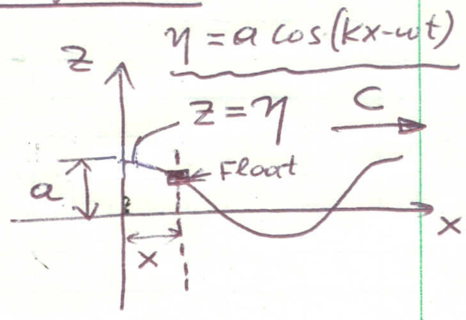
Applying kbc on the free surface:



For deep water
$$\begin{cases} \phi = A e^{kz} \cdot \sin(kx-\omega t) \\ \eta(x,t) = a \cdot \cos(kx-\omega t) \end{cases}$$

kbc: $\dfrac{\partial \phi}{\partial z} = \dfrac{\partial \eta}{\partial t}$ at $\underline{z=\eta}$ ← "exact" location of free-surface

$$\frac{\partial \phi}{\partial z} = A k e^{kz}\Big|_{z=\eta} \sin(kx-\omega t) = \underbrace{A \cdot k e^{k\eta} \sin(kx-\omega t)}$$

must be equal to each other

$$\frac{\partial \eta}{\partial t} = a\left[-\sin(kx-\omega t)\right](-\omega) = \overbrace{a\omega\sin(kx-\omega t)}$$

$$\Rightarrow A k e^{k\eta} \cancel{\sin(kx-\omega t)} = a\omega \cancel{\sin(kx-\omega t)} \Rightarrow$$

$$\Rightarrow A k e^{k\eta} = a\omega \Rightarrow A = \frac{a\omega}{k e^{k\eta}} \quad (1)$$

problem! This cannot be a constant since it depends on η!

Need to resort to linear wave theory assumption $\boxed{\text{that } a \ll L}$

$$e^{-k\eta} = 1 - k\eta \quad \left(\text{remember } e^x \approx 1+x \text{ for small } x\right) \text{ since } k\eta = \frac{2\pi a}{L}\cos(kx-\omega t) \ll 1$$

Thus from (1) $\Rightarrow A = \dfrac{a\omega}{k}(1-k\eta)$ ⟹ (multiply both sides by k^2)

$$\Rightarrow A k^2 = a\omega k(1-k\eta) = a\omega k - a\omega k^2 \eta =$$

$$= a\omega\frac{2\pi}{L} - a\omega\left(\frac{2\pi}{L}\right)^2 a\cos(kx-\omega t) =$$

$$= 2\pi\omega\left(\frac{a}{L}\right) - \underbrace{(2\pi)^2\omega\cos(kx-\omega t)\left(\frac{a}{L}\right)^2}_{\text{H.O.T.}} \Rightarrow$$

$$\Rightarrow A k^{\cancel{2}} = 2\pi\omega\left(\frac{a}{L}\right) = \frac{2\pi}{L}a\omega = \cancel{k}a\omega \Rightarrow \boxed{A = \frac{a\omega}{k}}$$

same result as if we plugin $\eta \approx 0$ in (1) or as if we $\boxed{\text{apply kbc at } z=0}$