The Random-Utility-Based Multiregional Input-Output Model: Solution Existence and Uniqueness

by

Yong Zhao
Graduate Student Researcher
The University of Texas at Austin
6.9 E. Cockrell Jr. Hall
Austin, TX 78712-1076
yzhao@mail.utexas.edu

and

Kara M. Kockelman
(Corresponding author)
Clare Boothe Luce Professor of Civil Engineering
The University of Texas at Austin
6.9 E. Cockrell Jr. Hall
Austin, TX 78712-1076
kkockelm@mail.utexas.edu
Phone: 512-471-0210
FAX: 512-475-8744

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ABSTRACT

A number of operational land use-transportation models make use of spatial input-output (SIO) models, some of which are based on random utility theory. The random-utility-based multiregional input-output (RUBMRIO) model has been solved in practice by iteratively applying a set of equations. Each of the model equations describes relationships among key model variables. This paper examines the existence and uniqueness of the RUBMRIO solution, which represents the spatial allocation of productive activities and commodity flows. Formulating the set of equations as a fixed-point problem illuminates these two key properties, and provides a general solution algorithm. Several numerical examples illustrate the solution uniqueness and algorithm convergence. These results are valuable for efficient application of such models to large-scale problems. By proving that a unique solution does exist and offering an algorithm that is guaranteed to converge, this work adds valuable support to the growing popularity of this integrated transportation-land use modeling framework.

KEYWORDS

spatial input-output models, random-utility-based multiregional input-output models, solution existence and uniqueness, integrated land use-transportation models

1. INTRODUCTION

A number of operational land use-transportation models make use of spatial (or interregional, interzonal) input-output (SIO) models, including Echenique and colleagues' MEPLAN (Hunt, 1993), de la Barra's TRANUS (1995), and Kim's model (1989). MEPLAN and TRANUS are random-utility-based, and thus may be referred to as random-utility-based multiregional input-output (RUBMRIO) models. These combine traditional SIO models with a multinomial logit (MNL) model for trade and travel choices to represent the distributed nature of commodity flow patterns. The RUBMRIO model is usually solved by iteratively applying a set of equations (Hunt, 1993). Each equation describes a key model variable. This paper examines the existence and uniqueness of the equilibrium solution, which represents the spatial allocation of activities and commodity flow patterns, by formulating the original set of equations into a fixed-point problem. A modified solution algorithm also is developed, and several numerical examples illustrate the convergence of the proposed algorithm and compare it to the original algorithm.

2. BACKGROUND

Originally proposed by Leontief (1941), input-output (IO) analysis is a macroeconomic approach focused on a single region's industries' interactions via business expenditure patterns. The analysis is driven by exogenous demand for regional goods (e.g., exports). In contrast to Lowry-type models, the IO approach endogenously determines interactions of basic and non-basic industrial activities. SIO analysis extends the classical IO model to include spatial disaggregations (Isard, 1960; Leontief and Strout, 1963). Entropy concepts were then proposed, to establish a connection between SIO models, entropy-maximizing theory, and random utility theory (Wilson, 1970; Anas, 1984). In this section, we review the original, single-region IO models and then summarize the RUBMRIO model based on the literature of Hunt (1993) and de la Barra (1995). While the RUBMRIO model has a connection to entropy theory, similarities between entropy maximization and RUBMRIO equations are not the focus of this research. Transport prices affect input prices here, in a manner very different from standard network equilibration approaches (as in, e.g., Oppenheim 1995 and Kim et al. 2002)).

2.1 Single Regional IO Models

IO models characterize the interactions between various market actors (typically producers of commodities and services). The actors are usually aggregated into sectors. If one has M industry sectors, the basic IO model identifies the flow of commodities and services x^{mn} between sectors within a single-region economy, where m is a producing sector and n is a purchasing sector (and $m,n=1,2,...,M$). For consistency of units, x^{mn} is the dollar value of sector m's output that is purchased by sector n.

The total output of any given sector of the single region economy, X^m, is given by

$$X^m = \sum_n x^{mn} + Y^m \qquad \forall m \tag{2.1}$$

where Y^m is the final demand for (or export of) sector m's output.

The direct purchase can be expressed as:

$$x^{mn} = a^{mn} X^n \qquad \forall m,n \tag{2.2}$$

where a^{mn} is a technical coefficient, representing the amount of sector *m* product required to produce one dollar of sector *n* product. So equation (2.1) can be written as:

$$X^m = \sum_n a^{mn} X^n + Y^m \qquad \forall m \tag{2.3}$$

This model assumes that equilibrium between total supply and total demand occurs, but substitution across inputs to production does not (i.e., one cannot substitute one input for another in producing any output; they are used in fixed ratios). The production technology does not change rapidly and is assumed constant over the period of model application.

In matrix notation, the IO model is as follows:

$$\begin{aligned} &\mathbf{X} = \mathbf{AX} + \mathbf{Y} \\ &\text{or} \quad (\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{Y} \end{aligned} \tag{2.4}$$

where **Y** represents a vector of final demand, **X** a vector of outputs, **A** the technical coefficient matrix $\{a^{mn}\}$, and **I** the identity matrix.

Assuming that the **(I-A)** matrix is nonsingular, it is possible to solve for production levels, given final demand:

$$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{Y} \tag{2.5}$$

Thus, the equilibrium solution for **X** is deterministic in equation (2.5), given **Y** and **A**. The only necessary condition is that the **(I-A)** matrix be nonsingular. Then x^{mn} is solved by equation (2.2).

2.2 Random-Utility-Based Multiregional Input-Output Models

The extension of the IO model to multiple regions was first proposed by Isard (1960), who introduced a spatial dimension into the intersectoral flow tables:

$$X_i^m = \sum_j x_{ij}^m \qquad \forall i, m \tag{2.6}$$

where x_{ij}^m is the flow of sector *m* from region *i* to region *j*, and $i, j = 1, 2, ..., J$. Equation (2.3) therefore becomes:

$$\sum_i x_{ij}^m = \sum_n a_j^{mn} \sum_k x_{jk}^n + Y_j^m \qquad \forall j, m \tag{2.7}$$

where $\{a_i^{mn}\}$ is the set of technical coefficients for production processes in region *i*, and $k = 1, 2, ..., J$.

Equation (2.7) describes the commodity balance condition, which requires that the flow of sector *m*'s goods into region *j* equals the use of that sector's goods for producing goods of other sectors (intermediate demand) plus any final demand. Of course, a region can acquire many or even all of its inputs locally (i.e., from itself), but this is not required.

As one can see, Equation (2.7) is insufficient for determining flows since it describes only the origin and destination flow totals. Denoting $C_j^m = \sum_i x_{ij}^m, \forall j, m$ as the total consumption of commodity *n* in region *j*, one can rewrite (2.7) as the following:

$$C_j^m = \sum_n a_j^{mn} X_j^n + Y_j^m \qquad \forall j, m \tag{2.8}$$

where X_j^n is defined as in equation (2.6).

