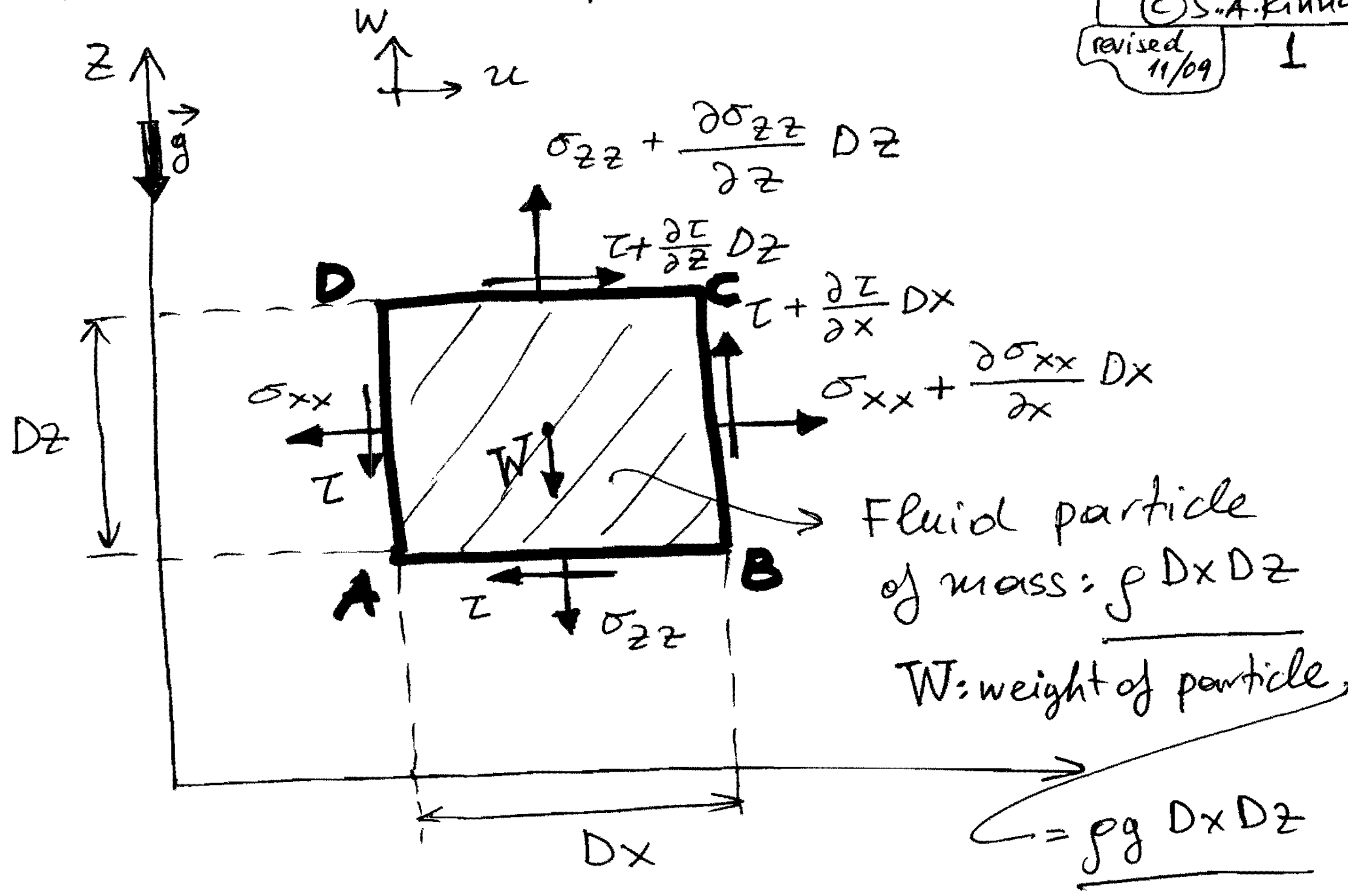
Navier-Stokes equations in 2-D



$\tau = \sigma_{xz} = \sigma_{zx}$ (shear stress)

$$\sum F_x = (\rho \, Dx \, Dz) \cdot a_x$$

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \rightarrow \text{fluid particle acceleration along } x$$

$$\sum F_x = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} Dx\right) \cdot \underbrace{BC}_{Dz} - \left(\sigma_{xx}\right) \underbrace{AD}_{Dz}$$

$$+ \left(\tau + \frac{\partial \tau}{\partial z} Dz\right) \cdot \underbrace{DC}_{Dx} - (\tau) \underbrace{AB}_{Dx} =$$

$$= \frac{\partial \sigma_{xx}}{\partial x} Dx \cancel{Dz} + \frac{\partial \tau}{\partial z} Dz \cancel{Dx} = \rho \cancel{Dx \, Dz} \, a_x$$

$$\Rightarrow \boxed{\rho a_x = \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right] = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau}{\partial z}} \quad (1)$$

