{dz\, b\, dx} =$$

$$= b.dx \int_{-d}^{\eta} \rho \left[g z + \frac{q^2}{2}\right] dz$$

$$\bar{E}_{slice} = \frac{T.E._{slice}}{b dx} = \int_{-d}^{\eta} \rho \left[g z + \frac{q^2}{2}\right] dz$$

(energy density
or specific energy

$b\,dx$ → area of footprint of the slice

WAVE ENERGY

$$\boxed{\overline{E}^k_{slice}} = \int_{-d}^{\eta} \rho \frac{q^2}{2} dz = \int_{-d}^{\eta} \rho \frac{a^2\omega^2}{2} e^{2kz} dz$$

In deep $H_2O \Rightarrow q = a\omega e^{kz}$

$$\rightarrow \rho \frac{a^2\omega^2}{2} \frac{1}{2k} \int_{-d}^{\eta} e^{2kz} d(2kz) = \rho \frac{a^2\omega^2}{4k} e^{2kz}\Big|_{-d}^{\eta} =$$

$$= \frac{\rho a^2\omega^2}{4k} \Big[\underbrace{e^{2k\eta}}_{=1} {}^{\approx 0} - \underbrace{e^{-2kd}}_{=0} \Big] = \frac{\rho a^2 \omega^2}{4k} = \boxed{\frac{\rho a^2 g}{4}}$$

$\omega^2/k = g$ (dispersion relationship for deep H_2O)

$$\overline{E}^{P}_{slice} = \int_{-d}^{\eta} \rho g z\, dz = \rho g \left.\frac{z^2}{2}\right|_{-d}^{\eta} = \rho g \left[\frac{\eta^2}{2} - \frac{(-d)^2}{2}\right] =$$

$$= \boxed{\rho g \frac{\eta^2}{2}} \boxed{-\rho g \frac{d^2}{2}}$$

due to waves

potential energy of standing water

$$\eta = a\cos(kx - \omega t)$$

$$E^{P}_{slice} = \rho g \frac{a^2 \cos^2(kx-\omega t)}{2}$$

Trig Id

$$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

$$= \rho g \frac{a^2}{2} \frac{1+\cos(2kx-2\omega t)}{2} =$$

$$= \rho g \frac{a^2}{4}\left[1 + \cos(2kx - 2\omega t)\right]$$

Let's take $x=0$ $\rightsquigarrow$ $E^{P}_{slice} = \rho g \frac{a^2}{4}\left[1+\cos(2\omega t)\right]$

$$= \rho g \frac{a^2}{4} + \rho g \frac{a^2}{4} \cos(2\omega t)$$

Average over time: $\Longleftarrow$

$$\boxed{\overline{E}^{P}_{slice} = \rho g \frac{a^2}{4}}$$

$$\bar{E}_{slice} = \bar{E}^{p}_{slice} + \bar{E}^{k}_{slice} = \rho g \frac{a^2}{2}$$

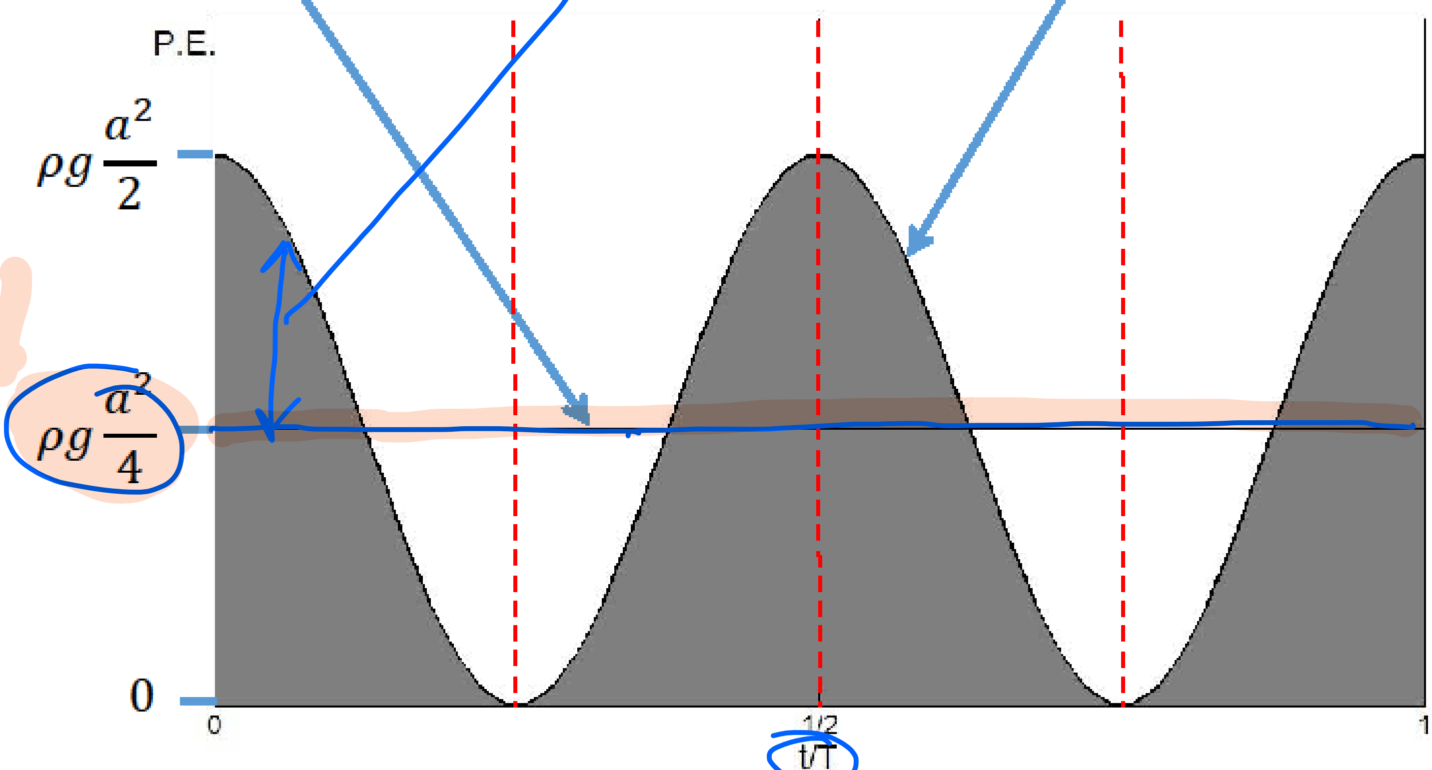
$$\boxed{\bar{E}_{slice} = (\rho g H^2)/8}$$

