FUNDAMENTALS OF OFFSHORE STRUCTURES AND DESIGN OF FIXED OFFSHORE PLATFORMS

NOTES ON FIFTH-ORDER GRAVITY WAVE THEORY

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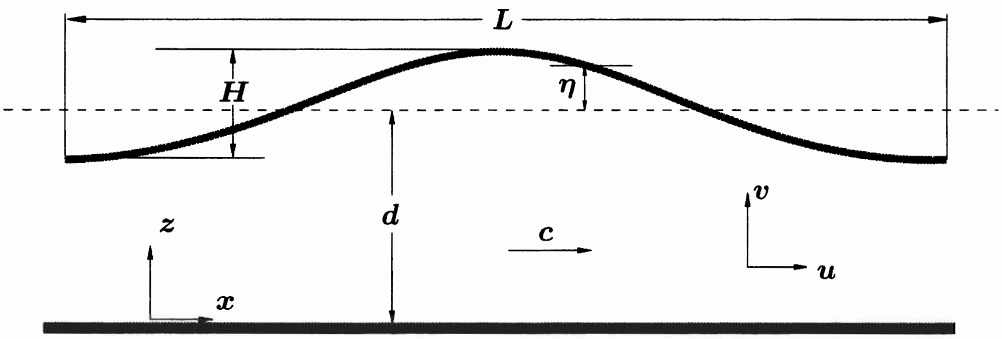

1 INTRODUCTION

Analytical forms of two alternate solutions for steady nonlinear waves based on the fifth-order Stokes theory of Skjelbreia and Hendrickson [1], and Fenton [2] are presented. The intent of the notes is to provide a brief outline of the wave characteristics without presenting the underlying theory. Please note the applicability of the models depend on the Ursell Parameter (for a complete description, refer to Dean[3])

$$\frac{L^2 H}{d^3} < \frac{8\pi^2}{3} \approx 26 \tag{1}$$

2 NOMENCLATURE

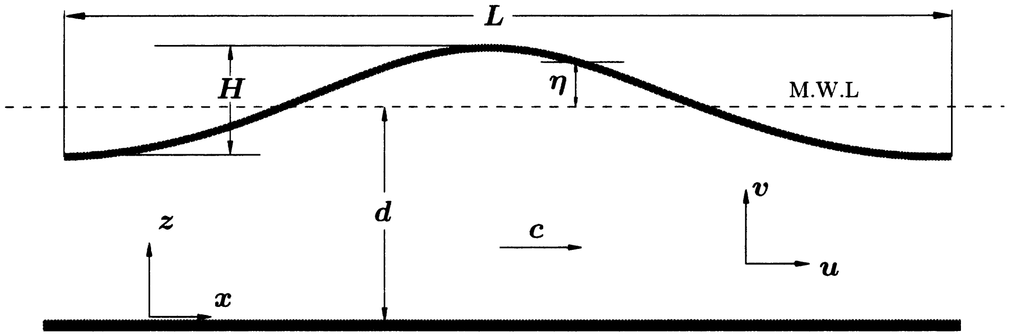

Figure 1: Important characteristics of a nonlinear wave

2.1 Wave Characteristics

H	:	wave height
L	:	wave length
T	:	wave period
c	:	wave speed
d	:	mean depth of water (distance from bottom surface to mean water line, M.W.L)
g	:	gravitational acceleration
k	:	wavenumber $= 2\pi/L$
(x, z)	:	Cartesian coordinates fixed to bottom surface through which waves move at speed c
$\eta \equiv \eta(x,t)$	:	wave elevation, elevation of free-surface above M.W.L
$\varphi \equiv \varphi(x,z,t)$	:	velociy potential with respect to (x, z) coordinate system
$u \equiv u(x,z,t)$	:	horizontal component of velocity with respect to (x, z) coordinate system
$v \equiv v(x,z,t)$	:	vertical component of velocity with respect to (x, z) coordinate system

3 SOLUTION I

The analytical form of nonlinear waves based on the theory of Skjelbreia and Hendrickson [1].

3.1 Wave Elevation

The wave elevation η is expressed as a Fourier series

$$\begin{aligned} k\eta &= \lambda \cos(\theta) \\ &+ (\lambda^2 B_{22} + \lambda^4 B_{24}) \cos(2\theta) \\ &+ (\lambda^3 B_{33} + \lambda^5 B_{35}) \cos(3\theta) \\ &+ \lambda^4 B_{44} \cos(4\theta) \\ &+ \lambda^5 B_{55} \cos(5\theta) \end{aligned} \tag{2}$$

where $\theta = k(x - ct)$. Equation (2) can be written in a more simplified form as

$$k\eta = \sum_{n=1}^{5} \lambda^n b_n \cos(n\theta) \tag{3}$$

where,

$$\begin{aligned} b_1 &= 1 \\ b_2 &= B_{22} + \lambda^2 B_{24} \\ b_3 &= B_{33} + \lambda^2 B_{35} \\ b_4 &= B_{44} \\ b_5 &= B_{55} \end{aligned} \tag{4}$$

In the above expansion, the quantity λ has no physical significance and may be interpreted as a length scale equal to the amplitude of the wave at lowest order (Fenton, [2]). The coefficients B_{ij} are dimensionless functions of d and L (refer to Appendix A).

