# Numerical Methods for Civil Engineers

Lecture Notes
CE 311K
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# Linear Equations

## Introduction

In many engineering applications it is necessary to solve systems of linear equations. Frequently, the number of equations will be equal to the number of unknowns. In such cases, we are usually able to solve for unique values of the variables. If there are more variables than equations, we expect, in general, to obtain an infinite number of solutions. Sometimes we have more equations than variables. However, not all the equations may be independent, that is, some of them can be derived from others. Under this circumstance, we try to find enough independent equations to be able to solve for all the variables.

Consider the linear system of equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

where $\boldsymbol{A}$ is an ($n$ x $n$) matrix, $\boldsymbol{b}$ is a column vector of constants, called the right-hand-side, and $\boldsymbol{x}$ is the vector of ($n$) unknown solution values to be determined. This system can be written out as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Performing the matrix multiplication and writing each equation out separately, we have

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

This system can also be written in the following manner

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad i = 1,\cdots n$$

A formal way to obtain a solution using matrix algebra is to multiply each side of the equation by the inverse of $\boldsymbol{A}$ to yield

$$\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

or, since

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

we have

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

Thus, we have obtained the solution to the system of equations. Unfortunately, this is not a very efficient way of solving the system of equations. We will discuss more efficient ways in the following sections.

**Example.** Consider the following two equations in two unknowns:

$$3x_1 + 2x_2 = 7$$
$$4x_1 + x_2 = 1$$

Solve the first equation for $x_2$

$$x_2 = \frac{-3}{2}x_1 + \frac{7}{2}$$

This equation represents a straight line with an intercept of 7/2 and a slope of (-3/2). Now, solve the second equation for $x_2$

$$x_2 = -4x_1 + 1$$

This is also a straight line, but with an intercept of 1 and a slope of (-4). These lines are plotted in the following Figure. The solution is the intersection of the two lines at $x_1 = -1$, $x_2 = 5$.

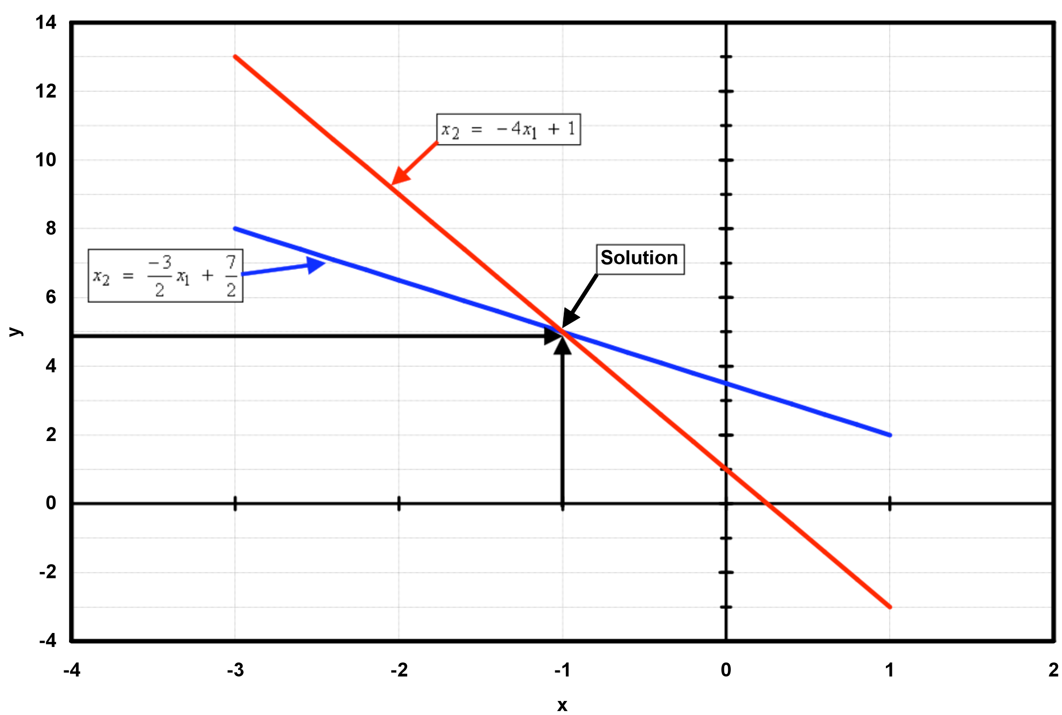



**Figure.** Graphical solution of two simultaneous linear equations.

Each linear equation

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

represents a hyperplane in an $n$-dimensional Euclidean space ($R^n$), and the system of $m$ equations