Similarly:

$$\sum F_z = (\rho\, Dx\, Dz)\, a_z$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \longrightarrow$$ fluid particle acceleration along z

$$\sum F_z = \left(\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial z} Dz\right)\underbrace{DC}_{Dx} - \sigma_{zz}\cdot\underbrace{AB}_{Dx} - W$$

$$+\left(\tau + \frac{\partial \tau}{\partial x} Dx\right)\underbrace{BC}_{Dz} - \tau\cdot\underbrace{AD}_{Dz} =$$

$$= \frac{\partial \sigma_{zz}}{\partial z} Dz\,\cancel{Dx} + \frac{\partial \tau}{\partial x} Dx\,\cancel{Dz} - \rho g\,\cancel{Dx\,Dz} =$$

$$= \rho\,\cancel{Dx\,Dz}\, a_z$$

$$\Rightarrow \boxed{\rho a_z = \rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right] = \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau}{\partial x} - \rho g} \quad (2)$$

Constitutive equations for Newtonian fluid:

$$\left.\begin{aligned} \sigma_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} \\ \sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z} \\ \tau &= \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \end{aligned}\right\} \quad 3$$

p: fluid pressure

μ: dynamic (or absolute) viscosity

Substituting σ_{xx}, σ_{zz}, and τ from (3) into (1) and (2) it can be shown that:

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

(4)

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right]-\rho g$$

Navier–Stokes equations in 2-D

The N-S equations can also be written in the following compact form:

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$

(4a)

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\mu\nabla^2 w-\rho g$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$ (Laplacian operator)

and

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}$$ (substantial derivative operator)

To the above equations we need to add the continuity equation: $\boxed{\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0}$ (5)

Equations 4a can be applied to <u>laminar</u> as well as <u>turbulent</u> flows. However in the case of turbulent flows the required spatial and temporal discretization must be very fine (very small $\Delta x, \Delta y, \Delta t$) and that requires powerful (parallel) computer with very large RAM and makes the computational method very slow (CPU Time ↑↑↑) even for simple 2-D geometries. We call methods that use equ. 4a for modeling of turbulent flows: <u>DNS</u> which stands for <u>D</u>irect (Navier Stokes) <u>N</u>umerical <u>S</u>imulations

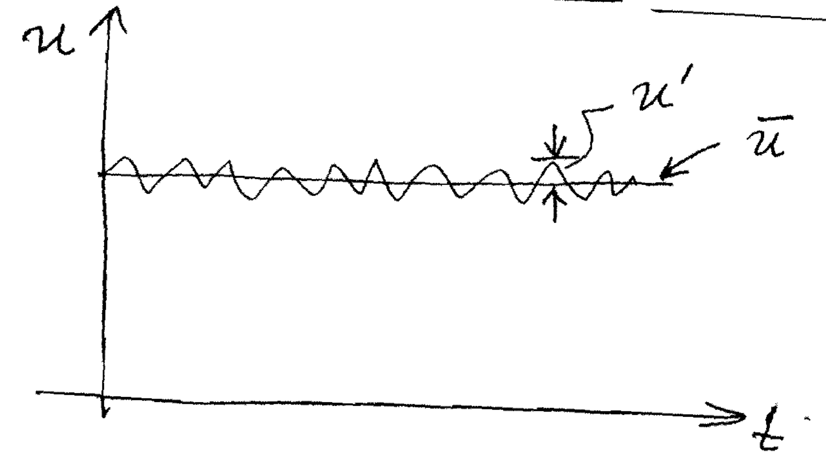
Turbulent flows can be dealt easier by averaging of the quantities of interest:

$$u = \bar{u} + u' \qquad \begin{cases} \bar{u} = \text{average value} \\ u' = \text{turbulent fluctuation} \end{cases}$$