3.2 Wave Potential

The velocity potential can be written as

$$\begin{aligned} \frac{k}{c}\varphi &= (\lambda A_{11} + \lambda^3 A_{13} + \lambda^5 A_{15}) \cosh(kz) \sin(\theta) \\ &+ (\lambda^2 A_{22} + \lambda^4 A_{24}) \cosh(2kz) \sin(2\theta) \\ &+ (\lambda^3 A_{33} + \lambda^5 A_{35}) \cosh(3kz) \sin(3\theta) \\ &+ (\lambda^4 A_{44}) \cosh(4kz) \sin(4\theta) \\ &+ (\lambda^5 A_{55}) \cosh(5kz) \sin(5\theta) \end{aligned} \tag{5}$$

Equation (5) can be written in a simplified form as follows

$$\frac{k}{c}\varphi = \sum_{n=1}^{5} \lambda^n a_n \cosh(nkz) \sin(n\theta) \tag{6}$$

where,

$$\begin{aligned}
a_1 &= A_{11} + \lambda^2 A_{13} + \lambda^4 A_{15} \\
a_2 &= A_{22} + \lambda^2 A_{24} \\
a_3 &= A_{33} + \lambda^2 A_{35} \\
a_4 &= A_{44} \\
a_5 &= A_{55}
\end{aligned} \tag{7}$$

The coefficients A_{ij} are dimensionless functions of d and L (refer to Appendix A).

3.3 Velocity Components

The horizontal and vertical velocity components are derived from the expression for the velocity potential, as given in Equation (6).

$$\frac{k}{c}\,u = \frac{k}{c}\frac{\partial \varphi}{\partial x} = \sum_{n=1}^{5} \lambda^n a_n \cosh(nkz)\cos(n\theta)\, n\, \frac{\partial \theta}{\partial x} \tag{8}$$

$\because\ \theta = k(x - ct)$, $\dfrac{\partial \theta}{\partial x} = k$. Thus

$$\frac{u}{c} = \sum_{n=1}^{5} \lambda^n\, n\, a_n \cosh(nkz)\cos(n\theta) \tag{9}$$

Similarly,

$$\frac{k}{c}\,v = \frac{k}{c}\frac{\partial \varphi}{\partial z} = \sum_{n=1}^{5} \lambda^n a_n \sinh(nkz)\sin(n\theta)\, n\, k \tag{10}$$

or,

$$\frac{v}{c} = \sum_{n=1}^{5} \lambda^n\, n\, a_n \sinh(nkz)\sin(n\theta) \tag{11}$$

3.4 Application

For a given H, d and T, the wave length L and the parameter λ are obtained by solving a set of nonlinear equations of the form

$$\begin{aligned}
\frac{\pi H}{d} &= \frac{L}{d}\left[\lambda + \lambda^3 B_{33} + \lambda^5 (B_{35} + B_{55})\right] \\
\frac{d}{L_0} &= \frac{d}{L}\tanh(kd)\left(1 + \lambda^2 C_1 + \lambda^4 C_2\right)
\end{aligned} \tag{12}$$

where C_1 and C_2 are coefficients that are functions of d and L (refer to Appendix A) and

$$L_0 = \frac{gT^2}{2\pi} \tag{13}$$

4 SOLUTION II

The analytical form of nonlinear waves based on the theory of Fenton [2].

4.1 Wave Elevation

The wave elevation η is expressed as a Fourier series

$$\begin{aligned} k\eta = & \left[\epsilon + \epsilon^3 B_{31} - \epsilon^5 (B_{53} + B_{55})\right] \cos(\theta) \\ & + (\epsilon^2 B_{22} + \epsilon^4 B_{42}) \cos(2\theta) \\ & + (-\epsilon^3 B_{31} + \epsilon^5 B_{53}) \cos(3\theta) \\ & + \epsilon^4 B_{44} \cos(4\theta) \\ & + \epsilon^5 B_{55} \cos(5\theta) \end{aligned} \tag{14}$$

where $\theta = k(x - ct)$. Equation (14) can be written in a more simplified form as

$$k\eta = \sum_{n=1}^{5} \epsilon^n b_n \cos(n\theta) \tag{15}$$

where,

$$\begin{aligned} b_1 &= 1 + \epsilon^2 B_{31} - \epsilon^4 (B_{53} + B_{55}) \\ b_2 &= B_{22} + \epsilon^2 B_{42} \\ b_3 &= -B_{31} + \epsilon^2 B_{53} \\ b_4 &= B_{44} \\ b_5 &= B_{55} \end{aligned} \tag{16}$$

In the above expansion, the quantity $\epsilon = kH/2$ represents the dimensionless wave height. The coefficients B_{ij} are dimensionless functions of d and L (refer to Appendix B).