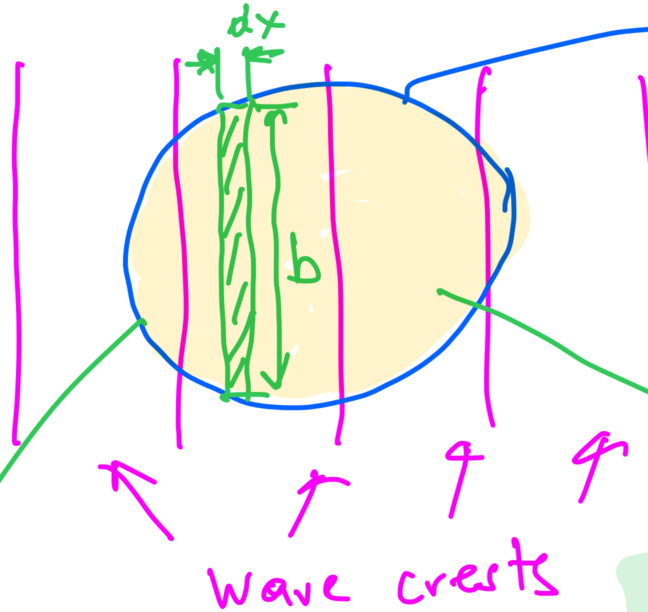
$$\bar{E}^{p}_{slice} = \bar{E}^{k}_{slice} = \rho g \frac{a^2}{4}$$

The potential energy (P.E.) under a sinusoidal wave

mean value of $P.E. = \rho g \frac{a^2}{4}$

$P.E.(t) = \rho g \frac{a^2}{4}\cos(2\omega t) + \rho g \frac{a^2}{4}$





top view of a column of water going from bottom to top

$$\bar{E}_{slice} = \rho g H^2/8 = \frac{E_{slice}}{b\,dx}$$

$$E_{column} = \sum E_{slice} = \sum (\rho g H^2/8)\, b\,dx = \rho g H^2/8 \boxed{\sum b\,dx} = \frac{\rho g H^2}{8} \boxed{A}$$