Random utility can then be adopted to describe how "industries" (including households) choose where to acquire their inputs, in a utility-maximizing or cost-minimizing way, subject to certain constraints. For example, MEPLAN and TRANUS determine trade volumes essentially based on the following disutility function[1]:

$$-u_{ij}^{n} = b_{i}^{n} + d_{ij}^{n} + \varepsilon_{ij}^{n} \qquad \forall i,j,n \tag{2.9}$$

where u_{ij}^{n} is the utility of purchasing one unit (one dollar) of sector n's goods from region i for use as inputs in region j[2]; b_{i}^{n} is the price of producing a unit of n in region i; d_{ij}^{n} is the price of transporting a unit of n from i to j (which may be a logsum term, from lower-order mode choice, time-of-day choice, and/or transport choices within a nested logit model framework [see, e.g., Ben-Akiva and Lerman, 1979]), and ε_{ij}^{n} is a random error term. If ε_{ij}^{n} follows the i.i.d. Gumbel distribution (McFadden, 1974), then the trade volume of sector n from i to j is given by:

$$x_{ij}^{n} = C_{j}^{n} \frac{\exp(\lambda^{n} v_{ij}^{n})}{\sum_{k} \exp(\lambda^{n} v_{kj}^{n})} \qquad \forall i,j,n \tag{2.10}$$

where λ^{n} is a dispersion parameter (inversely related to the standard deviation of the Gumbel error terms) and $v_{ij}^{n} = -(b_{i}^{n} + d_{ij}^{n})$, the systematic utility.

Here we assume the final demand of each zone, Y_{j}^{m}, is known, which may be hard to determine and ideally should be endogenous (especially over the long run). One possible model improvement is to define a certain number of export zones (ports, airports, etc.) whose export amounts are observable, and use another logit model to distribute the export demands across production zones. This has been done by Jin, Kockelman and Zhao (2002), and the utility function is similar to (2.9). In addition, one may simply regard the export zones as regular (producing) zones but assume practically infinite interzonal transportation costs from these export zones, to prevent them selling/producing any products.

Since Leontief technology is linear, the average cost of input n in region j is taken to be the weighted average (across input origins, i) of purchase prices (b_{i}^{n}) plus the transportation prices (d_{ij}^{n}) to region j.

$$c_{j}^{n} = \frac{\sum_{i} x_{ij}^{n}(b_{i}^{n} + d_{ij}^{n})}{\sum_{i} x_{ij}^{n}} \qquad \forall j,n \tag{2.11}$$

The sales price of a good produced by sector n in region j, b_{j}^{n}, is assumed equal to its manufacture cost[3], which is given by the following:

$$b_{j}^{n} = \sum_{m} a_{j}^{mn} \times c_{j}^{m} \qquad \forall j,n \tag{2.12}$$

In practice, the transportation costs, d_{ij}^{n}, are given exogenously (implying a non-congestible network)[4]. The dispersion parameters, λ^{n}, are generally estimated *a priori*, based on trade observations (for example, those included in the Commodity Flow Survey [BTS, 2001]).

It is proven here that simultaneously solving equations (2.7) through (2.12) produces a unique spatial equilibrium solution for trade volumes. An equilibrium is characterized here as a situation that satisfies all equations. In general, solving this complex set of equations requires iterative calculations. The standard algorithm, as suggested by Hunt (1993), can be summarized as follows:

Original RUBMRIO Algorithm, as Applied in Practice:

Given Y_i^m, a_j^{mn}, d_{ij}^n, and λ^n, solve for x_{ij}^n, b_j^n, and c_i^n, for all i, j, m, n.

Step 0: Initialization. Set all x_{ij}^n, b_i^n, and c_i^n to initial values (usually zeros).

Step 1: Calculate all utilities u_{ij}^n from equation (2.8); calculate production levels X_i^m from (2.6) and consumption levels C_j^n from equation (2.10).

Step 2: Update all x_{ij}^n using equation (2.9).

Step 3: Update all c_i^n using equation (2.11) and b_j^n using equation (2.12).

Step 4: Convergence test. Check the predefined convergence criterion. (For example, $\max(\left|x_{ij}^{n(t)} - x_{ij}^{n(t-1)}\right|) < 0.01 x_{ij}^{n(t-1)}, \forall i, j, n$, where t is the iteration number.) If the convergence criterion is met, then stop and the current solution $\{x_{ij}^n\}$ is taken to be the equilibrium solution; otherwise, go to step 1.

This iterative process is not clearly convergent. And it does not indicate whether its solution is unique (or whether it even exists). If the solution is not unique, a number of issues arise, such as which solution(s) could represent the system equilibrium and how the initial values should be chosen to obtain such a solution. The following section formulates the RUBMRIO model as a fixed-point problem. The fixed-point formulation reveals that prices are based on the exogenous transportation prices and other parameters. In addition, the fixed-point formulation suggests that there is a unique solution for prices when transportation costs and other parameters are known. Thus, commodity flows are unique, once prices are determined. Such information is crucial to successful implementation of RUBMRIO models, since non-existence and/or non-uniqueness present serious problems for applications and predictions.

3. A FIXED-POINT APPROACH TO THE RUBMRIO MODEL

The fixed-point approach is a major mathematical tool for numerical analysis. It has been extensively used to demonstrate the existence and uniqueness of solution concepts in game theory and economics (Border, 1985). Within the discipline of transportation engineering and planning, a number of studies make use of fixed-point formulations for trip assignment to networks. Dafermos (1980) first proposed a fixed-point model for equilibrium assignment on road networks with fixed demand; her work also provided results for convergence analysis. More recently, the fixed-point approach has been adopted as a general framework to define user equilibrium (UE) and stochastic user equilibrium (SUE) problems and develop solution algorithms (Cantarella and Cascetta, 1995; Cantarella, 1997).

A fixed-point formulation of the RUBMRIO model satisfies two objectives. First, it illuminates fundamental relationships among key variables to improve model understanding. Second, it allows one to determine solution existence and uniqueness, as well as specify convergent algorithms, suitable for large-size problems.

3.1 The Fixed-Point RUBMRIO Formulation

In order to simplify the equations, we denote P_{ij}^m as the probability (as defined in [2.10]) that region *j* purchases input *m* from region *i*:

$$P_{ij}^m = \frac{\exp(\lambda^m v_{ij}^m)}{\sum_k \exp(\lambda^m v_{kj}^m)} = \frac{\exp[-\lambda^m (b_i^m + d_{ij}^m)]}{\sum_k \exp[-\lambda^m (b_k^m + d_{kj}^m)]} \tag{3.1}$$

Then, by substituting (2.11) for c_j^m and combining the result with (2.10), equation (2.12) can be rewritten as follows:

$$\begin{aligned}
b_j^n &= \sum_m a_j^{mn} c_j^m = \sum_m a_j^{mn} \frac{\sum_i x_{ij}^m (b_i^m + d_{ij}^m)}{\sum_i x_{ij}^m} \\
&= \sum_m a_j^{mn} \frac{\sum_i C_j^m \frac{\exp(\lambda^m v_{ij}^m)}{\sum_k \exp(\lambda^m v_{kj}^m)} (b_i^m + d_{ij}^m)}{\sum_i x_{ij}^m} \\
&= \sum_m a_j^{mn} \frac{C_j^m \sum_i \frac{\exp(\lambda^m v_{ij}^m)}{\sum_k \exp(\lambda^m v_{kj}^m)} (b_i^m + d_{ij}^m)}{C_j^n} \\
&= \sum_m a_j^{mn} \sum_i \frac{\exp(\lambda^m v_{ij}^m)}{\sum_k \exp(\lambda^m v_{kj}^m)} \cdot (b_i^m + d_{ij}^m) \\
&= \sum_m a_j^{mn} \sum_i \frac{\exp[-\lambda^m (b_i^m + d_{ij}^m)]}{\sum_k \exp[-\lambda^m (b_k^m + d_{kj}^m)]} (b_i^m + d_{ij}^m) \\
&= \sum_m a_j^{mn} \sum_i P_{ij}^m \cdot (b_i^m + d_{ij}^m)
\end{aligned} \tag{3.2}$$

In equation (3.2), the prices $\{b_j^n\}$ are clearly defined as functions of themselves, if the exogenous transportation prices $\{d_{ij}^m\}$, dispersion parameters $\{\lambda^m\}$, and technical coefficients $\{a_j^{mn}\}$ are known. One should notice that the prices $\{b_j^n\}$ are *not* a function of the commodity flows $\{x_{ij}^n\}$ or consumption levels when written in this way. This suggests perfectly elastic supply, thanks to constant-rate IO technologies and an implicit lack of resource constraints.