$$\boldsymbol{Ax} = \boldsymbol{b}$$

or

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

represents $m$ hyperplanes. The solution of the system of equations is the intersection of all of the $m$ hyperplanes, and can be

- the empty set (no solution)
- a point (unique solution)
- a line (non-unique solution)
- a plane (non-unique solution)

# Direct Methods for Solving Linear Systems

## Gauss Elimination

The method of Gaussian Elimination is based on the approach to the solution of a pair of simultaneous equations whereby a multiple of one equation is subtracted from the other to eliminate one of the two unknowns (a *forward elimination* step). The resulting equation is then solved for the remaining unknown, and its value is substituted back into the original equations to solve for the other (a *back-substitution* step).

Consider the two equations from the previous example:

$$3x_1 + 2x_2 = 7$$
$$4x_1 + x_2 = 1$$

Divide the first equation by 3, multiply it by 4 and subtract it from the second equation, yielding the new system of equations

$$3x_1 + 2x_2 = 7$$
$$-\frac{5}{3}x_2 = -\frac{25}{3}$$

Now we can solve the second equation for $x_2 = 5$. Substituting this back into the first equation, we have

$$\mathrm{x}_1 = \frac{7 - 2(5)}{3} = -1$$

This approach can be extended to large sets of equations through the development of a systematic approach to *forward elimination* and *back-substitution*. Consider the following set of $n$ equations in $n$ unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots & \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

The solution vector, $\boldsymbol{x}$, for this system of equations remains unchanged if either of the following fundamental row operations is performed:

(1) Multiplication or division of any equation by a constant.

(2) Replacement of any equation by the sum (or difference) of that equation and any other equation.

Gauss elimination is a sequential application of these basic row operations. To begin, (assuming that $a_{11} \neq 0$) the first equation is multiplied by $a_{21}/a_{11}$ and subtracted from the second equation, yielding the new system:

$$
\begin{array}{rcccccccccl}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
0 & + & a'_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a'_{2n}x_n & = & b'_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
 & & & & & & & & & \vdots & \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & + & a_{nn}x_n & = & b_n
\end{array}
$$

The primes denote elements which have been changed from their original values, e.g., $a'_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} a_{12}$. The first equation can now be multiplied by $a_{31}/a_{11}$ and subtracted from the third equation, and so on, until the last equations is multiplied by $a_{n1}/a_{11}$ and subtracted from the last equation. During these operations, the first row is termed the *pivot row* and $a_{11}$ is termed the *pivot element*. The entire first column below $a_{11}$ has now been cleared (reduced) to zero and the set of equations appears as:

$$
\begin{array}{rcccccccccl}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 & & a'_{22}x_2 & + & a'_{23}x_3 & + & \cdots & + & a'_{2n}x_n & = & b'_2 \\
 & & a'_{32}x_2 & + & a'_{33}x_3 & + & \cdots & + & a'_{3n}x_n & = & b'_3 \\
 & & & & & & & & & \vdots & \\
 & & a'_{n2}x_2 & + & a'_{n3}x_3 & + & \cdots & + & a'_{nn}x_n & = & b'_n
\end{array}
$$

The second row now becomes the pivot row and $a'_{22}$ the pivot element. Multiplication of the second equation by $a'_{32}/a'_{22}$ and subtraction from the third equation, and so on, until the last equations is multiplied by $a'_{n2}/a'_{22}$ and subtracted from the last equation, clearing (reducing) the second column below the main diagonal to zero:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a''_{n3}x_3 + \cdots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