wave energy inside column

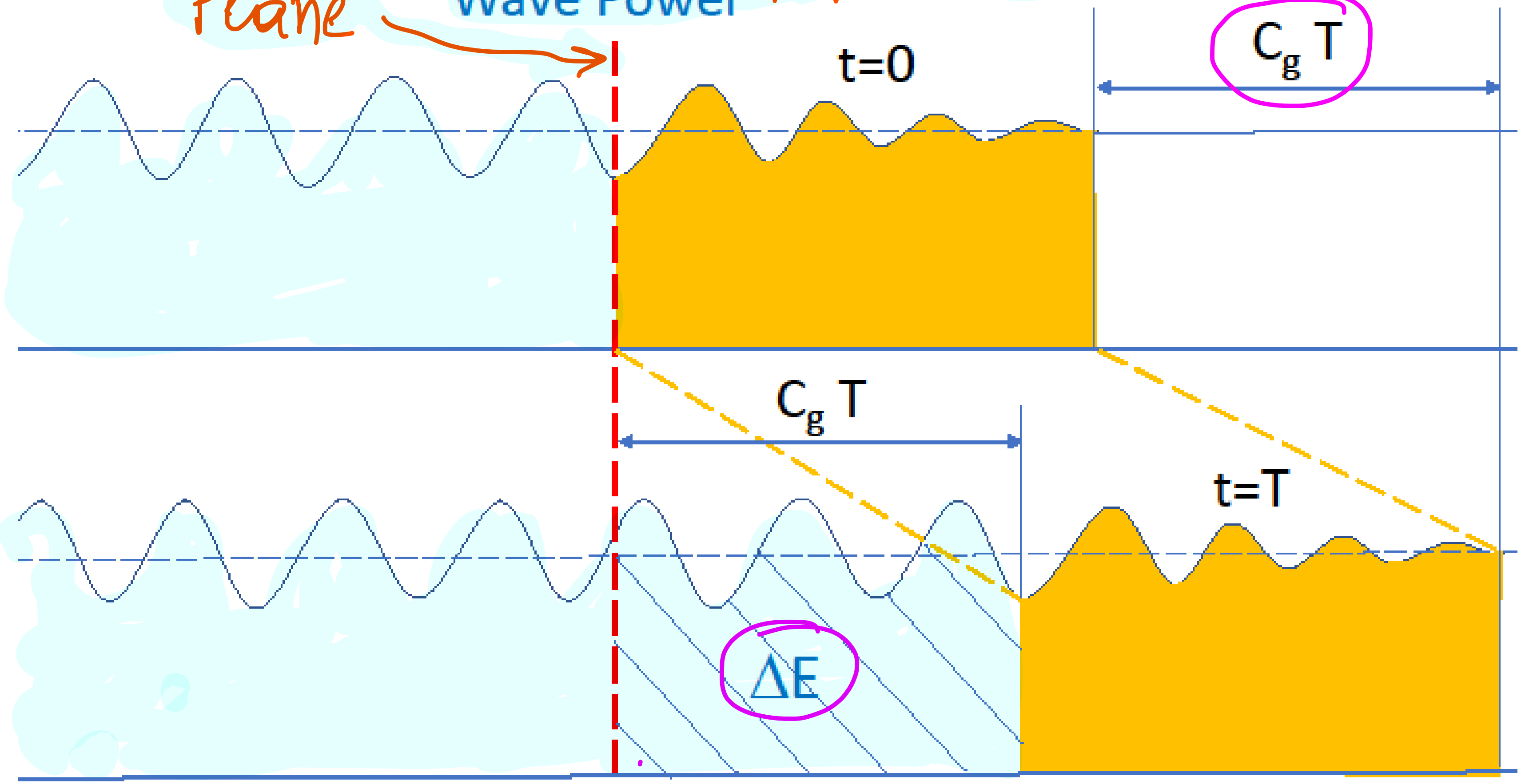
A = sectional area of column of water

$$\frac{E_{column}}{A} \Rightarrow \bar{E} \text{ (specific energy or Energy density)} = \frac{\rho g H^2}{8}$$

The formula $\bar{E} = \frac{\rho g H^2}{8}$ applies to transitional and shallow water as well.

WAVE POWER

Wave Power = power of wave crossing fixed plane.



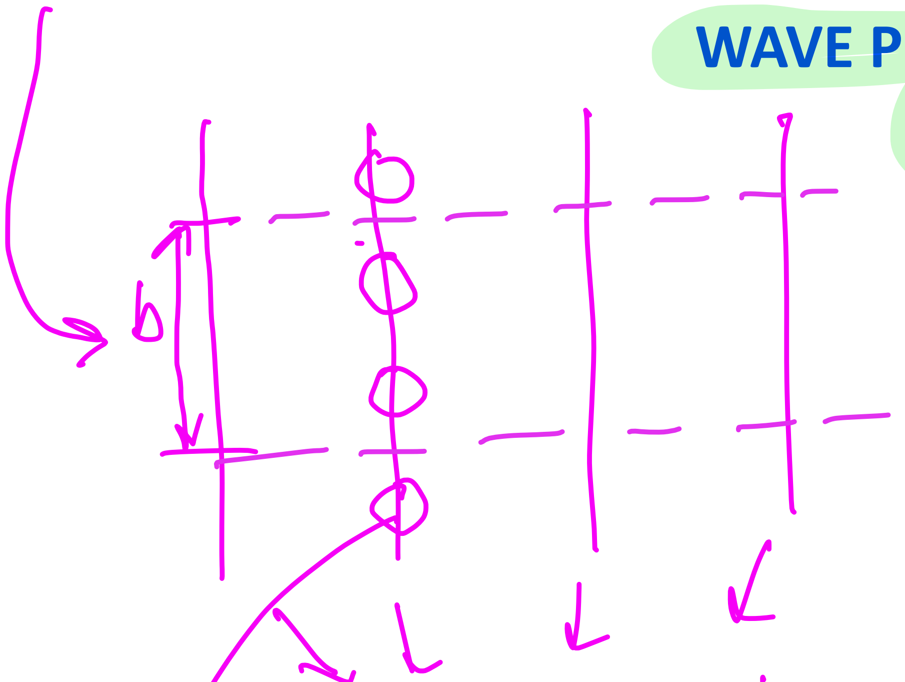
b= wave crest width

Power: $P = \Delta E/T = \bar{E}\, C_g \cancel{T}\, b / \cancel{T} = \bar{E}\, C_g\, b$

$\rho g H^2/8$

C_g = group velocity

WAVE POWER



wave crests

(top view)