Denote $\vec{b} = \{b_j^n\}$, and let

$$P_{ij}^m(\vec{b}) = \frac{\exp[-\lambda^m (b_i^m + d_{ij}^m)]}{\sum_k \exp[-\lambda^m (b_k^m + d_{kj}^m)]} \tag{3.3}$$

And, from equation (3.2), let

$$f_j^n(\vec{b}) = \sum_m a_j^{mn} \sum_i P_{ij}^m(\vec{b}) \cdot (b_i^m + d_{ij}^m) \tag{3.4}$$

Therefore, one has a fixed-point problem from (3.2) as follows:

$$\vec{b} = \vec{f}(\vec{b}) \tag{3.5}$$

And the elements of the function $\vec{f}$ are defined by (3.4).

To guarantee solution existence, first impose a rather weak condition on the feasible set *t*. Let $K_b = \{b_{ij}^n \mid 0 \le b_{ij}^n \le b_{ij}^{n*}, \forall i,j,n\}$, where $\{b_{ij}^{n*}\}$ are upper bounds which we assume can be determined *a priori*. (In practice, this is usually the case.) Then K_b is a bounded and closed convex subset (i.e., a compact set) on the space R^{MJ}. One easily can observe that, if the prices are bounded, the function $\vec{f}$ also can be considered bounded, since it is a convex combination of prices (plus transportation costs) across space (i.e., $\sum_i P_{ij}^m = 1$) and across economic sectors (i.e., $\sum_m a_j^{mn} \le 1$). If one assumes that the $\vec{f}$'s upper bounds also are $\{b_{ij}^{n*}\}$, one essentially assumes that the upper bounds are large enough to accommodate the transportation prices' contributions to $\vec{f}$. Then, $\vec{f}$ is a mapping $K_b \to K_b$, and it is continuous. Following Brouwer's theorem (Khamsi and Kirk, 2001), the following condition emerges:

Existence Condition for Price Solution

The fixed-point problem (3.5) provides at least one solution if and only if there exist positive constants $\{b_{ij}^{n*}\}$ such that the fixed-point problem (3.5) provides a solution in K_b.

Sufficient conditions for the uniqueness of a fixed-point problem solution are given by Banach's theorem (Border, 1985) which requires that the function be contractive over a complete set, or the function be quasi-contractive (implying monotonicity) over a compact set. For purposes of definition, a function $\vec{f}$ provides contractive mapping of $\vec{b}$ if the following holds:

$$\left\|\vec{f}(\vec{b}) - \vec{f}(\vec{b}\,')\right\| \le \varphi \left\|\vec{b} - \vec{b}\,'\right\|, \vec{b} \ne \vec{b}\,', 0 < \varphi < 1 \tag{3.6}$$

where $\|\cdot\|$ denotes the norm of the vector, which is a measure of distance on the vector space (see Golub and Van Loan, 1989). Due to the mean-value theorem (see Khamsi and Kirk, 2001),

$$\vec{f}(\vec{b}) - \vec{f}(\vec{b}\,') = \nabla\vec{f}(\vec{\delta})(\vec{b} - \vec{b}\,') \tag{3.7}$$

where $\vec{\delta}$ lies between $\vec{b}$ and $\vec{b}\,'$. So one only needs to study the norm of the Jacobian matrix (a measure of distance on the matrix space). In other words, if $\left\|\nabla\vec{f}(\vec{b})\right\| < 1$, then the fixed-point problem has a unique solution and the sequence $\vec{b}^{(t+1)} = \vec{f}(\vec{b}^{(t)})$ converges on the unique solution $\vec{b} = \vec{f}(\vec{b})$, if $\vec{b}^{(o)} \in K_b$. This property is illustrated for four general cases in Figure 3.1.

We first consider a simplified case where the probabilities $\{P_{ij}^m\}$ are fixed (i.e., they are not a function of prices). The Jacobian matrix is:

$$\nabla \vec{f}(\vec{b}) = \begin{bmatrix} \frac{\partial f_1^1(\vec{b})}{\partial b_1^1} & \frac{\partial f_1^1(\vec{b})}{\partial b_2^1} & \cdots & \frac{\partial f_1^1(\vec{b})}{\partial b_J^M} \\ \frac{\partial f_2^1(\vec{b})}{\partial b_1^1} & \frac{\partial f_2^1(\vec{b})}{\partial b_2^1} & \cdots & \frac{\partial f_2^1(\vec{b})}{\partial b_J^M} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_J^M(\vec{b})}{\partial b_1^1} & \frac{\partial f_J^M(\vec{b})}{\partial b_2^1} & \cdots & \frac{\partial f_J^M(\vec{b})}{\partial b_J^M} \end{bmatrix} = \begin{bmatrix} a_1^{11} P_{11}^1 & a_1^{11} P_{21}^1 & \cdots & a_1^{M1} P_{J1}^M \\ a_2^{11} P_{12}^1 & a_2^{11} P_{22}^1 & \cdots & a_2^{M1} P_{J2}^M \\ \vdots & \vdots & \vdots & \vdots \\ a_J^{1M} P_{1J}^1 & a_J^{1M} P_{2J}^1 & \cdots & a_J^{MM} P_{JJ}^M \end{bmatrix} \tag{3.8}$$

There are two properties of this Jacobian matrix: first, it is a positive matrix (i.e., all elements of it are strictly positive); and second, the *row* sums are the following:

$$\sum_{i,m} \frac{\partial f_j^n(\vec{b})}{\partial b_i^m} = \sum_i \sum_m a_j^{mn} P_{ij}^m = \sum_m a_j^{mn} (\sum_i P_{ij}^m) = \sum_m a_j^{mn}, \quad \forall j,n \tag{3.9}$$

Therefore, we calculate the Jacobian matrix's norm[5] as follows (Golub and Van Loan, 1989):

$$\left\| \nabla \vec{f}(\vec{b}) \right\| = \max_{\substack{1 \le j \le J \\ 1 \le n \le M}} \left| \sum_{i,m} \frac{\partial f_j^n(\vec{b})}{\partial b_i^m} \right| = \max_{\substack{1 \le j \le J \\ 1 \le n \le M}} (\sum_m a_j^{mn}) \tag{3.10}$$

We also note that the technical coefficients have the following property:

$$\sum_m a_j^{mn} < 1 \forall j,n, \tag{3.11}$$

because the total value of inputs required to produce one dollar of sector *n* product should be less than one dollar. If not, final demand effects will multiply infinitely through an IO model with any column that sums to one (and the matrix **(I-A)** will not be invertible). In practice, equation (3.11)'s constraint is met through import or profit leakages[6], since labor is generally endogenous and represents an "industry" that can absorb all profits. Thus, if $\left\| \nabla \vec{f}(\vec{b}) \right\| < 1$, $\vec{f}$ is contractive on $\vec{b}$, and there exists a unique solution for the simplified, fixed-probability problem.