Similar operations with the remaining rows as pivot rows finally yields

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a^{\cdots}_{nn}x_n &= b^{\cdots}_n
\end{aligned}
$$

The number of primes indicates the number of times that a row was modified in the forward elimination process. The final equation in the last system of equations now yields directly the value of $x_n$ as

$$x_n = \frac{b^{\cdots}_n}{a^{\cdots}_n}$$

Backing up, the previous equation is

$$a^{\cdots}_{n-1,n-1}x_{n-1} + a^{\cdots}_{n-1,n}x_n = b^{\cdots}_{n-1}$$

Since $x_n$ is known from the previous, this value may be substituted in and the equation solved for $x_{n-1}$ to yield

$$x_{n-1} = \frac{b^{\cdots}_{n-1} - a^{\cdots}_{n-1,n}x_n}{a''^{\cdots}_{n-1,n-1}}$$

Repeated back substitution, moving upwards, yields one new unknown from each equation, and the unknown vector will have been completely determined when the top equation is solved for $x_1$.

$$x_1 = \frac{b_1 - (a_{12}x_2 \ + \ a_{13}x_3 + \ \cdots \ + \ a_{1n}x_n)}{a_{11}}$$

$$= \frac{b_1 - (\sum_{k=2}^{n} a_{1k}x_k)}{a_{11}}$$

In general we can write

$$x_i = \frac{b_i - (\sum_{k=i+1}^{n} a_{ik}x_k)}{a_{ii}}$$

A flow chart for Gauss elimination is shown in the following three Figures.

## Gauss Elimination

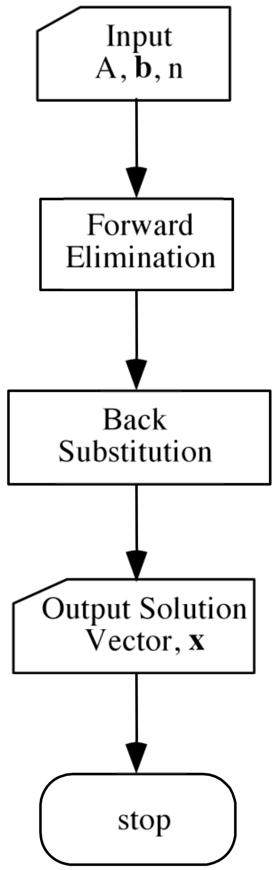
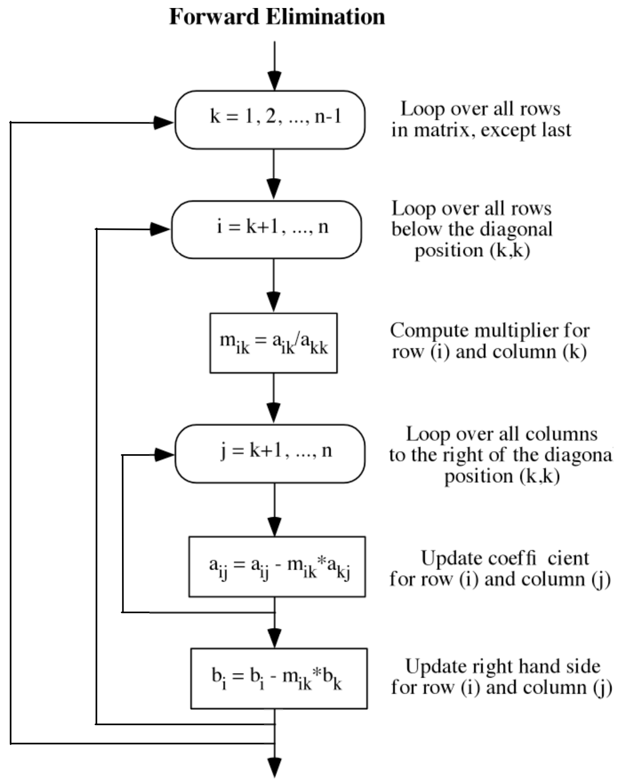
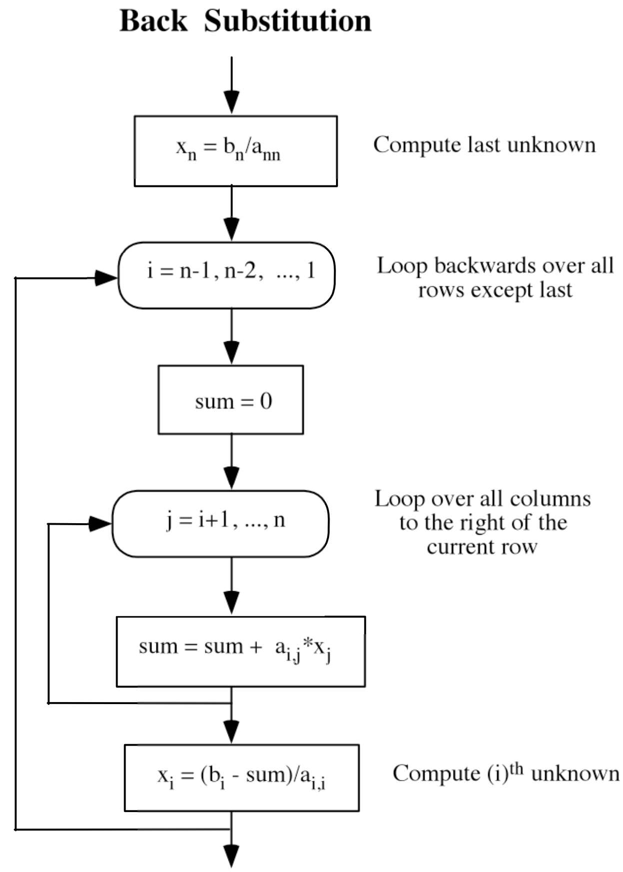