WEC's

Wave Energy Converters

$$P = \bar{E}\, C_g\, b = \bar{P}\, b$$

Power per unit crest width

$$\bar{P} = \frac{P}{b} = \bar{E} \cdot C_g = \frac{\rho g H^2}{8} C_g$$

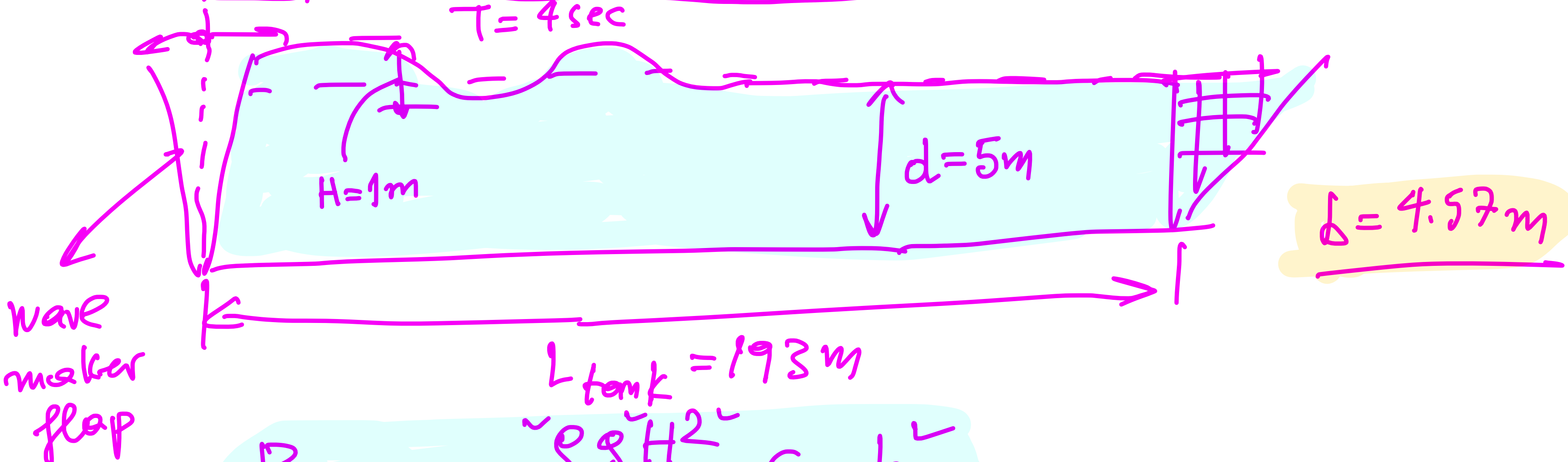
$$\left(\frac{W}{m}\right)$$

units of $\bar{P}$

$\rho = 1{,}025\ kg/m^3$

WAVE POWER/EXAMPLE

A sea-water wave tank at the U.S. Army Coastal Engineering Research Center is 193 m long, 4.57 m wide, and 5 m deep. Assume that the wave energy is fully absorbed at the opposite end of the wave generator. What power is required to generate 4-second waves which are 1m high, assuming that the wave generator is 40% efficient?



$$P_{waves} = \frac{\rho g H^2}{8} \cdot C_g \cdot b$$

We need: $\boxed{C_g = n\,C}$

$T = 4sec \rightsquigarrow L_o = \frac{gT^2}{2\pi} = 24.88m \Rightarrow \frac{d}{L_o} = \frac{5}{24.88} = 0.2002$

Table C-1 $\longrightarrow \frac{d}{L} = 0.2252 \longrightarrow \boxed{L = 22.2m}$

WAVE POWER/EXAMPLE

A sea-water wave tank at the U.S. Army Coastal Engineering Research Center is 193 m long, 4.57 m wide, and 5 m deep. Assume that the wave energy is fully absorbed at the opposite end of the wave generator. What power is required to generate 4-second waves which are 1m high, assuming that the wave generator is 40% efficient?