Following the same process, consider now the general situation, where the probabilities are determined by the disutility which depends on prices.

$$\frac{\partial f_j^n(\vec{b})}{\partial b_i^m} = \frac{\partial}{\partial b_i^m} [\sum_m a_j^{mn} \sum_k P_{kj}^m(\vec{b}) \cdot (b_k^m + d_{kj}^m)] = a_j^{mn} \frac{\partial}{\partial b_i^m} \left[\sum_k P_{kj}^m(\vec{b}) \cdot (b_k^m + d_{kj}^m) \right]$$

(3.12)

where

$$\frac{\partial}{\partial b_i^m}\left[\sum_k P_{kj}^m(\vec{b})\cdot(b_k^m+d_{kj}^m)\right]=\sum_{k\neq i}\frac{\partial}{\partial b_i^m}\left[P_{kj}^m(\vec{b})\cdot(b_k^m+d_{kj}^m)\right]+\frac{\partial}{\partial b_i^m}\left[P_{ij}^m(\vec{b})\cdot(b_i^m+d_{ij}^m)\right]$$
$$=\sum_{k\neq i}(b_k^m+d_{kj}^m)\frac{\partial}{\partial v_{ij}^m}P_{kj}^m(v_{ij}^m)\cdot\frac{\partial}{\partial b_i^m}(v_{ij}^m)+(b_i^m+d_{ij}^m)\frac{\partial}{\partial v_{ij}^m}P_{ij}^m(v_{ij}^m)\cdot\frac{\partial}{\partial b_i^m}(v_{ij}^m)+P_{ij}^m$$
$$=\sum_{k\neq i}(b_k^m+d_{kj}^m)P_{kj}^m\cdot(-P_{ij}^m)\cdot(-\lambda^m)+(b_i^m+d_{ij}^m)P_{ij}^m\cdot(1-P_{ij}^m)(-\lambda^m)+P_{ij}^m$$
$$=P_{ij}^m\cdot\{1-\lambda^m[(b_i^m+d_{ij}^m)(1-P_{ij}^m)-\sum_{k\neq i}P_{kj}^m\cdot(b_k^m+d_{kj}^m)]\}$$
$$=P_{ij}^m\cdot\{1-\lambda^m[(b_i^m+d_{ij}^m)-\sum_{k}P_{kj}^m\cdot(b_k^m+d_{kj}^m)]\}$$
$$=P_{ij}^m\cdot\{1-\lambda^m[(b_i^m+d_{ij}^m)-c_j^m]\} \tag{3.13}$$

Then, equation (3.12) becomes:

$$\frac{\partial f_j^n(\vec{b})}{\partial b_i^m}=a_j^{mn}P_{ij}^m[1-\lambda^m(b_i^m+d_{ij}^m-c_j^m)] \tag{3.14}$$

Notice that the second property of the Jacobian matrix in the simplified situation holds true here as well:

$$\begin{aligned}\sum_{i,m}\frac{\partial f_j^n(\vec{b})}{\partial b_i^m}&=\sum_i\sum_m a_j^{mn}P_{ij}^m\cdot[1-\lambda^m(b_i^m+d_{ij}^m-c_j^m)]\\&=\sum_m a_j^{mn}\{\sum_i P_{ij}^m-\lambda^m[\sum_i P_{ij}^m\cdot(b_i^m+d_{ij}^m)-c_j^m\sum_i P_{ij}^m]\}\\&=\sum_m a_j^{mn}[1-\lambda^m(c_j^m-c_j^m)]\\&=\sum_m a_j^{mn},\quad\forall j,n\end{aligned} \tag{3.15}$$

If one wants to apply the finding from the simplified situation (i.e., with fixed probabilities), one needs to determine whether all Jacobian matrix elements are positive. There exist three specific situations, which can be characterized as the following:

(i) $b_i^m+d_{ij}^m<c_j^m \quad \forall i,j,m$. The economic meaning of this situation is that the sales price at region *i* plus the transportation price to region *j* is less than the average input cost for that good *m* in region *j*. Under this condition, region *j* will purchase a positive amount of sector *m* from region *i*, so the derivative in equation (3.14) is positive (for any solution that satisfies this inequality).

(ii) $b_i^m+d_{ij}^m=c_j^m \quad \forall i,j,m$. This situation is similar to a "spatial price equilibrium", where the sales price at region *i* plus the transportation price to region *j* equals the average cost of input *m* in region *j*, and only under this condition there exist flows from *i* to *j* (see Nagurney, 1999). Then, equation (3.14)'s derivative (for any solution that satisfies this condition) is positive.

(iii) $b_i^m+d_{ij}^m>c_j^m \quad \forall i,j,m$. The economic interpretation of this situation is that the sales price at region *i* plus the transportation price to region *j* exceeds the average cost of input *m* in region *j*. If the prices satisfy a "spatial price equilibrium," as defined here, there will be no purchase from region *i* (Nagurney, 1999). However, in the RUBMRIO model, the commodity flow distribution is based on random utility theory, so there is a certain (small) amount of any commodity that will be purchased from any origin region whose sales price plus transportation cost exceeds the average input cost at the destination region. Under this scenario, the condition to ensure that the Jacobian matrix elements are all positive is the following:

$$\lambda^m < \frac{1}{\max\limits_{1\le i,j\le J}(b_i^m + d_{ij}^m - c_j^m)} \quad \forall m \tag{3.16}$$

Inequality (3.16) describes a situation wherein the dispersion parameters $\{\lambda^m\}$ are sufficiently small: i.e., the commodity purchases are reasonably well spread over all regions. A uniqueness condition summarizes the above three cost situations:

(Restrictive) Uniqueness Condition for Price Solution.

The fixed-point problem (3.5) results in at most one equilibrium price solution if the dispersion parameters $\{\lambda^m\}$ *are sufficiently small such that inequality (3.16) holds.*

This condition is rather restrictive for the dispersion parameters. So we next discuss conditions under which dispersion parameters are relatively large. If the $\{\lambda^m\}$ are sufficiently large, then the commodity flows become local and concentrated (i.e., the origin regions offering minimum total cost [sales price plus transportation cost] will dominate the flow to the destination region). The flows (or the probabilities[7]) from all other regions to this destination will be close to zero. The average cost then tends to be very close to the dominant, minimum (total) price. This satisfies the above situation (ii). Therefore, the Jacobian matrix will have rows where the only positive elements tie to the dominant regions for each sector. And equation (3.15) holds true here as well. Then $\vec{f}$ is contractive on $\vec{b}$, and the fixed-point problem (3.5) provides a unique equilibrium price solution.

Since the problem (3.5) has a unique price solution under the conditions that the $\{\lambda^m\}$ are either sufficiently small or sufficiently large, it is natural to suspect that the uniqueness property holds with other, regular $\{\lambda^m\}$ values (and that is the common case for the dispersion parameters in practice). Suppose we specify the origin index as the following:

$$b_1^m + d_{1j}^m \le b_2^m + d_{2j}^m \le \ldots \le b_{i^*}^m + d_{i^*j}^m \le c_j^m \le b_{i^*+1}^m + d_{i^*+1,j}^m \le \ldots \le b_J^m + d_{Jj}^m \quad \forall j, m \tag{3.17}$$

Thus, a lower origin index indicates a lower total cost. And the equal signs in (3.17) do not all hold in general; otherwise, one has situation (ii) above (and there is a unique solution). The probability in (3.1) can be rewritten as:

$$P_{ij}^m = \frac{\exp[-\lambda^m(b_i^m + d_{ij}^m - b_1^m - d_{1j}^m)]}{\sum\limits_k \exp[-\lambda^m(b_k^m + d_{kj}^m - b_1^m - d_{1j}^m)]} \tag{3.18}$$

Clearly, the largest fraction of a commodity will be purchased from the region offering the lowest total cost (i.e., sales price plus transportation cost), and the purchase probabilities from other regions depend on differences between their total costs and the lowest total cost. If the difference is large enough, the probabilities approach zero. For example, if $\lambda^m = 10$ and the difference is 2, then

$$P_{ij}^m = \frac{\exp(-20)}{1 + \sum\limits_{k\ne 1} \exp[-\lambda^m(b_k^m + d_{kj}^m - b_1^m - d_{1j}^m)]} < \exp(-20) = 2.06\times 10^{-9} \approx 0 \tag{3.19}$$

If for those regions satisfying situation (iii), where sales prices plus transportation costs exceed average cost (and therefore exceed the lowest cost), cost differences are so great that

$$P_{ij}^{m} \approx 0, for\ i > i^{*}, \forall j, m \ , \tag{3.20}$$

the resulting purchase probabilities can be ignored, where i^{*} is the region with the closest (and lower) sales prices plus transportation cost to average cost. Under this assumption (and the rest regions satisfying situation (i) or (ii)), it is easy to obtain the following result:

$$\sum_{i,m} \left| \frac{\partial f_j^n(\vec{b})}{\partial b_i^m} \right| < 1 \tag{3.21}$$

Then, *almost* certainly, there exists a *unique* price solution for the RUBRIO model for all dispersion parameter levels. This proof neglects purchase probabilities from regions whose total costs exceed average cost (for a given destination region); however, this omission is only viable for the fixed-point price-solution proof. These small probabilities cannot be neglected in the following commodity flow calculation, since the products of small probabilities and large commodity volumes can still result in large (monetary) flows.

