Numerical Methods for Civil Engineers

Lecture Notes
CE 311K
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Nonlinear Equations

Introduction

In this section we consider methods for solving nonlinear equations. Given a nonlinear function $f(x)$, we seek a value of x for which

$$f(x) = 0 \tag{1}$$

Such a solution value for x is called a root of the equation, and a zero of the function f. An example of a nonlinear equation in one variable is

$$f(x) = x^2 - 4\sin(x) = 0$$

As can be seen from Figure 1, the function has roots at 0.0 and near 1.9. Graphical examination of a function is often a good way to find the neighborhood of roots.

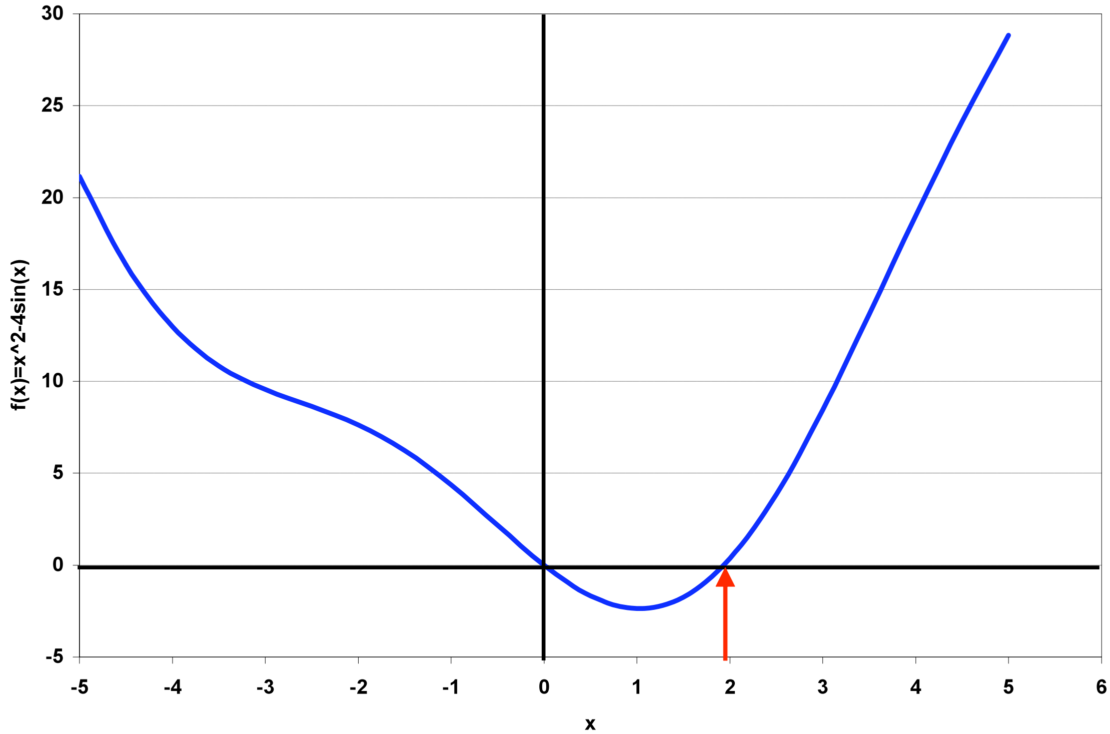


Figure 1. Graphical representation of $f(x) = x^2 - 4\sin(x)$.

Fixed Point Iteration

Usually a formula for finding the root of an equation can be found by rearranging Equation (1) to be:

$$x = g(x) \tag{2}$$

and then using the computation formula

$$x_{i+1} = g(x_i) \tag{3}$$

to solve for successively more accurate approximations of the root. Consider our previous example

$$f(x) = x^2 - 4\sin(x) = 0$$

which can be rearranged

$$x = g(x) = 4\frac{\sin(x)}{x}$$

So, the computational formula is

$$x_{i+1} = g(x_i) = 4\frac{\sin(x_i)}{x_i}$$

We could solve the equation for x by starting with an initial guess of $x = x_0$ and computing x_1 as

$$x_1 = 4\frac{\sin(x_0)}{x_0}$$

we would continue this process for several iterations until the value of x_{i+1} was sufficiently close to the previous value of x_i. That is we would continue our iterations until the relative approximate error was less than an assigned accuracy ε:

$$e_A = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| < \varepsilon$$

Examples:

1. Consider the quadratic equation

$$f(x) = x^2 - 2x + 3 = 0$$

which can be rearranged to give the fixed point iteration formula

$$x_{i+1} = g(x_i) = \frac{x_i^2 + 3}{2}$$

2. Consider the transcendental equation

$$f(x) = \sin(x) = 0$$

which can be rearranged to give the fixed point iteration formula

$$x_{i+1} = g(x_i) = \sin(x_i) + x_i$$

3. Use fixed-point iteration to locate a root of

$$f(x) = \bar{e}^{x} - x$$

The function is separated and expressed in the form

$$x_{i+1} = g(x_i) = e^{-x_i}$$

Thus

$$x_1 = g(x_0) = e^{-x_o} = e^0 = 1$$

$$x_2 = g(x_1) = e^{-x_1} = e^{-1} = 0.367898$$

$$x_3 = g(x_2) = e^{-x_2} = e^{-0.367898} = ...$$

Table 1. Fixed point iteration method for finding the root of $f(x) = \bar{e}^{x} - x$.

Iteration i	Approximate root x_i	Approximate error $e_A = \left\lvert \frac{x_{i+1} - x_i}{x_{i+1}} \right\rvert x100$

0	0	
1	1.000000	100.0
2	0.367898	171.8
…	…	…
9	0.571143	1.93
10	0.564879	1.11

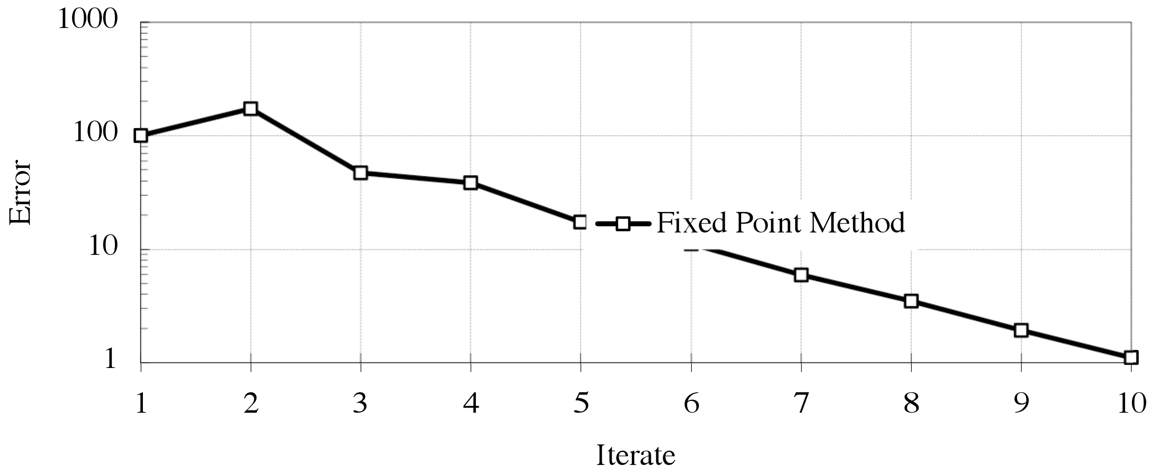


Figure 2. Approximate relative error for each iteration in the fixed point iteration method of finding a root of the function $f(x) = e^{-x} - x$.

Bisection Method

The sign of a function *f(x)* changes on opposite sides of a root. Suppose the continuous real valued function *f(x)* has one root in the interval between $x = a$ and $x = c$, or [*a,c*], as shown in the Figure below. The bisection method is based on the fact that when an interval [*a,c*] contains a root, the sign of the function at the two ends (*f(a)* and *f(c)*) are opposite each other, namely