$$C = \frac{L}{T} = \frac{22.2}{4} = 5.55 \ m/s$$

Table C-1 with $\frac{d}{L_0} = 0.2002 \Longrightarrow n = 0.6675$

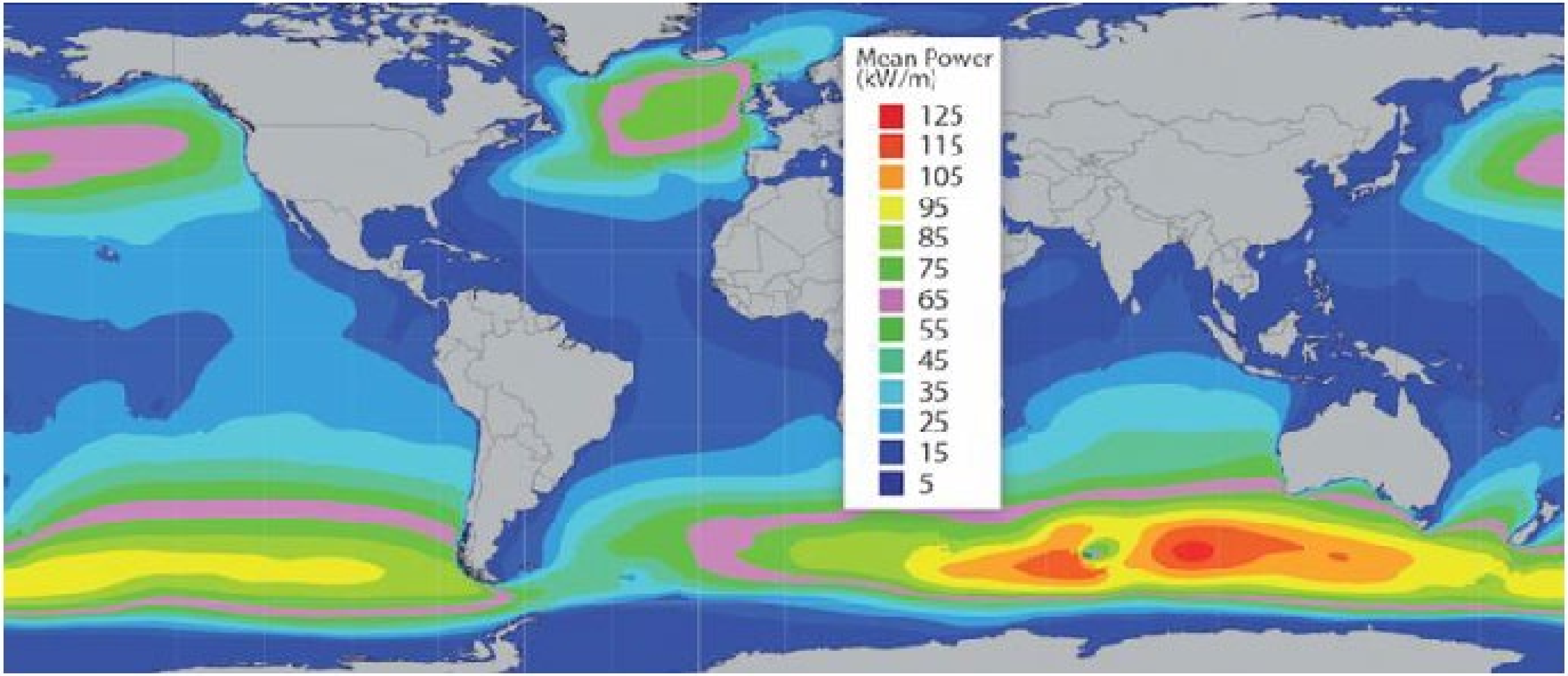
$$C_g = n \cdot C = 0.6675 \times 5.55 = 3.705 \ m/s$$

$$\boxed{P_{waves}} = \frac{1{,}025 \times 9.81 \times 1^2}{8} \times 3.705 \times 4.57 = 21{,}282 \ W = \boxed{21.3 \ kW}$$

$$0.4 = \text{efficiency} = \frac{\text{usefull power}}{\text{given power}} = \frac{P_{waves}}{P_{gen}}$$

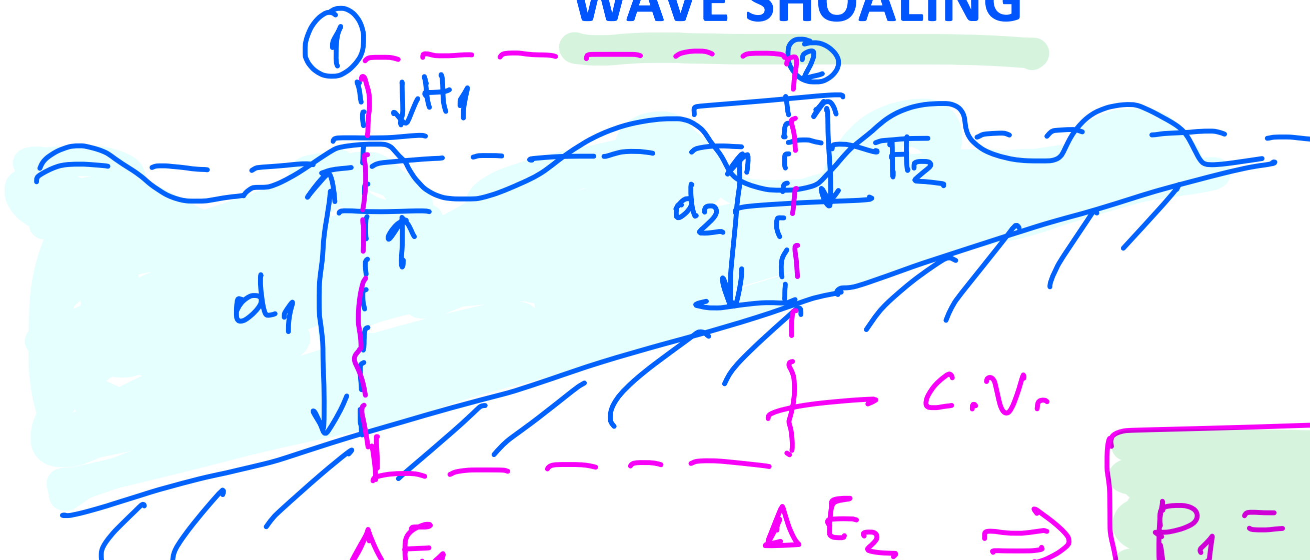
$$\boxed{P_{gen}} = \frac{P_{waves}}{0.4} = \frac{21.3}{0.4} = \boxed{53.2 \ kW}$$

Mean Wave Power (KW/m) – $\bar{P}$



Lewis, A., S. Estefen, J. Huckerby, W. Musial, T. Pontes, J. Torres-Martinez, 2011: Ocean Energy. In IPCC Special Report on Renewable Energy Sources and Climate Change Mitigation [O. Edenhofer, R. Pichs-Madruga, Y. Sokona, K. Seyboth, P. Matschoss, S. Kadner, T. Zwickel, P. Eickemeier, G. Hansen, S. Schlömer, C. von Stechow (eds)], Cambridge University Press. Figure 6.1