Once the cost vector, $\vec{b}$, is known, the probability vector $\vec{P}$ can be computed easily. From equation (3.2), one recognizes that $\vec{b}$ can be written independently of commodity flows; in addition, from equation (3.3), it is clear that $\vec{P}$ is not implicitly a function of commodity flows.

From equations (2.9) and (2.10), one has the following:

$$x_{ij}^{m} = P_{ij}^{m} \sum_{i} x_{ij}^{m} = P_{ij}^{m} \cdot (\sum_{n} a_j^{mn} \sum_{k} x_{jk}^{n} + Y_j^{m}), \ \forall i, j, m \tag{3.22}$$

Denoting:

$$g_{ij}^{m}(\vec{x}) = P_{ij}^{m} \cdot (\sum_{n} a_j^{mn} \sum_{k} x_{jk}^{n} + Y_j^{m}) \tag{3.23}$$

produces another fixed-point problem:

$$\vec{x} = \vec{g}(\vec{x}) \tag{3.24}$$

Similar to earlier descriptions, we first impose a weak condition on the feasible set to guarantee solution existence: Let $K_x = \left\{ x_{ij}^{m} \mid 0 \le x_{ij}^{m} \le x_{ij}^{m*}, \forall i, j, m \right\}$, where $\left\{ x_{ij}^{m*} \right\}$ are upper bounds. Then, K_x is a bounded, closed, and convex subset of R^{MIJ} . Also, assume that $\vec{g}$ maps $K_x \to K_x$ and is continuous. Following Brouwer's theorem (Khamsi and Kirk, 2001), one has the following condition:

Existence Condition for Flow Solution.
The fixed-point problem (3.24) permits at least one flow solution if and only if there exist positive constants $\left\{ x_{ij}^{m*} \right\}$*, such that the problem permits a solution in* K_x.

Again, we study the contractiveness of $\vec{g}$ over K_x in order to obtain sufficiency conditions for uniqueness of the flow solution. The elements of the Jacobian matrix of $\vec{g}$ are:

$$\frac{\partial}{\partial x_{kl}^{n}} g_{ij}^{m}(\vec{x}) = \begin{cases} P_{ij}^{m} a_j^{mn} & if\ k = j \\ 0 & otherwise \end{cases} \quad \forall i, j, k, l, m, n \tag{3.25}$$

The Jacobian matrix of $\vec{g}$ is:

$$\nabla \vec{g}(\vec{x}) = \begin{bmatrix} \frac{\partial g_{11}^1(\vec{x})}{\partial x_{11}^1} & \frac{\partial g_{11}^1(\vec{x})}{\partial x_{21}^1} & \cdots & \frac{\partial g_{11}^1(\vec{x})}{\partial x_{JJ}^M} \\ \frac{\partial g_{21}^1(\vec{x})}{\partial x_{11}^1} & \frac{\partial g_{21}^1(\vec{x})}{\partial x_{21}^1} & \cdots & \frac{\partial g_{21}^1(\vec{x})}{\partial x_{JJ}^M} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_{JJ}^M(\vec{x})}{\partial x_{11}^1} & \frac{\partial g_{JJ}^M(\vec{x})}{\partial x_{21}^1} & \cdots & \frac{\partial g_{JJ}^M(\vec{x})}{\partial x_{JJ}^M} \end{bmatrix} = \begin{bmatrix} P_{11}^1 a_1^{11} & 0 & \ldots & 0 \\ P_{21}^1 a_1^{11} & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & P_{12}^1 a_2^{11} & \ldots & 0 \\ 0 & P_{22}^1 a_2^{11} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ P_{J1}^M a_1^{MM} & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & P_{J2}^M a_2^{MM} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & P_{J-1,J}^M a_J^{MM} \\ 0 & 0 & \ldots & P_{JJ}^M a_J^{MM} \end{bmatrix} \quad (3.26)$$

There are two properties of this Jacobian matrix: first, it is nonnegative (i.e., all elements are equal to or larger than zero); second, the *column* sums are the following:

$$\sum_{i,j,m} \frac{\partial g_{ij}^m(\vec{x})}{\partial x_{kl}^n} = \sum_i \sum_m P_{ij}^m a_j^{mn} = \sum_m a_j^{mn} (\sum_i P_{ij}^m) = \sum_m a_j^{mn}, \quad \forall k(=j), l, n \quad (3.27)$$

Therefore, we have the following norm of the Jacobian (Golub and Van Loan, 1989):

$$\|\nabla \vec{g}(\vec{x})\| = \max_{\substack{1 \le k,l \le J \\ 1 \le n \le M}} \left| \sum_{i,j,m} \frac{\partial g_{ij}^m(\vec{x})}{\partial x_{kl}^n} \right| = \max_{\substack{1 \le k,l \le J \\ 1 \le n \le M}} (\sum_m a_j^{mn}) < 1 \quad (3.28)$$

This implies that $\vec{g}$ is contractive on $\vec{x}$, and so there exists a unique solution for the fixed-probability problem (3.24), producing the following condition:

Uniqueness Condition for Flow Solution

The fixed-point problem (3.24) results in at most one equilibrium flow solution. Thus, the existence and uniqueness of solution flows are very general, once prices and probabilities are known.

In summary, through a general fixed-point approach, one can easily find an interesting RUBMRIO model relationship where prices are independent of flows. Moreover, the price solutions exist and are unique under the conditions described above. In the next section, we make use of the fixed-point approach to verify the convergence of the original RUBMRIO model, and we propose a modified solution algorithm to efficiently apply the fixed-point formulation's properties.

3.2 A Modified RUBMRIO Algorithm

Assuming that the conditions for price solution existence and uniqueness hold, the sequence generated by the iterative function $\vec{b}^{(t+1)} = \vec{f}(\vec{b}^{(t)})$ converges on the unique solution $\vec{b} = \vec{f}(\vec{b})$, if $\vec{b}^{(o)} \in K_b$ (Khamsi and Kirk, 2001). This price convergence does require that prices be bounded. But one easily can construct lower bounds of zero (since prices should be non-negative) and upper bounds as very large numbers. Once the unique price solution is obtained (through the fixed-point sequence), a similar sequence for computation of flows can be generated as: $\vec{x}^{(t+1)} = \vec{g}(\vec{x}^{(t)})$, if $\vec{x}^{(o)} \in K_x$.

Convergence of the fixed-point sequence also suggests that the original RUBMRIO solution algorithm is convergent, since its sequence of iterative price vectors is similar to the fixed-point sequence. The coincidence is not surprising because nearly all iterative solution algorithms rely on the fixed-point approach. However, only when solution existence and uniqueness are assured can this sequence be guaranteed to converge to the correct solution. Otherwise, different initial values could lead to different results, or the iterative process may never converge.