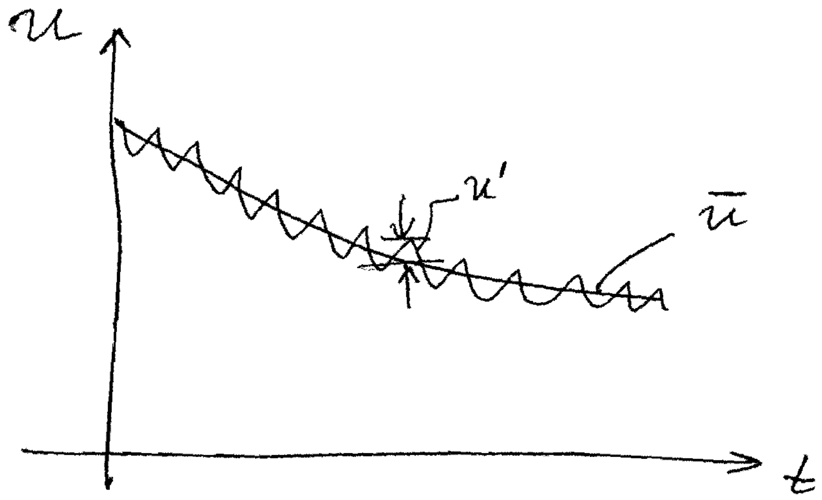
$$W = \bar{W} + w'$$

$$P = \bar{P} + P'$$

...



Steady turbulent flow



Unsteady turbulent flow

It can be shown that equs (1) and (2) become:

$$\rho\left[\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=\frac{\partial \bar{\sigma}_{xx}}{\partial x}+\frac{\partial \bar{\tau}}{\partial z} \quad (1t)$$

and

$$\rho\left[\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right]=\frac{\partial \bar{\sigma}_{zz}}{\partial z}+\frac{\partial \bar{\tau}}{\partial x}-\rho g \quad (2t)$$

and the continuity equation

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{w}}{\partial z}=0 \quad (5t)$$

In other words the equations apply for the averaged flow as well

However, the constitutive equations become:

$$\bar{\sigma}_{xx} = -\bar{p} + 2\mu\frac{\partial\bar{u}}{\partial x} - \rho\overline{(u')^2}$$

$$\bar{\sigma}_{zz} = -\bar{p} + 2\mu\frac{\partial\bar{w}}{\partial z} - \rho\overline{(w')^2} \qquad (3t)$$

$$\bar{\tau} = \mu\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right) - \rho\overline{(u'w')}$$

The additional terms in the RHS of (3t) are the <u>Reynolds (turbulent) stresses</u>.

Note that $\overline{u'} = 0$, $\overline{w'} = 0$, but $\overline{(u')^2}, \overline{(w')^2}, \overline{(u'w')} \neq 0$

<u>Turbulence modeling</u> attempts to express the Reynolds stresses in terms of the mean quantities $(\bar{u}, \bar{w}, \bar{p}, \ldots)$.

Equations (1t), (2t), (5t) with (3t) consist the <u>Reynolds Averaged Navier Stokes</u> (<u>RANS</u>) equations.

There are many <u>turbulence models</u>, some simplified, some more elaborate. A very popular model is the <u>k-ε</u> <u>model</u>. <u>k</u> is the turbulence kinetic energy (per unit mass):

$$k = \frac{1}{2}\left[\overline{(u')^2} + \overline{(w')^2}\right]$$

and ε is the turbulent dissipation (per unit mass):

$$\varepsilon = \underbrace{\boxed{\frac{\mu}{\rho}}}_{=\nu}\left[\overline{\frac{\partial u'}{\partial x}\cdot\frac{\partial u'}{\partial x}} + \overline{\frac{\partial w'}{\partial z}\cdot\frac{\partial w'}{\partial z}} + \overline{\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}} + \overline{\frac{\partial w'}{\partial x}\cdot\frac{\partial w'}{\partial x}}\right]$$

in Equ. (3)

We then replace μ with $\mu + \mu_t$ (*)
(in addition to some other modifications)

where $\boxed{\mu_t}$ = turbulent (or eddy) viscosity $= \boxed{\rho C_\mu \frac{k^2}{\varepsilon}}$

$C_\mu = 0.09$

(*) For a detailed description you may look in: "Turbulence Modeling for CFD" by David C. Wilcox (DCW 1994)