**Figure.** Flowchart for Gauss elimination.




| **Flowchart for forward elimination.** | **Flowchart for back substitution.** |
|---|---|

One computational difficulty can arise with the standard Gauss elimination technique. The pivot element in each row is the element on the main diagonal. By the time any given row becomes the pivot row, the diagonal element in that row will have been modified from its original value, with the elements in the lower rows being recomputed the most times. Under certain circumstances, the diagonal element can become very small in magnitude compared with the rest of the elements in the pivot row, as well as perhaps being quite inaccurate. The computation step

$$a'_{ij} = a_{ij} - \frac{a_{ik}}{a_{kk}} a_{kj}$$

is the source of round-off error in the algorithm, since $a_{kj}$ must be multiplied by $a_{ik} / a_{kk}$ which will be large if the pivot element $a_{kk}$ is small relative to $a_{ik}$. Improvements to the basic Gauss Elimination algorithm include careful selection of the pivot equation and pivot element. The problem can be effectively treated by interchanging columns of the matrix to shift the largest element (in magnitude) in the pivot row into the diagonal position. This largest element then becomes the pivot element. This operation is repeated with each new pivot row as necessary. Every column interchange also means an interchange of the locations of the unknowns in the solution vector. The logic necessary to accomplish these column interchanges is rather complex and is implemented in most standard Gauss elimination computer codes.

**Example.** Use Gauss Elimination to solve

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 16 \\ 3x_1 + 2x_2 + x_3 &= 10 \\ x_1 + 3x_2 + 3x_3 &= 16 \end{aligned}$$

*Forward elimination*: Multiply the first equation by 3/2 and subtract this from the second, resulting in

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 16 \\ \frac{1}{2}x_2 - 5x_3 &= \text{-}14 \\ x_1 + 3x_2 + 3x_3 &= 16 \end{aligned}$$

Multiply the first equation by 1/2 and subtract this from the third, to obtain

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 16 \\ \frac{1}{2}x_2 - 5x_3 &= \text{-}14 \\ \frac{5}{2}x_2 + x_3 &= 8 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x_1 + x_2 + 4x_3 &= 16 \\ x_2 - 10x_3 &= \text{-}28 \\ \frac{5}{2}x_2 + x_3 &= 8 \end{aligned}$$

Multiply the second equation by 5/2 and subtract this from the third, resulting in

$$2x_1 + x_2 + 4x_3 = 16$$
$$x_2 - 10x_3 = -28$$
$$26x_3 = 78$$

*Back substitution*:

$$x_3 = \frac{78}{26} = 3$$

This result is back substituted into the second equation to give

$$x_2 = -28 + 10(3) = 2$$

Now, back substituting again

$$x_1 = \frac{16 - 2 - 4(3)}{2} = 1$$

This solution ($x_1$=1, $x_2$=2, $x_3$=3) can be checked by substituting these values back into the original equations and comparing the left and right-hand sides of the equations.