4.2 Wave Potential

The velocity potential can be written as

$$\begin{aligned} \frac{1}{C_0}\sqrt{\frac{k^3}{g}}\,\varphi = & (\epsilon A_{11} + \epsilon^3 A_{31} + \epsilon^5 A_{51}) \cosh(kz) \sin(\theta) \\ & + (\epsilon^2 A_{22} + \epsilon^4 A_{42}) \cosh(2kz) \sin(2\theta) \\ & + (\epsilon^3 A_{33} + \epsilon^5 A_{53}) \cosh(3kz) \sin(3\theta) \\ & + (\epsilon^4 A_{44}) \cosh(4kz) \sin(4\theta) \\ & + (\epsilon^5 A_{55}) \cosh(5kz) \sin(5\theta) \end{aligned} \tag{17}$$

Equation (17) can be written in a simplified form as follows

$$\frac{1}{C_0}\sqrt{\frac{k^3}{g}}\,\varphi = \sum_{n=1}^{5} \epsilon^n a_n \cosh(nkz) \sin(n\theta) \tag{18}$$

where,

$$\begin{aligned} a_1 &= A_{11} + \epsilon^2 A_{31} + \epsilon^4 A_{51} \\ a_2 &= A_{22} + \epsilon^2 A_{42} \\ a_3 &= A_{33} + \epsilon^2 A_{53} \\ a_4 &= A_{44} \\ a_5 &= A_{55} \end{aligned} \tag{19}$$

The coefficients C_0 and A_{ij} are dimensionless functions of d and L (refer to Appendix B).

4.3 Velocity Components

The horizontal and vertical velocity components are derived from the expression for the velocity potential, as given in Equation (18).

$$\frac{1}{C_0}\sqrt{\frac{k^3}{g}}\,u = \frac{1}{C_0}\sqrt{\frac{k^3}{g}}\,\frac{\partial\varphi}{\partial x} = \sum_{n=1}^{5} \epsilon^n a_n \cosh{(nkz)} \cos{(n\theta)}\, n\,\frac{\partial\theta}{\partial x} \tag{20}$$

$\because\ \theta = k(x - ct)$, $\dfrac{\partial\theta}{\partial x} = k$. Thus

$$\frac{1}{C_0}\sqrt{\frac{k}{g}}\,u = \sum_{n=1}^{5} \epsilon^n\, n\, a_n \cosh{(nkz)} \cos{(n\theta)} \tag{21}$$

Similarly,

$$\frac{1}{C_0}\sqrt{\frac{k^3}{g}}\,v = \frac{1}{C_0}\sqrt{\frac{k^3}{g}}\,\frac{\partial\varphi}{\partial z} = \sum_{n=1}^{5} \epsilon^n a_n \sinh{(nkz)} \sin{(n\theta)}\, n\, k \tag{22}$$

or,

$$\frac{1}{C_0}\sqrt{\frac{k}{g}}\,v = \sum_{n=1}^{5} \epsilon^n\, n\, a_n \sinh{(nkz)} \sin{(n\theta)} \tag{23}$$

4.4 Application

For a given H, d and T, the wave length L can be obtained by solving the dispersion relation

$$C_0 + \epsilon^2 C_2 + \epsilon^4 C_4 = \frac{2\pi}{T\sqrt{gk}} \tag{24}$$

where C_0, C_2 and C_4 are dimensionless functions of d and L (refer to Appendix B).

Appendix A

$$S = \sinh(kd)$$

$$C = \cosh(kd)$$

$$A_{11} = 1/S$$

$$A_{13} = -C^2(5C^2+1)/(8S^5)$$

$$A_{15} = -(1184C^{10} - 1440C^8 - 1992C^6 + 2641C^4 - 249C^2 + 18)/(1536S^{11})$$

$$A_{22} = 3/(8S^4)$$

$$A_{24} = (192C^8 - 424C^6 - 312C^4 + 480C^2 - 17)/(768S^{10})$$

$$A_{33} = (13 - 4C^2)/(64S^7)$$

$$A_{35} = (512C^{12} + 4224C^{10} - 6800C^8 - 12808C^6 + 16704C^4 - 3154C^2 + 107) / [4096S^{13}(6C^2 - 1)]$$

$$A_{44} = (80C^6 - 816C^4 + 1338C^2 - 197)/[1536S^{10}(6C^2 - 1)]$$

$$A_{55} = -(2880C^{10} - 72480C^8 + 324000C^6 - 432000C^4 + 163470C^2 - 16245) / [61440S^{11}(6C^2 - 1)(8C^4 - 11C^2 + 3)]$$