WAVE SHOALING



$$\frac{\Delta E_1}{\Delta t} = \frac{\Delta E_2}{\Delta t} \Rightarrow \boxed{P_1 = P_2}$$

$$\frac{\cancel{\rho} g H_1^2}{\cancel{8}} c_{g_1} \cancel{b_1} = \frac{\cancel{\rho g} H_2^2}{\cancel{8}} c_{g_2} \cancel{b_2} \qquad (b_1 = b_2)$$

$$\boxed{H_1^2 c_{g_1} = H_2^2 c_{g_2}} \Rightarrow \boxed{H_2 = H_1 \sqrt{\frac{c_{g_1}}{c_{g_2}}}}$$

In practice $P_2 = x P_1$ $\quad 0 < x < 1$

↳ If losses are important

WAVE SHOALING

Special application:

$H_1 \rightarrow H_0$

$H_2 \rightarrow H$

$$H = H_0\sqrt{\frac{C_{g0}}{C_g}} = H_0 K_S$$

$$\boxed{K_S = \frac{H}{H_0} = \sqrt{\frac{C_{g0}}{C_g}} = \sqrt{\frac{C_0}{2nC}}}\;;\quad C_{g0} = \frac{C_0}{2},\quad C_g = nC$$

$\boxed{\text{shoaling coefficient}}$

WAVE SHOALING COEFFICIENT

$P = \rho g \eta K_z - \rho g z$

$= K_S$

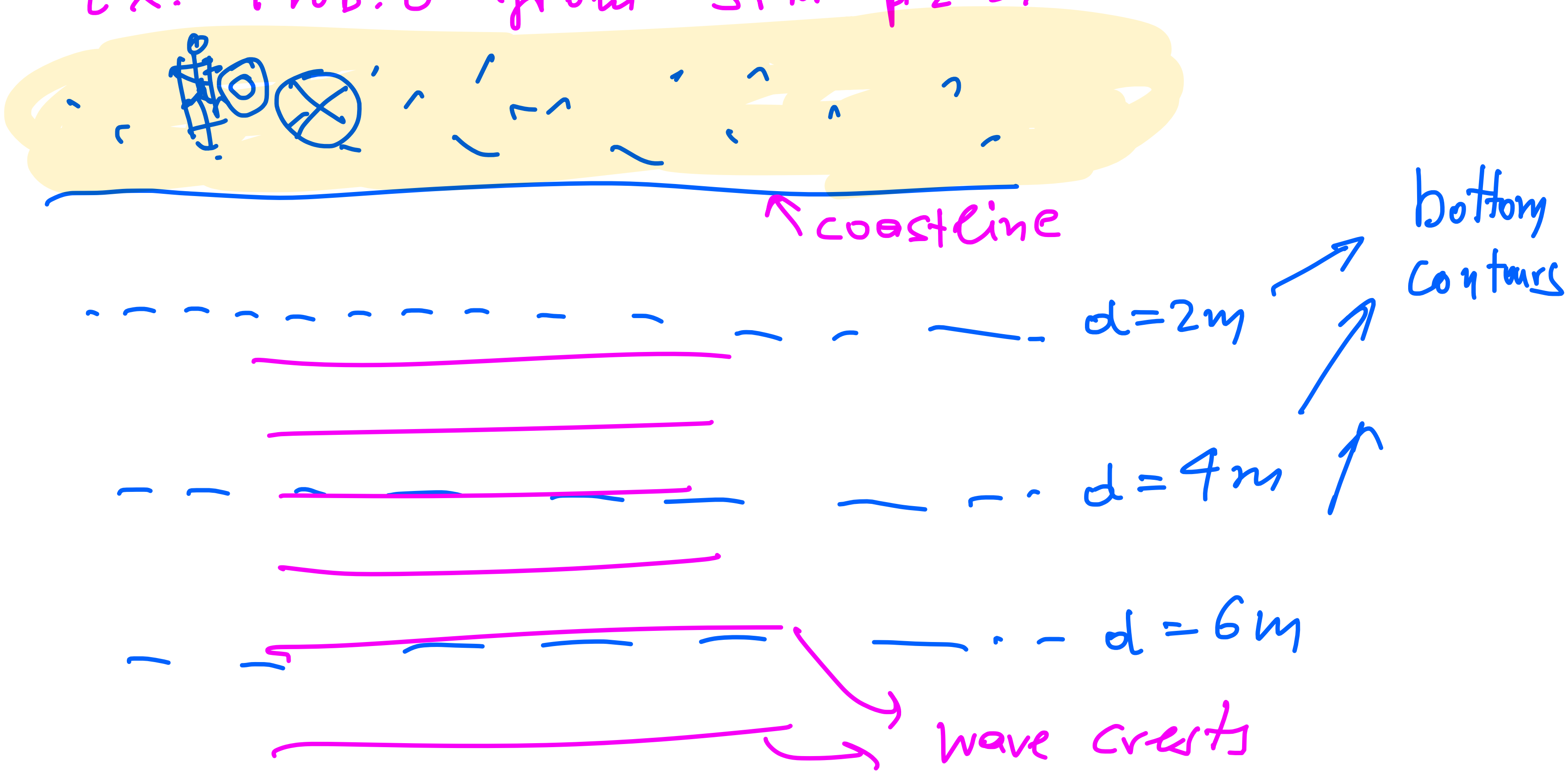
This is NOT K_S it is K_z at $z=-d$

Table C-1. Continued.