# Iterative Methods for Solving Linear Systems

Sometimes when solving engineering problems systems of equations will results which involve large numbers of equations and unknowns (100,000s to 1,000,000s). For large systems of equations, Gauss Elimination is inefficient and prone to large roundoff errors. In this case it is often more convenient to use a solution method that involves a sequential process of generating solutions that converge on the true solutions as the number of steps in the sequence increases.

$$\boldsymbol{Ax} = \boldsymbol{b}$$

or

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Iterative methods of solution, as distinct from direct methods such as Gauss Elimination, begin by rearranging the system of equations so that one unknown is isolated on the left-hand side of each equation

$$
\begin{aligned}
x_1 &= \frac{b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)}{a_{11}} \\
x_2 &= \frac{b_2 - (a_{21}x_1 + \cdots + a_{2n}x_n)}{a_{22}} \\
&\vdots \\
x_n &= \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn-1}x_{n-1})}{a_{nn}}
\end{aligned}
$$

In general, we have

$$
x_i = \frac{b_i - (\sum_{j \neq i} a_{ij}x_j)}{a_{ii}}
$$

Now, if an initial guess $x_1^0, x_2^0, \cdots, x_n^0$ for all the unknowns was available, we could substitute these values into the right-hand side of the previous set of equations and compute an updated guess for the unknowns, $x_1^1, x_2^1, \cdots, x_n^1$. There are several ways to accomplish this, depending on how you use the most recently computed guesses.

## Jacobi Method

In the Jacobi method, all of the values of the unknowns are updated before any of the new information is used in the calculations. That is, starting with the initial guess $x_1^0, x_2^0, \cdots, x_n^0$, compute the next approximation of the solution as

$$x_1^1 = \frac{b_1 - (a_{12}x_2^0 + \cdots + a_{1n}x_n^0)}{a_{11}}$$

$$x_2^1 = \frac{b_2 - (a_{21}x_1^0 + \cdots + a_{2n}x_n^0)}{a_{22}}$$

$$\vdots$$

$$x_n^1 = \frac{b_n - (a_{n1}x_1^0 + a_{n2}x_2^0 + \cdots + a_{nn-1}x_{n-1}^0)}{a_{nn}}$$

or, after *k* iterations of this process, we have

$$x_1^{k+1} = \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}$$

$$x_2^{k+1} = \frac{b_2 - (a_{21}x_1^k + \cdots + a_{2n}x_n^k)}{a_{22}}$$

$$\vdots$$

$$x_n^{k+1} = \frac{b_n - (a_{n1}x_1^k + a_{n2}x_2^k + \cdots + a_{nn-1}x_{n-1}^k)}{a_{nn}}$$

More generally

$$x_i^{k+1} = \frac{b_i - (\sum_{j \neq i} a_{ji} x_j^k)}{a_{ii}}$$

**Example.** Consider the system of equations

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 3x_2 - x_3 &= 8 \\ -x_2 + 2x_3 &= -5 \end{aligned}$$

A convenient initial guess is usually $x_1^0 = 0, x_2^0 = 0, x_3^0 = 0$. In this case we have

$$\begin{aligned} x_1^{k+1} &= \frac{1-(-x_2^k)}{2} = \frac{1+x_2^k}{2} \\ x_2^{k+1} &= \frac{8-(-x_1^k - x_3^k)}{3} = \frac{8+x_1^k+x_3^k}{3} \\ x_3^{k+1} &= \frac{-5-(-x_2^k)}{2} = \frac{-5+x_2^k}{2} \end{aligned}$$

or, after 1 iteration of the method we have

$$\begin{aligned} x_1^1 &= \frac{1+x_2^0}{2} = \frac{1+0}{2} = 0.5 \\ x_2^1 &= \frac{8+x_1^0+x_3^0}{3} = \frac{8+0+0}{3} = 2.667 \\ x_3^1 &= \frac{-5+x_2^0}{2} = \frac{-5+0}{2} = -2.5 \end{aligned}$$