The original RUBMRIO algorithm calculates both prices and flows at each iteration. However, we have shown prices to be independent of the flows; so there is no need to calculate flows before the prices (or to compute prices after their convergence) in order to achieve the unique solution for prices and flows. A modified algorithm is now presented, which efficiently applies the fixed-point approach for price and flow calculations.

The Modified RUBMRIO Algorithm

Step 0: Initialization. Set $\vec{x}^{(o)} \in K_x$ and $\vec{b}^{(0)} \in K_b$; let $t = 1$.

Step 1: Computation of prices. Calculate the prices $\{b_j^n\}$ using the following fixed-point equation:

$$\vec{b}^{(t+1)} = \vec{f}(\vec{b}^{(t)}) \tag{3.29}$$

where $\vec{f}(\vec{b})$ is defined in (3.4).

Step 2: Verification of prices convergence. If $\max(\left|b_j^{n(t)} - b_j^{n(t-1)}\right|) < \tau, \forall j, n$, with a pre-specified tolerance $\tau > 0$, then go to step 3; else, set $t = t+1$, and go to step 1.

Step 3: Computation of probabilities. Compute probabilities using equation (3.1).

Step 4: Computation of flows. Set $t = 1$, and calculate the flows using the following fixed-point equation:

$$\vec{x}^{(t+1)} = \vec{g}(\vec{x}^{(t)}) \tag{3.30}$$

where $\vec{g}(\vec{x})$ is defined in (3.23).

Step 5: Convergence test of flows. If $\max(\left|x_{ij}^{n(t)} - x_{ij}^{n(t-1)}\right|) < \tau,\ \forall i, j, n$, then stop; the current solution, $\{x_{ij}^n\}$ is the set of equilibrium solutions. Otherwise, set $t = t+1$, and go to step 4.

4. NUMERICAL EXAMPLE

In this section a numerical example demonstrates RUBMRIO model solution existence and uniqueness and compares the original and modified algorithms. We consider a simple case with only two regions and two commodity sectors; the exogenous variables' values are shown in Table 4.1. Dispersion parameters are arbitrarily set to $\lambda^1 = 15$ and $\lambda^2 = 0.2$, with the larger value implying less dispersed flows.

4.1 Convergence to the Unique Solution

Given this example, we first examined convergence patterns for prices and flows using the original algorithm. The convergence criterion requires that absolute values of prices and flows between two successive iterations differ by no more than 0.0001. We tested two scenarios with different initial values: the first started with zero values (which are common start points), and the second used some randomly generated positive values.

The original algorithm converged after 138 iterations for the first scenario and 111 iterations for the second scenario. Both converged to the same solution, as depicted in Figure 4.1. The second scenario used non-zero initial values and converged faster, which suggests that the traditional start values (zeros) probably are not the best choice. Price patterns were evaluated as well, and these are shown in Figure 4.2. These actually converged after just 86 iterations for the first scenario and 95 iterations in the second scenario. Finally, similar runs of both scenarios were made using the modified algorithm's fixed-point sequence. These are discussed here now.

4.2 Algorithms Comparison

It is rather natural and efficient to adopt the modified algorithm to eliminate unnecessary computations in the original algorithm. Table 4.2 compares the original and modified algorithms for the RUBMRIO model. Both algorithms converge to the same, unique solutions. But the modified algorithm reduces computational effort, especially in the computation of prices.

5. CONCLUSIONS AND EXTENSIONS

In this paper, a fixed-point formulation of a random-utility-based spatial input-output (RUBMRIO) model was constructed, in order to examine the properties of solutions to many integrated land use-transportation models. This formulation and the problem properties allowed us to develop existence and uniqueness conditions for the RUBMRIO model solutions.

Under weak conditions regarding sales prices, the set of solution prices were shown to be unique. Once prices and spatial purchase probabilities are known, commodity flows also are found to be unique. The fixed-point formulation established here verifies that the common/original RUBMRIO iterative algorithm always converges. However, a modified algorithm was demonstrated to be more efficient.

Several other, related issues are worthy of further research. For example, when applying the RUBMRIO model over a congestible transportation network, one needs to compute transportation costs as a function of commodity flows, rather than treating them as exogenous variables. One practical way to achieve this is to link the RUBMRIO model with a UE or SUE assignment model. It has been shown that there exist unique solutions to general UE and SUE problem specifications (e.g., Sheffi, 1985; Cantarella and Cascetta, 1995), and we demonstrated here that the RUBMRIO model solution is unique. Thus, the only gap is a theoretical analysis of the uniqueness of the overall/integrated congestible system solution. However, we fully expect that this exists (based on proofs of congestible travel demand model solution uniqueness using fixed-point approach [Cantarella, 1997]).

Additionally, permitting substitution across inputs to production (through continuous production functions and explicit recognition of competing input prices) will make the problem more realistic. However, it also will make the problem much more difficult. Purchase decisions will become functions of all prices across all regions, rather than across just regions, and the straightforward matrix algebra of the IO model will disappear. Moreover, calibration of such production processes, as functions of the variables tracked in these models is highly unlikely (due to anonymity, cost, and other issues). However, progress is being made (e.g., Abraham and Hunt, 1999) and model improvements are expected. In future extensions to the RUBMRIO model, the fixed-point approach may continue to be very useful for proving solution existence and uniqueness, and for evaluating solution algorithms.

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ENDNOTES:

[1] In practice, there are other items in the disutility function, such as the "excess profit" made when producing a dollar of a commodity in a region, the region-specific (constant) disutility associated with producing a dollar of a commodity in a region, and the "size term" (proportional to the log of the number of sites available to a commodity in a region; see Hunt, 1993). The introduction of these exogenously determined components does not affect the findings in this paper.

[2] Note that this specification is independent of the user/consumer of the good. With better data (for example, a Commodity Flow Survey that specifies producing and consuming sectors, for each commodity shipped), one could make these equations user-dependent. As practiced in existing models (such as MEPLAN and TRANUS), however, they are independent.

[3] One can include profits here, if one wishes.

[4] This can be made more realistic by allowing travel-time feedbacks which update travel costs, d_{ij}^{n}. However, the proofs of trade-solution uniqueness and existence would then require more mathematics; these proofs are not provided here.

[5] According to the Norm-Equivalence Theorem (see Ortege and Rheinboldt, 1970), all norms on R^n are equivalent. Here we use the l_∞-norm (and, later, the l_1-norm) to obtain the sufficiency result.

[6] Money is spent outside the region, so no multiplying effects are locally or regionally generated by those expenditures.

[7] A formal discussion about the dominate probability due to the sufficiently large dispersion parameter can be found in Wehr and Xin (1996, Proposition 2.1, p. 5).

[8] Both UE and SUE solution uniqueness require that flows on distinct links do not interact.

References:

Abraham, J.E. and J.D. Hunt (1999). "Semi-Automated Calibration of the MEPLAN Model of Sacramento", 6th Computers in Urban Planning and Urban Management Conference, Venice, Italy.

Anas, A. (1984). "Discrete choice theory and the general equilibrium of employment, housing, and travel networks in a Lowry-type model of the urban economy". *Environment and Planning,* 16(A), pp.1489-1502.

Ben-Akiva, M. and S. Lerman (1979). "Disaggregate Travel and Mobility Choice Models and Measures of Accessibility", in D. A. Hensher and P. R. Stopher (Eds.) *Behavioral Travel Modeling*. London: Croom Helm.

Border, K. C. (1985). *Fixed Point Theorems with Applications to Economics and Game Theory*. New York: Cambridge University Press.

Bureau of Transportation Statistics (BTS) (2001). 1997 Commodity Flow Survey Data CD-ROM, Washington, D.C.

