<u>A more rigorous way for going from equ.(21) → (23.)</u>
(under section on "Waves in Infinite Depth Water")

<u>Eq.(21):</u> $$-\cancel{\rho}\frac{a\omega^2}{k}\cos(kx-\omega t)+\frac{\cancel{\rho}}{2}a^2\omega^2+\cancel{\rho}\, g\, a\cos(kx-\omega t)=0$$

Replace: $\boxed{a=\varepsilon L}$ where $\varepsilon \ll 1$ (according to linear wave theory assumption)

Then we get:

$$-\frac{\varepsilon \cancel{L}\omega^2}{k}\cos(kx-\omega t)+\frac{\varepsilon^2 L^{\cancel{2}}\omega^2}{2}+g\,\varepsilon\,\cancel{L}\cos(kx-\omega t)=0 \quad \rightarrow (21a)$$

$$\Downarrow$$

$$-\varepsilon\frac{\omega^2}{k}\cos(kx-\omega t)+\cancel{\frac{\varepsilon^2\omega^2 L}{2}}+g\,\varepsilon\cos(kx-\omega t)=0$$

H.O.T. (since $\varepsilon \ll 1$)

$$\Downarrow$$

$$-\cancel{\varepsilon}\frac{\omega^2}{k}\cancel{\cos(kx-\omega t)}+g\,\cancel{\varepsilon}\,\cancel{\cos(kx-\omega t)}=0$$

$$\Downarrow$$

<u>Eq. (23):</u> $$\boxed{\frac{\omega^2}{k}=g}$$ dispersion relationship for deep water

<u>Note:</u> Equ.(21) has resulted from equ.(13) by taking $e^{k\eta}\approx 1$ and $e^{2k\eta}\approx 1$, since

$$k\eta=\frac{2\pi}{L}\eta=\frac{2\pi}{L}a\cdot\cos(kx-\omega t)=2\pi\,\varepsilon\cos(kx-\omega t)\ll 1$$

(or taking all exponential at $z=0$ instead of $z=\eta$)

For example, the first term in equ. (13):

$$\rho \frac{\partial \phi}{\partial t}\Big|_{z=\eta} = \rho \frac{a\omega}{k} e^{k\eta} \cos(kx-\omega t)\cdot(-\omega) =$$

$$= -\rho \frac{a\omega^2}{k} e^{k\eta} \cos(kx-\omega t) =$$

$$= -\rho \frac{\varepsilon L \omega^2}{k} \boxed{e^{2\pi\varepsilon\cos(kx-\omega t)}} \cos(kx-\omega t)$$

since $k\eta = 2\pi\varepsilon\cos(kx-\omega t) \ll 1$

and $e^x \approx 1+x$ when $x \ll 1$

$$\approx -\rho \frac{\varepsilon L\omega^2}{k} \boxed{\left[1 + 2\pi\varepsilon\cos(kx-\omega t)\right]} \cos(kx-\omega t)$$

$$= -\rho \frac{\varepsilon L\omega^2}{k} \cdot \cos(kx-\omega t) \underbrace{- \rho \frac{\varepsilon L\omega^2}{k} 2\pi \varepsilon \cos^2(kx-\omega t)}_{\sim \varepsilon^2 \Rightarrow \text{H.O.T.!}}$$

$$\approx -\rho \frac{\varepsilon L\omega^2}{k} \cdot \cos(kx-\omega t) = \underbrace{-\rho \frac{a\omega^2}{k} \cos(kx-\omega t)}$$

→ same as the 1^{st} term on the (RHS) of equ. (21)