Then, after 2 iterations of the method

$$\begin{aligned} x_1^2 &= \frac{1+x_2^1}{2} = \frac{1+2.6667}{2} = 1.833335 \\ x_2^2 &= \frac{8+x_1^1+x_3^1}{3} = \frac{8+0.5+(-2.5)}{3} = 2 \\ x_3^2 &= \frac{-5+x_2^1}{2} = \frac{-5+1.833335}{2} = -1.1667 \end{aligned}$$

Continuing this process ultimately leads, after 20 iterations, to

$$
\begin{aligned}
x_1 &= 2\\
x_2 &= 3\\
x_3 &= -1
\end{aligned}
$$

## Gauss Seidel Method

When applying the Jacobi method, one may realize that information is being made available at each step of the algorithm. That is the current approximation of one of the unknowns is available for use after each step. This information could be used immediately in the calculation of the next unknown. If we implement this, our method would look like

$$
\begin{aligned}
x_1^1 &= \frac{b_1 - (a_{12}x_2^0 + \cdots + a_{1n}x_n^0)}{a_{11}}\\
x_2^1 &= \frac{b_2 - (a_{21}x_1^1 + \cdots + a_{2n}x_n^0)}{a_{22}}\\
&\vdots\\
x_n^1 &= \frac{b_n - (a_{n1}x_1^1 + a_{n2}x_2^1 + \cdots + a_{nn-1}x_{n-1}^1)}{a_{nn}}
\end{aligned}
$$

or, after *k* iterations of this process, we have

$$
\begin{aligned}
x_1^{k+1} &= \frac{b_1 - (a_{12}x_2^k + \cdots + a_{1n}x_n^k)}{a_{11}}\\
x_2^{k+1} &= \frac{b_2 - (a_{21}x_1^{k+1} + \cdots + a_{2n}x_n^k)}{a_{22}}\\
&\vdots\\
x_n^{k+1} &= \frac{b_n - (a_{n1}x_1^{k+1} + a_{n2}x_2^{k+1} + \cdots + a_{nn-1}x_{n-1}^{k+1})}{a_{nn}}
\end{aligned}
$$

More generally

$$x_i^{k+1} = \frac{b_i - (\sum_{\substack{j<i \\ j\neq i}} a_{ji} x_j^{k+1} + \sum_{\substack{j>i \\ j\neq i}} a_{ji} x_j^k)}{a_{ii}}$$

**Example.** Consider again the system of equations

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 3x_2 - x_3 &= 8 \\ -x_2 + 2x_3 &= -5 \end{aligned}$$

A convenient initial guess is usually $x_1^0 = 0, x_2^0 = 0, x_3^0 = 0$. Using the Gauss Siedel method, we have

$$\begin{aligned} x_1^{k+1} &= \frac{1 + x_2^k}{2} \\ x_2^{k+1} &= \frac{8 + x_1^{k+1} + x_3^k}{3} \\ x_3^{k+1} &= \frac{-5 + x_2^{k+1}}{2} \end{aligned}$$

or, after 1 iteration of the method, we have

$$\begin{aligned} x_1^1 &= \frac{1 + x_2^0}{2} = \frac{1+0}{2} = 0.5 \\ x_2^1 &= \frac{8 + x_1^1 + x_3^0}{3} = \frac{8 + 0.5 + 0}{3} = 2.8333 \\ x_3^1 &= \frac{-5 + x_2^1}{2} = \frac{-5 + 2.83333}{2} = -1.08333 \end{aligned}$$

Then, after 2 iterations we have

$$x_1^2 = \frac{1 + x_2^1}{2} = \frac{1 + 0.5}{2} = 1.9167$$
$$x_2^2 = \frac{8 + x_1^2 + x_3^1}{3} = \frac{8 + 1.9167 + (-1.0833)}{3} = 2.9444$$
$$x_3^2 = \frac{-5 + x_2^2}{2} = \frac{-5 + 2.9444}{2} = -1.0278$$

Continuing this process ultimately leads, after 9 iterations, to

$$x_1 = 2$$
$$x_2 = 3$$
$$x_3 = -1$$

Notice that this method converges to the solution much faster than the Jacobi method.