d/L_o	d/L	$2\pi\, d/L$	TANH $2\pi\, d/L$	SINH $2\pi\, d/L$	COSH $2\pi\, d/L$	H/H_o'	K	$4\pi\, d/L$	SINH $4\pi\, d/L$	COSH $4\pi\, d/L$	n	C_G/C_o	M
.1500	.1833	1.152	.8183	1.424	1.740	.9133	.5748	2.303	4.954	5.054	.7325	.5994	7.369
.1510	.1841	1.157	.8200	1.433	1.747	.9133	.5723	2.314	5.007	5.106	.7311	.5994	7.339
.1520	.1850	1.162	.8217	1.442	1.755	.9132	.5699	2.324	5.061	5.159	.7296	.5995	7.309
.1530	.1858	1.167	.8234	1.451	1.762	.9132	.5675	2.335	5.115	5.212	.7282	.5996	7.279
.1540	.1866	1.173	.8250	1.460	1.770	.9132	.5651	2.345	5.169	5.265	.7268	.5996	7.250
.1550	.1875	1.178	.8267	1.469	1.777	.9131	.5627	2.356	5.225	5.320	.7254	.5997	7.221
.1560	.1883	1.183	.8284	1.479	1.785	.9130	.5602	2.366	5.283	5.376	.7240	.5998	7.191
.1570	.1891	1.188	.8301	1.488	1.793	.9129	.5577	2.377	5.339	5.432	.7226	.5999	7.162
.1580	.1900	1.194	.8317	1.498	1.801	.9130	.5552	2.387	5.398	5.490	.7212	.5998	7.134
.1590	.1908	1.199	.8333	1.507	1.809	.9130	.5528	2.398	5.454	5.544	.7198	.5998	7.107
.1600	.1917	1.204	.8349	1.517	1.817	.9130	.5504	2.408	5.513	5.603	.7184	.5998	7.079
.1610	.1925	1.209	.8365	1.527	1.825	.9130	.5480	2.419	5.571	5.660	.7171	.5998	7.052
.1620	.1933	1.215	.8381	1.536	1.833	.9130	.5456	2.429	5.630	5.718	.7157	.5998	7.026
.1630	.1941	1.220	.8396	1.546	1.841	.9130	.5432	2.440	5.690	5.777	.7144	.5998	7.000
.1640	.1950	1.225	.8411	1.555	1.849	.9130	.5409	2.450	5.751	5.837	.7130	.5998	6.975
.1650	.1958	1.230	.8427	1.565	1.857	.9131	.5385	2.461	5.813	5.898	.7117	.5997	6.949
.1660	.1966	1.235	.8442	1.574	1.865	.9132	.5362	2.471	5.874	5.959	.7103	.5996	6.924
.1670	.1975	1.240	.8457	1.584	1.873	.9132	.5339	2.482	5.938	6.021	.7090	.5996	6.900
.1680	.1983	1.246	.8472	1.594	1.882	.9133	.5315	2.492	6.003	6.085	.7076	.5995	6.876
.1690	.1992	1.251	.8486	1.604	1.890	.9133	.5291	2.503	6.066	6.148	.7063	.5994	6.853

WAVE SHOALING - EXAMPLE

Ex. Prob. 6 from SPM p. 2-27



We know $H_0 = 2m$, $T = 10\,sec$

a) Find H at d=3m

$$\frac{H}{H_0} = K_s \quad ; \quad L_0 = \frac{gT^2}{2\pi} = 156\,m$$

WAVE SHOALING - EXAMPLE

$$\frac{d}{L_0} = \frac{3}{156} = 0.01923 \xrightarrow[C-1]{\text{Table}} \frac{H}{H_0'} = 1.237 = K_S \left(H_0 = H_0'\right)$$

$$\boxed{H} = 1.237 \times 2 = \boxed{2.474\,m}$$

b) Determine the rate at which energy per unit crest width is transported toward the coastline

They ask for $\overline{P}$!

One way of doing this part.

$$\boxed{\overline{P} = \frac{\rho g H^2}{8} C_g}$$

We can apply formula above at $d = 3m$

We have $H = 2.474m$, but we need $C_g = n\,C$

I can do that by using Table C-1 to find L, $C = L/T$, and $n \Rightarrow C_g = n \cdot C$

Remember (if no losses) $\overline{P}_0 = \overline{P}$

$\overline{P}_0 = \frac{\rho g H_0^2}{8} \cdot C_{g_0}$; $C_{g_0} = \frac{C_0}{2}$ (deep H_2O)

$C_0 = L_0/T = \frac{156}{10} = 15.6\ \frac{m}{s}$

$\boxed{\overline{P}} = \boxed{\overline{P}_0} = \frac{1{,}025 \times 9.81 \times 2^2}{8} \times \frac{15.6}{2} = \boxed{39.2\ \frac{kW}{m}}$

Easier way to solve the part of the example problem above. This way we know $C_{g_0} = \frac{C_0}{2}$ (deep H_2O) and $C_0 = L_0/T$. This way requires less calculations.