Cantarella, G. E. (1997). "A General Fixed-Point Approach to Multimode Multi-User Equilibrium Assignment with Elastic Demand", *Transportation Science*, 31(2), pp. 107-128.

Cantarella, G. E., and E. Cascetta (1995). "Dynamic Processes and Equilibrium in Transportation Networks: Towards a Unifying Theory", *Transportation Science*, 29, pp. 305-329.

Dafermos, S., (1980). "Traffic Equilibrium and Variational Inequalities", *Transportation Science*, 14(1), pp. 43-54.

De la Barra T., (1994). *Integrated Land Use and Transport Modeling.* New York: Cambridge University Press.

Hunt, J. D. (1993). "A description of the MEPLAN framework for land use and transport interaction modeling". Paper presented at the 73rd Annual Transportation Research Board Meetings January 9-13, Washington, D.C.

Isard, W. (1960). *Methods of Regional Analysis: An Introduction to Regional Science*. Cambridge, MA: M.I.T. Press and Wiley (New York).

Golub, G. H. and C. F. Van Loan (1983). *Matrix Computations*, Baltimore: The Johns Hopkins University Press.

Khamsi, M. A., and W. A. Kirk (2001). *An Introduction to Metric Spaces and Fixed Point Theory*. New York: John Willy and Sons, INC. Press.

Kim, T. J., (1989). *Integrated Urban System modeling: Theory and Practice*. Norwell, MA: Martinus Nijhoff.

Kim, T. J., H. Ham and D. E. Boyce (2002) Economic impacts of transportation network changes: Implementation of a combined transportation and input-output model, *Papers in Regional Science* 81, 223-246.

Leontief, W. W. (1941). *The Structure of American Economy, 1919-1929*. Cambridge, MA: Harvard University 8.

Leontief, W. W., and A. Strout (1963). "Multiregional Input-Output Analysis." In *Structural Interdependence and Economic Development*, edited by T. Barna, pp.119-150. New York: Macmillan.

Lin, J., K. Kockelman, and Y. Zhao, (2002). "Tracking Land Use, Transport, and Industrial Production using Random-Utility Based Multizonal Input-Output Models: Applications for Texas Trade." Working Paper submitted to the 82nd Annual Meeting of the Transportation Research Board, January 2002 in Washington, D.C.

McFadden, D. (1974). "Conditional Logit Analysis of Qualitative Choice Behavior," *Frontiers in Econometrics*, edited by Zarembka, pp.105-142, New York: Academic Press.

Nagurney A. (1999). *Network Economics: A Variational Inequality Approach*, Revised second edition, Boston: Kluwer Academic Publishers.

Oppenheim, N. (1985) *Urban Travel Demand Modeling*. New York: Wiley.

Ortege, J. M., and W. C. Rheinboldt, (1970). *Iterative Solution of Nonlinear Equations in Several Variables*, New York: Academic Press.

Sheffi, Y. (1985) *Urban Transportation Network*, Englewood Cliffs, NJ: Prentice-Hall..

Wilson, A. G. (1970). “Interregional Commodity Flows: Entropy Maximizing Procedures”, *Geographical Analysis*, 2, pp. 255-282.

Wehr, J. and J. Xin (1996). “White Noise Perturbation of the Viscous Shock Fronts of the Burgers Equation”, *Common Mathematics and Physics*, 181, pp. 183-203.

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TABLE 4.1. Exogenous Values for Numerical Example (model specification)

Variables	Values	Variables	Values	Variables	Values
Transportations prices ($)		Technical Coef.		Final Demand ($)	
d_{11}	2	a^{11}	0.2	Y_1^1	100
d_{12}	10	a^{12}	0.8	Y_2^1	200
d_{21}	10	a^{21}	0.7	Y_1^2	20
d_{22}	1	a^{22}	0.1	Y_2^2	50

TABLE 4.2. Comparison of the original and modified RUBMRIO algorithms

Scenario	Variables	Computation effort	
		Original Algorithm	Modified Algorithm
1 (zero initial values)	Prices	138 iterations	86 iterations
	Flows	138 iterations	137 iterations
2 (arbitrary initial values)	Prices	111 iterations	95 iterations
	Flows	111 iterations	110 iterations

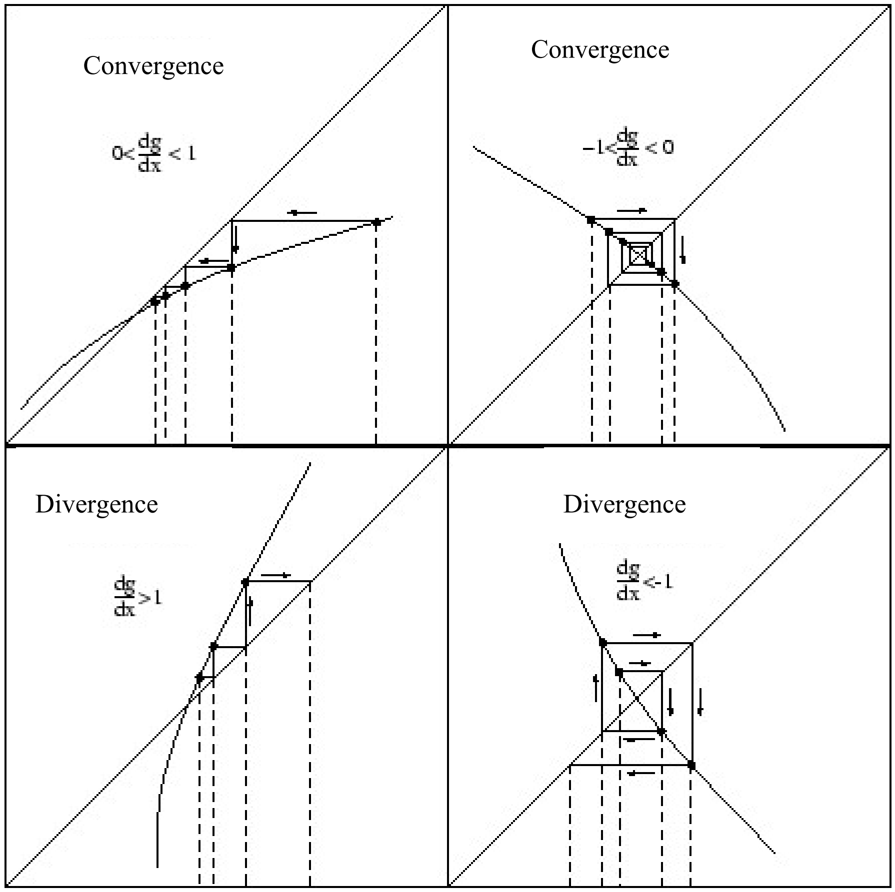


FIGURE 3.1 Examples of fixed-point problem convergence to unique solutions

Note: Fixed-point iteration for a general function $g(x)$ for four cases of interest. Positive-slope cases are shown on the left. Negative-slope cases are shown on the right. The solution sequences converge to the fixed-point solution only when the slopes lie between -1 and 1.