## Exercises

1. Given the system of equations

$$0.77x_1 + 6x_2 = 14.25$$
$$1.2x_1 + 1.7x_2 = 20$$

   a. Solve graphically
   b. On the basis of the graphical solution, what do you expect regarding the condition of the system?
   c. Solve by elimination of unknowns.

2. Solve the following system of equations graphically:

$$2x_1 - 6x_2 = -18$$
$$-x_1 + 8x_2 = 40$$

3. Use Gauss Elimination to solve the following systems of linear equations. Show all steps in the computation. Be sure to substitute your answers into the original equation to check your answers.

a.
$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 7 \\ 2x_1 + x_2 + x_3 &= 4 \\ x_1 + 3x_2 + 5x_3 &= 2 \end{aligned}$$

b.
$$\begin{aligned} -12x_1 + x_2 - x_3 &= -20 \\ -2x_1 - 4x_2 + 2x_3 &= 10 \\ x_1 + 2x_2 + 2x_3 &= 25 \end{aligned}$$

c.
$$\begin{aligned} -12x_1 + x_2 - 8x_3 &= -80 \\ x_1 - 6x_2 + 4x_3 &= 13 \\ -2x_1 - x_2 + 10x_3 &= 90 \end{aligned}$$

d.
$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 2x_2 + x_3 &= 10 \\ -2x_1 + 3x_2 - 2x_3 &= -2 \end{aligned}$$

4. The following systems of equations is designed to determine the concentrations (the $c$'s in $g/m^3$) in a series of coupled reactors as a function of the amount of mass input to each reactor (right-hand sides in g/day)

$$\begin{aligned} 17c_1 - 2c_2 - 3c_3 &= 500 \\ -5c_1 + 21c_2 - 2c_3 &= 200 \\ -5c_1 - 5c_2 + 22c_3 &= 30 \end{aligned}$$

Solve this system using the Gauss-Seidel method to a stopping tolerance of 5%. Use an initial guess of $c_1 = c_2 = c_3 = 0$.

5. The series expansion for sine $x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Starting with the simplest version, $\sin x \approx x$, add terms, one at a time in order to estimate $\sin(\frac{\pi}{2})$. After each new term is added, compute the true error and the approximate relative error. Add terms until the absolute value of the approximate relative error falls below a stopping criterion of 0.001%. Use a spreadsheet.

6. If $x$ is approximated by $\tilde{x}$ = 123.456 and the relative error is 0.1, then what is the possible range of values for $x$?

7. An engineer needs 4800, 5810, and 5690 m3 of sand, fine gravel, and coarse gravel, respectively, at a construction site. There are three sources where these materials can be obtained and the composition of the material from these sources is

| | % Sand | % Fine Gravel | % Coarse Gravel |
|---|---|---|---|
| Source 1 | 52 | 30 | 18 |
| Source 2 | 20 | 50 | 30 |
| Source 3 | 25 | 20 | 55 |

How many cubic meters must be hauled from each source in order to meet the engineer's needs?