$$f(a)f(c) < 0$$

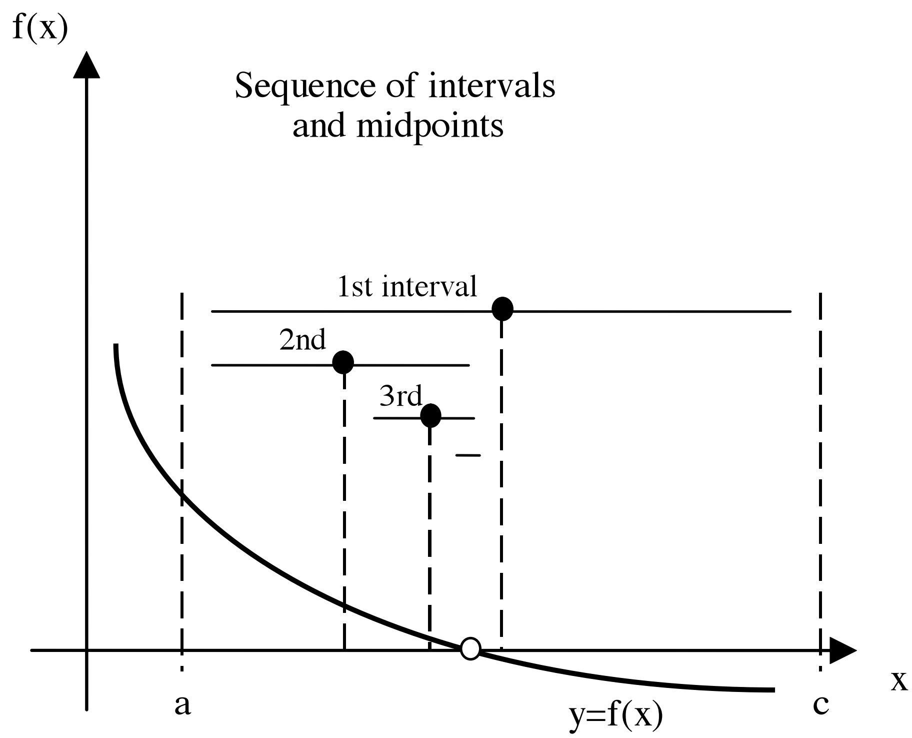


Figure 3. Illustration of the Bisection Method.

The first step in the bisection method is to bisect the interval [*a*,*c*] into two halves, namely [*a*,*b*] and [*b*,*c*], where

$$b = \frac{a+c}{2}$$

By checking the signs of $f(a)*f(b)$ and $f(b)*f(c)$, the half-interval containing the root can be identified. If $f(a)*f(b) < 0$, the interval [*a*,*b*] has the root, otherwise the other interval [*b*,*c*] has the root. Then the new interval containing the root is bisected again. As the procedure is repeated the interval becomes smaller and smaller. At each step, the midpoint of the interval containing the root is taken as the best approximation of the root. The iterative procedure is stopped when the half-interval size is less than a prespecified size.

The interval size after n iterations is

$$\frac{(c\text{-}a)_0}{2^{n-1}}$$

where the numerator $(c\text{-}a)_0$ is the initial interval size. This represents the error when the root is approximated by the midpoint of the interval after the nth iteration. That is, the root could actually lie anywhere inside the interval. Therefore, if the tolerance for the error is ε, the number of iteration steps required is the smallest integer satisfying

$$\frac{(c\text{-}a)_0}{2^{n}} < e$$

which can be solved for an estimate of the number of iterations required to achieve the given tolerance. The error can be computed as

$$|e_A| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$$

where x_{i+1} is the new (current) approximation of the root and x_i is the approximation from the previous iteration.

Example: Use the Bisection method to determine the drag coefficient c needed for a parachutist of mass m = 68.1 kg to have a velocity of v = 40 m/s after free-falling for time t = 10 s. Let the acceleration of gravity be g = 9.8 m/s^2. Recall the equation for free-fall velocity of an object:

$$v(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]$$

This equation can be rearranged to yield

$$f(c) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right] - v = 0$$

The true value of *c* for this example is *c* = 14.7802.