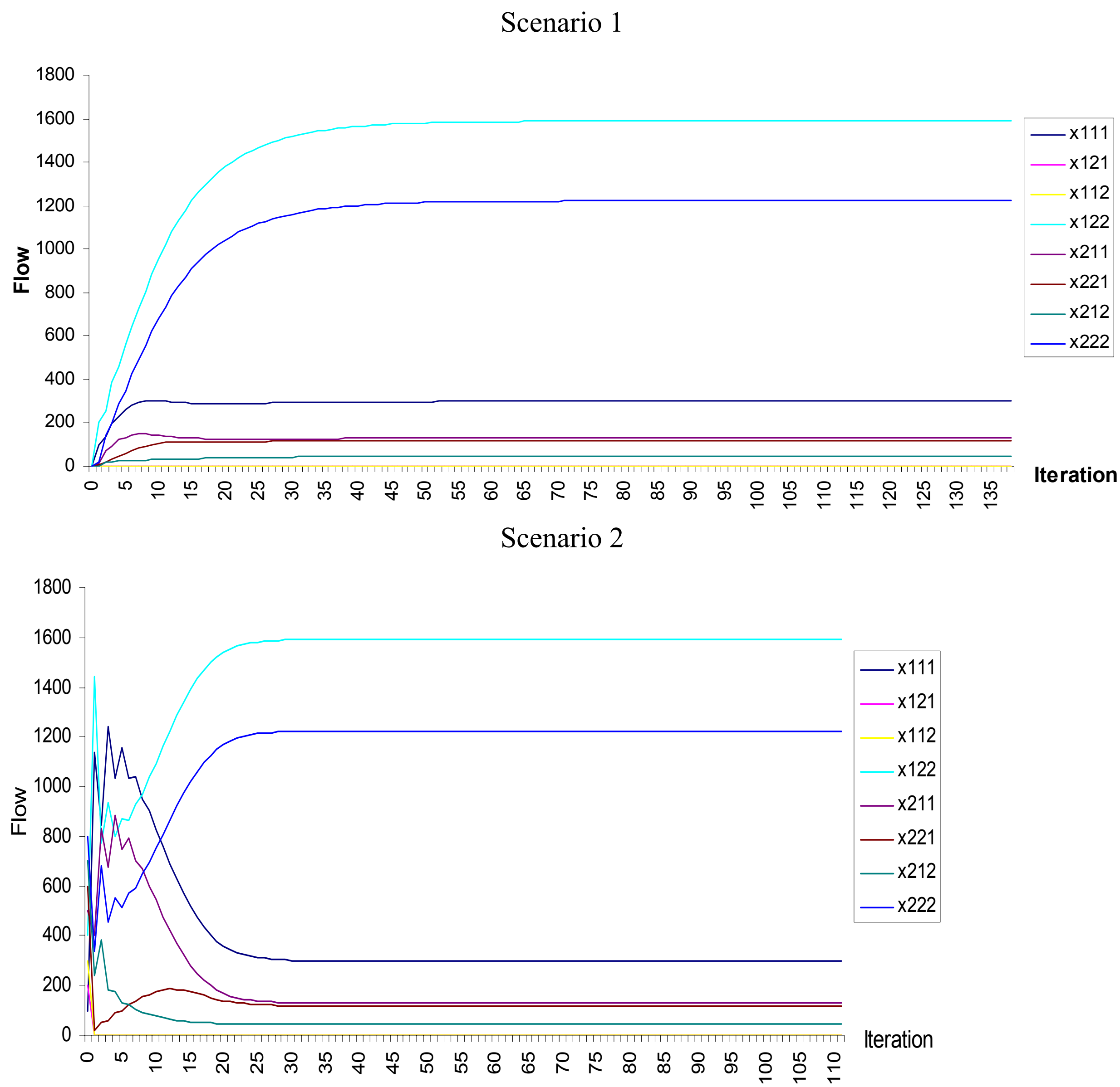
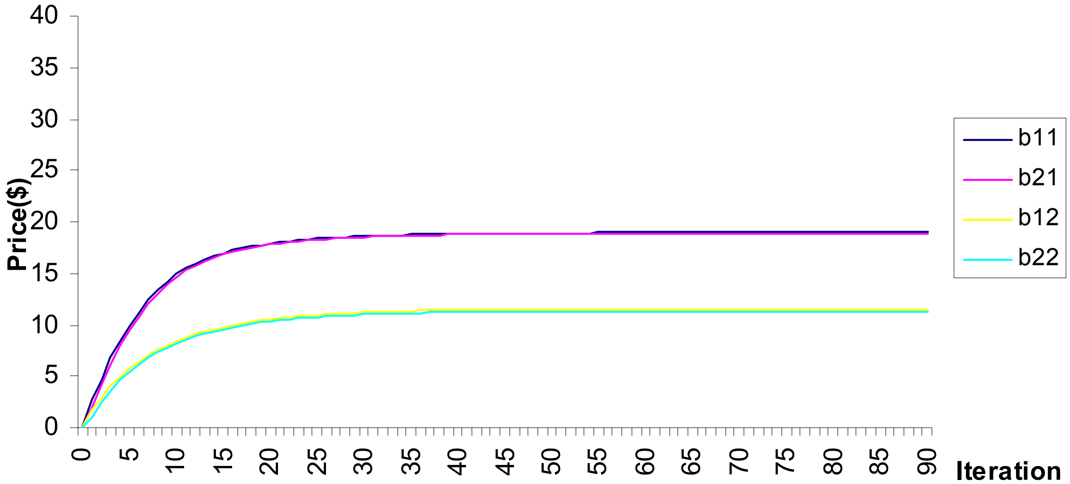


FIGURE 4.1. Convergence of Flows

Note: The number label of the flow indicates in the order of sector, origin, and destination. For example, x121 stands for trade flow of sector 1 from zone 2 to 1.

Scenario 1



Scenario 2

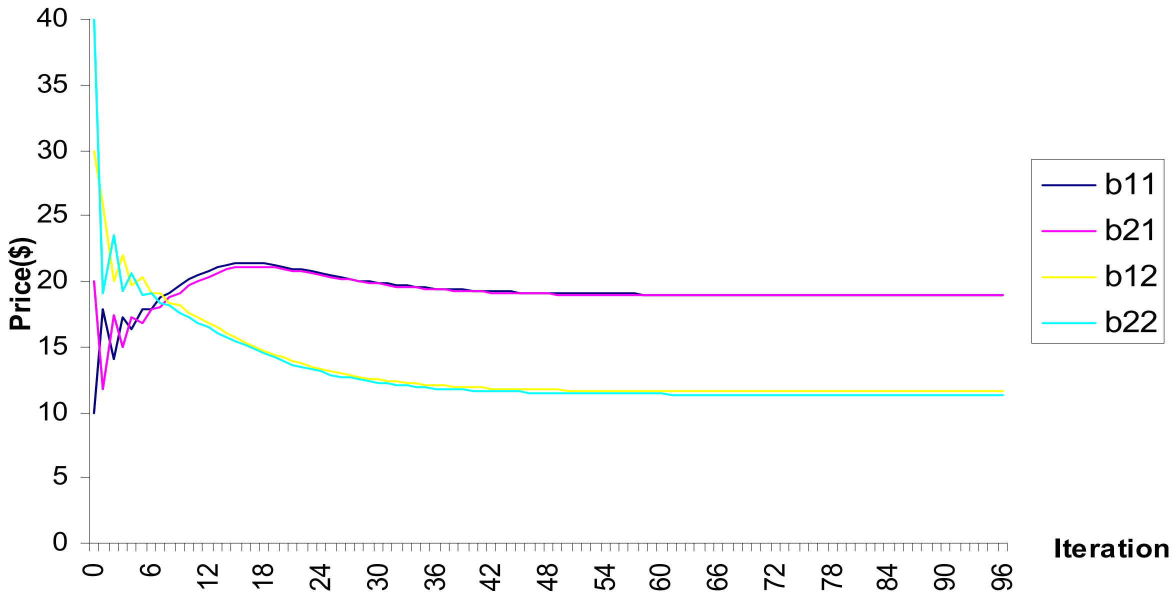


FIGURE 4.2. Convergence of Prices

Note: The number label of the price indicates in the order of sector and zone. For example, b12 stands for sector 1's price at zone 2.