8. Write the following systems of equations in matrix-vector form (***Ax*** = ***b***). Find the transpose of the matrix ***A***.

$$30 = 2x_2 + 6x_3$$
$$20 = 3x_2 + 8x_1$$
$$10 = x_1 + x_3$$

9. Solve the following system of equations by the Gauss-Siedel method. Use an accuracy of e = 0.001.

$$\begin{aligned} 2x_1 - x_2 &= 124 \\ -x_1 + 2x_2 - x_3 &= 4 \\ -x_2 + 2x_3 &= 14 \end{aligned}$$

10. Solve the following system of equations by the Gauss-Siedel method. Use an accuracy of e = 0.001.

$$
\begin{aligned}
17x_1 - 2x_2 - 3x_3 &= 500 \\
-5x_1 + 21x_2 - 2x_3 &= 200 \\
-5x_1 - 5x_2 + 22x_3 &= 30
\end{aligned}
$$

11. Start from an initial guess of $x_1^0 = x_2^0 = x_3^0 = 1.0$, show the first 2 iterations of the Gauss Seidel method for the solution of this system of equations. Compute the error after the second iteration.

$$
\begin{aligned}
3x_1 \quad - 0.1x_2 \quad - 0.2x_3 &= 7.85 \\
0.1x_1 \quad + 7x_2 \quad - 0.3x_3 &= -19.3 \\
0.3x_1 \quad - 0.2x_2 \quad + 10x_3 &= 71.4
\end{aligned}
$$

**12.** What is meant by ill conditioning of a set of linear equations?

The slopes of the equations are so similar that the computing method can not distinguish between the solutions.

**13.** The following figure shows three reactors linked by pipes. As indicated, the rate of transfer of chemicals through each pipe is equal to a flow rate (*Q*, with units of cubic meters per second) multiplied by the concentration of the reactor from which the pipe originates (*c*, with units of milligrams per cubic meter). If the system is at a steady state, the transfer into each reactor will balance the transfer out. Develop mass balance equations for the reactors and use Gauss Elimination to solve the three simultaneous linear algrbraic equations for the unknown concentrations, $c_1$, $c_2$, and $c_3$. Show your work.

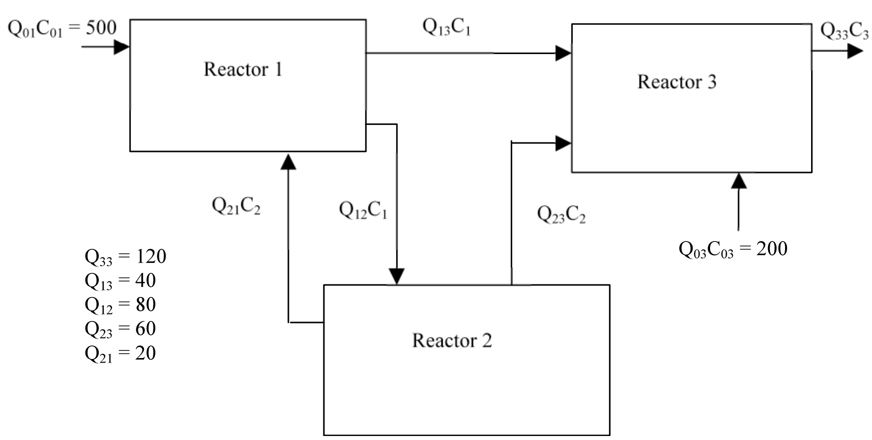



**Figure.** Three reactors linked by pipes. The rate of mass transfer through each pipe is equal to the product of flow *Q* and concentration *c* of the reactor from which the flow originates.

**14.** Solve the following set of equations usgin the Gauss-Seidel iterative method

$$
\begin{aligned}
5x + y + 2z &= 17 \\
x + 3y + z &= 8 \\
2x + y + 6z &= 23
\end{aligned}
$$

use the starting values $x^0 = 1$, $y^0 = 1$, and $z^0 = 1$. Show the computations for the first 2 iterations of the Gauss Seidel method. Be sure to show all equations for the computations.

**15.** Given the system of equations

$$
\begin{aligned}
-12x_1 + x_2 - x_3 &= -20 \\
-2x_1 - 4x_2 + 2x_3 &= 10 \\
x_1 + 2x_2 + 2x_3 &= 25
\end{aligned}
$$

Make 2 iteration of the solution of this system of equations by the Gauss Seidel method. Show all steps in the computation. Start from an initial guess of $x_1^0 = x_2^0 = x_3^0 = 0.0$. Compute the error after the second iteration.