$$B_{22} = C(2C^2+1)/(4S^3)$$

$$B_{24} = C(272C^8 - 504C^6 - 192C^4 + 322C^2 + 21)/(384S^9)$$

$$B_{33} = 3(8C^6+1)/(64S^6)$$

$$B_{35} = (88128C^{14} - 208224C^{12} + 70848C^{10} + 54000C^8 - 21816C^6 + 6264C^4 - 54C^2 - 81) / [12288S^{12}(6C^2 - 1)]$$

$$B_{44} = C(768C^{10} - 448C^8 - 48C^6 + 48C^4 + 106C^2 - 21)/[384S^9(6C^2 - 1)]$$

$$B_{55} = (192000C^{16} - 262720C^{14} + 83680C^{12} + 20160C^{10} - 7280C^8 + 7160C^6 - 1800C^4 - 1050C^2 + 225) / [12288S^{10}(6C^2 - 1)(8C^4 - 11C^2 + 3)]$$

$$C_1 = (8C^4 - 8C^2 + 9)/(8S^4)$$

$$C_2 = (3840C^{12} - 4096C^{10} - 2592C^8 - 1008C^6 + 5944C^4 - 1830C^2 + 147) / [512S^{10}(6C^2 - 1)]$$

Appendix B

$$S = \operatorname{sech}(2kd) = \frac{1}{\cosh(2kd)}$$

$$C = 1 - S$$

$$A_{11} = 1/\sinh(kd)$$

$$A_{22} = 3S^2/(2C^2)$$

$$A_{31} = (-4 - 20S + 10S^2 - 13S^3)/[8\sinh(kd)C^3]$$

$$A_{33} = (-2S^2 + 11S^3)/[8\sinh(kd)C^3]$$

$$A_{42} = (12S - 14S^2 - 264S^3 - 45S^4 - 13S^5)/(24C^5)$$

$$A_{44} = (10S^3 - 174S^4 + 291S^5 + 278S^6)/[48(3 + 2S)C^5]$$

$$A_{51} = (-1184 + 32S + 13232S^2 + 21712S^3 + 20940S^4 + 12554 * S^5 - 500S^6 - 3341S^7 - 670S^8) /[64\sinh(kd)(3 + 2S)(4 + S)C^6]$$

$$A_{53} = (4S + 105S^2 + 198S^3 - 1376S^4 - 1302S^5 - 117S^6 + 58S^7) /[32\sinh(kd)(3 + 2S)C^6]$$

$$A_{55} = (-6S^3 + 272S^4 - 1552S^5 + 852S^6 + 2029S^7 + 430S^8) /[64\sinh(kd)(3 + 2S)(4 + S)C^6]$$

$$B_{22} = \coth(kd)(1 + 2S)/(2C)$$

$$B_{31} = -3(1 + 3S + 3S^2 + 2S^3)/(8C^3)$$

$$B_{42} = \coth(kd)(6 - 26S - 182S^2 - 204S^3 - 25S^4 + 26S^5)/[6(3 + 2S)C^4]$$

$$B_{44} = \coth(kd)(24 + 92S + 122S^2 + 66S^3 + 67S^4 + 34S^5)/[24(3 + 2S)C^4]$$

$$B_{53} = 9(132 + 17S - 2216S^2 - 5897S^3 - 6292S^4 - 2687S^5 + 194S^6 + 467S^7 + 82S^8) /[128(3 + 2S)(4 + S)C^6)$$

$$B_{55} = 5(300 + 1579S + 3176S^2 + 2949S^3 + 1188S^4 + 675S^5 + 1326S^6 + 827S^7 + 130S^8) /[384(3 + 2S)(4 + S)C^6]$$

$$C_0 = \sqrt{\tanh(kd)}$$

$$C_2 = C_0(2 + 7S^2)/(4C^2)$$

$$C_4 = C_0(4 + 32S - 116S^2 - 400S^3 - 71S^4 + 146S^5)/(32C^5)$$

References

[1] L. Skjelbreia and J. A. Hendrickson. Fifth order gravity wave theory. *Proceedings 7th Coastal Engineering Conference, The Hague*, pages 184–196, 1960.

[2] J. D. Fenton. A fifth-order stokes theory for steady waves. *Journal of Waterway, Port, Coastal and Ocean Engineering*, 111(2):216–234, 1985.

[3] R. G. Dean and R. A. Dalrymple. Water Wave Mechanics for Engineers and Scientists. *Prentice-Hall, Inc.*, 1984.