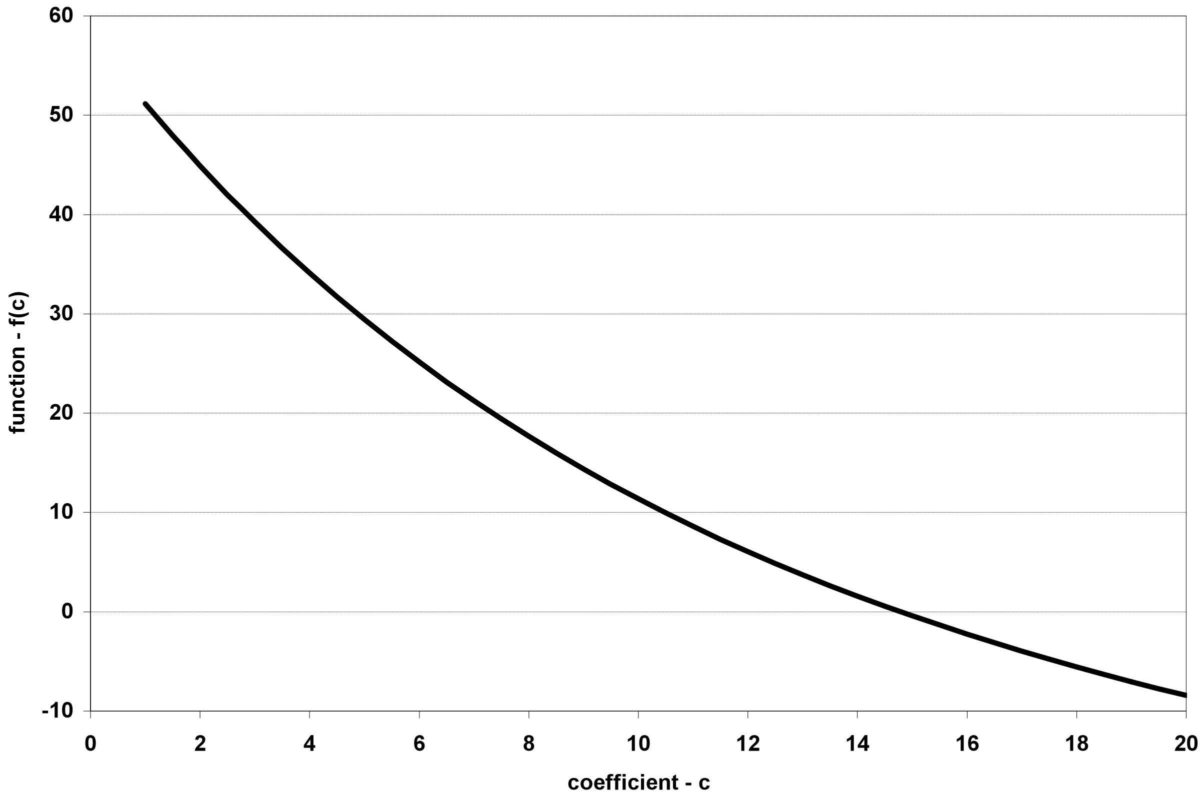


Figure 4. Plot of $f(c) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right] - v$ vs c

The first step is to select two values of *c* where *f(c)* has different signs. From the Figure we see that the function changes sign between 12 and 16. Therefore, we can take x_a = 12 and x_b= 16.

Iteration 1. The estimate of the root is the midpoint of this interval

$$x_c = \frac{x_a + x_b}{2} = \frac{12 + 16}{2} = 14$$

The true relative error for this iterate is

$$|e| = \left|\frac{x_{true} - x_c}{x_{true}}\right| 100\% = \left|\frac{14.7802 - 14}{14.7802}\right| 100\% = 5.3\%$$

Next, compute the product of the function values at the midpoint and lower bound of the interval

$$f(x_a) f(x_c) = f(12) f(14) = 9.517 > 0$$

Thus, no sign change takes place in the left-hand portion of the interval. So, create a new interval by redefining *a*= *c*.

Iteration 2. A new approximate root is computed

$$x_c = \frac{x_a + x_b}{2} = \frac{14 + 16}{2} = 15$$

and the true error is $|e| = 1.5\%$. This process is repeated to find refined estimates of the root. That is, compute the product of the function values at the midpoint and lower bound of the interval

$$f(x_a) f(x_c) = f(14) f(15) = -0.666 < 0$$

Thus, a sign change takes place in the left-hand portion of the interval. So, create a new interval by redefining *b*= *c*.

Iteration 3. A new approximate root is computed

$$x_c = \frac{x_a + x_b}{2} = \frac{14 + 15}{2} = 14.5$$

and the true error is $|e| = 1.9\%$. The approximate error form the first two iterations can be computed as

$$|e_A| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\% = \left|\frac{15 - 14}{15}\right| 100\% = 6.667\%$$

Note that the approximate error is always greater than the true error. This is an attractive characteristic since we can't always compute the true error.