TSUNAMIS

Harbor

Wave (in Japanese)

if $L = 100\,km$ in $d = 4\,km$

it is a shallow H_2O wave

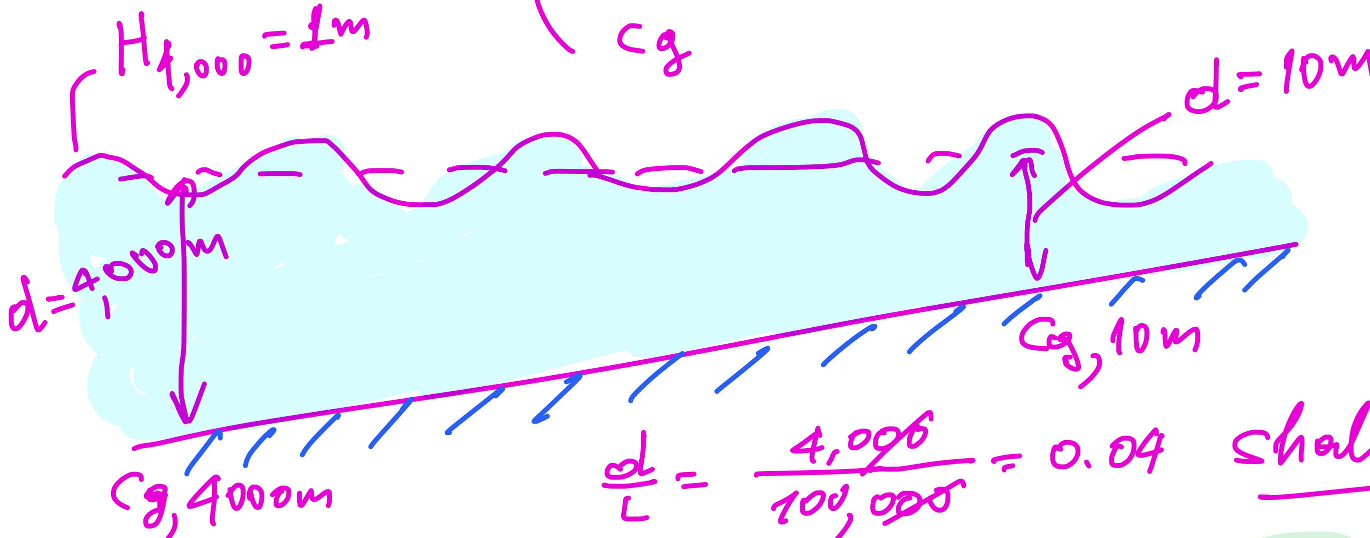
since: $\frac{d}{L} = \frac{4,000\,m}{100,000\,m} = 0.04$

TSUNAMIS - EXAMPLE

$d = 4{,}000\ m$ $\quad L = 100{,}000\ m$

An earthquake at a water depth of 4 km generates a tsunami with a length of 100 km and a height of 1m. Determine the speed of the front (in km/h and mph) of this tsunami at the depth of 4 km and at a depth of 10 m. What is the period (in minutes), the speed of its front, and the height of this tsunami at a depth of 10 m? Ignore refraction and reflection.

$H_{4{,}000}$



$T = ?$

$H_{10} = ?$

$\frac{d}{L} = \frac{4{,}000}{100{,}000} = 0.04$ shallow H_2O

$$c_{g,4000} = \sqrt{g\,d} = \sqrt{9.81 \times 4{,}000} = 198\ \frac{m}{s} = 713\ \frac{km}{h} = 444\ mph$$

Very high speed!

TSUNAMIS TRAVEL VERY FAST!!

TSUNAMIS - EXAMPLE

An earthquake at a water depth of 4 km generates a tsunami with a length of 100 km and a height of 1m. Determine the speed of the front (in km/h and mph) of this tsunami at the depth of 4 km and at a depth of 10 m. What is the period (in minutes), the speed of its front, and the height of this tsunami at a depth of 10 m? Ignore refraction and reflection.

Assume that wave is shallow at $d=10m$

↳ must VERIFY assumption at the end!!

$$C_{g,10} = \sqrt{g\,d} = \sqrt{9.81 \times 10} = 9.9\,\frac{m}{s}$$

We need to check if $\frac{d}{L} < 0.04$ at 10m

$$C_{4,000} = C_{g,4,000} = 198\,m/s \quad (\text{since shallow } C_g = C)$$

$$T = \frac{L}{C} = \frac{100,000}{198} = 505\,sec = 8.41\,min$$

$$C_{10} = C_{g,10} = 9.9\,\frac{m}{s} \quad (\text{since we assumed shallow } H_2O)$$

$$L_{10} = C_{10} \cdot T = 9.9 \times 505 = \underline{5,000\,m}$$

at 10 m:

$$\frac{d}{L} = \frac{10}{5{,}000} = 2\times10^{-3} < 0.04$$

Indeed shallow H$_2$O..!!

$$\bar{P}_{10} = \bar{P}_{4{,}000}$$

$$\frac{\cancel{\rho g} H_{10}^2}{\cancel{8}} c_{g,10} = \frac{\cancel{\rho g} H_{4{,}000}^2}{\cancel{8}} c_{g,4000}$$

$$H_{10} = H_{4{,}000}\sqrt{\frac{c_{g,4000}}{c_{g,10}}} = 1m\sqrt{\frac{198}{9.9}} \Rightarrow$$

wave height at $d = 10m$

$$\boxed{H_{10} = 4.5m}$$

significant magnification of wave height!