Table 2. Example of Bisection method

Iteration	x_a	x_b	x_c	$\|e_A\|\%$	$\|e\|\%$
1	12	16	14		5.279
2	14	16	15	6.667	1.487
3	14	15	14.5	3.448	1.896
4	14.5	15	14.75	1.695	0.204
5	14.75	15	14.875	0.840	0.641
6	14.75	14.875	14.8125	0.422	0.219

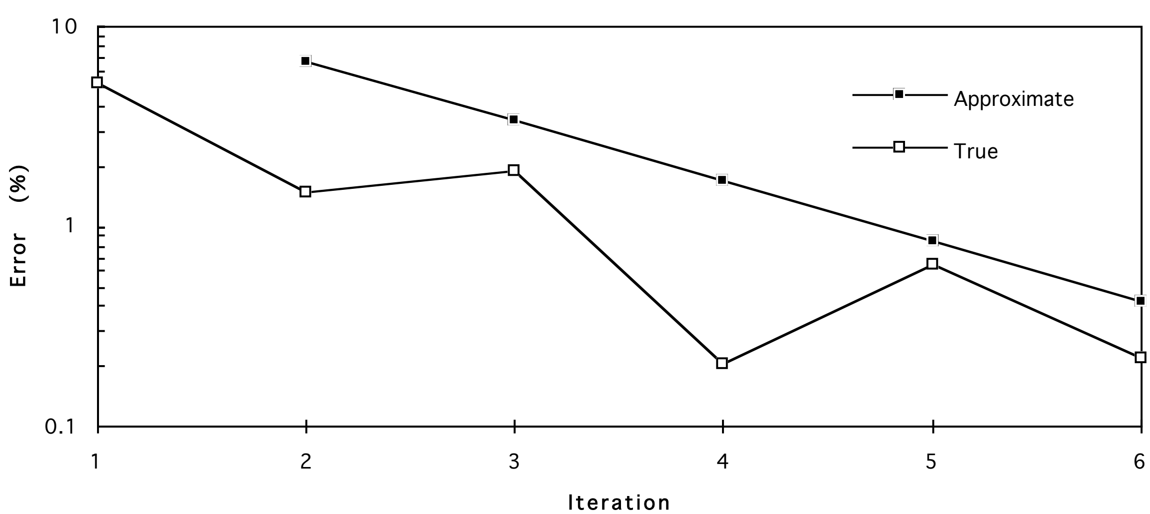


Figure 5. Example of Bisection method

Newton's Method

In Newton's method a tangent line is extended from the current approximation of the root, [x_i, $f(x_i)$] to the point where the tangent crosses the x axis. Newton's method can be derived either form a geometrical argument or a Taylor series approach. The numerical approximation of the derivative of the function $f(x)$ at the current approximation x_i is

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

So the next Newton approximation (iterate) is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

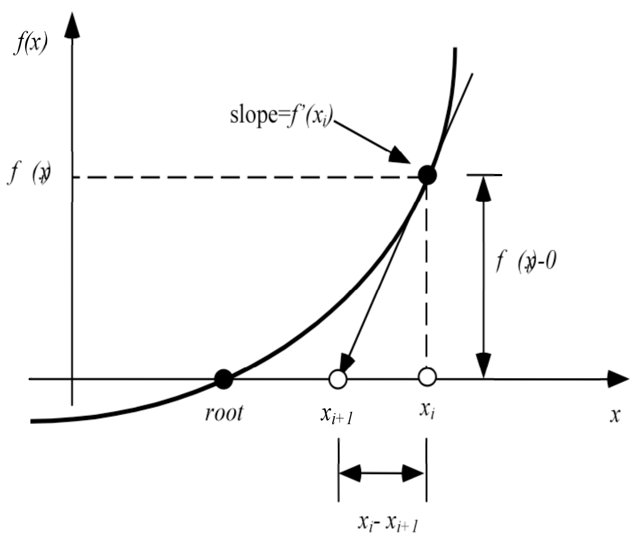


Figure 6. Graphical depiction of Newton's method.

Example: Use Newton's method to locate a root of

$$f(x) = e^{-x} - x$$

Table 3. Newton's method for finding the root of $f(x) = e^{-x} - x$.

Iteration I	Approximate value x_i	Approximate error $\lvert\varepsilon_A\rvert$, %
0	0	100.0
1	0.500000	11.8
2	0.566311003	0.147
3	0.567243265	0.000220
4	0.567143290	< 10^{-8}

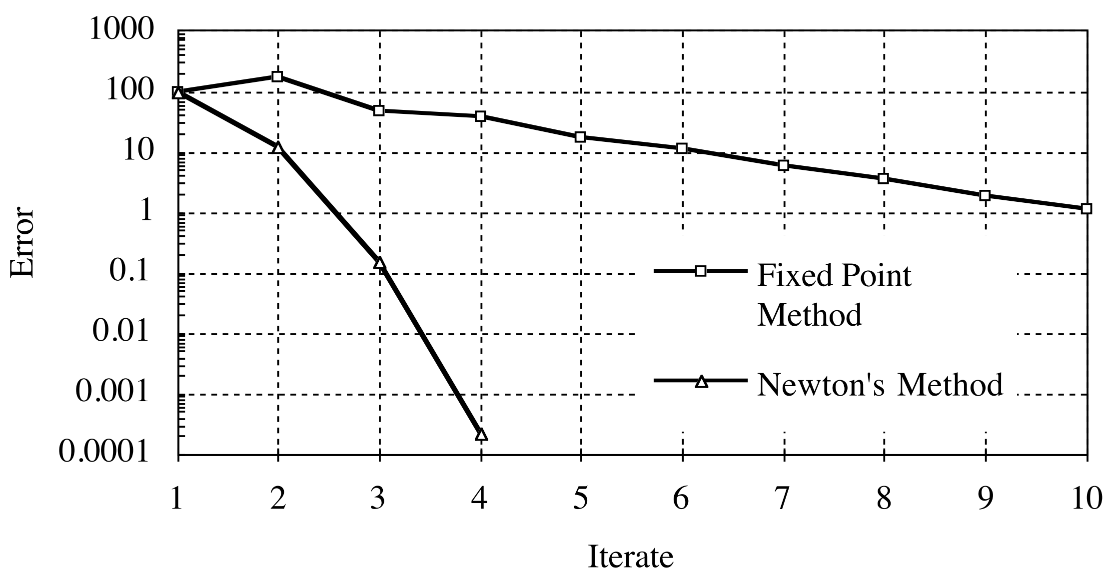


Figure 7. Relative error for each iterate of finding a root of the function $f(x) = e^{-x} - x$ by fixed point iteration and Newton's methods.

Why does Newton's method work so well? It has a quadratic convergence rate, rather than linear. To see this, consider the Taylor series approximation of the function

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots$$

Now, if x_{i+1} is the true root of the equation, then $f(x_{i+1}) = 0$, and

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

or, rearranging

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

In Newton's method we don't use all of the terms in the Taylor series. We only use the first and second terms to find the slope of the curve. If we solve for the *next* term in the Taylor series, this will give us an estimate of the error that we are incurring by truncating the series in our method. This estimate is

$$E_{t,i+1} = \frac{-f''(x_r)}{2f'(x_r)}E_t^2$$

where x_r is the true root. This formula shows us the error is proportional to the square of the previous error, and that if these errors are small (<1) then they are decreasing rapidly form one iteration to the next.

In summary, Newton's method iteratively uses tangential lines that pass through the consecutive approximations of the root. The method needs a good initial guess, or else the iterative solution may diverge or converge to an alternate root. The convergence rate is high when it works.

Systems of Nonlinear Equations

Consider the system of n nonlinear equations

$$
\begin{aligned}
&f_1(x_1, x_2, \ldots, x_n) = 0 \\
&f_2(x_1, x_2, \ldots, x_n) = 0 \\
&\vdots \\
&f_n(x_1, x_2, \ldots, x_n) = 0
\end{aligned}
$$

Example:

$$
\begin{aligned}
f_1(x_1, x_2) &= x_1^2 + x_1 x_2 - 10 = 0 \\
f_2(x_1, x_2) &= x_2 + 3x_1 x_2^2 - 57 = 0
\end{aligned}
$$

Several methods are available to solve systems of nonlinear equations, e.g., fixed-point iteration and Newton's methods. We will concentrate on Newton's method here. The first-order Taylor series for functions of two independent variables is

$$
\begin{aligned}
f_1(x_{1,i+1}, x_{2,i+1}) &= f_1(x_{1,i}, x_{2,i}) + \Delta x_1 \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{1,i}, x_{2,i})} + \Delta x_2 \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{1,i}, x_{2,i})} \\
f_2(x_{1,i+1}, x_{2,i+1}) &= f_2(x_{1,i}, x_{2,i}) + \Delta x_1 \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_{1,i}, x_{2,i})} + \Delta x_2 \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_{1,i}, x_{2,i})}
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
\Delta x_1 &= x_{1,i+1}, - x_{1,i} \\
\Delta x_2 &= x_{2,i+1}, - x_{2,i}
\end{aligned}
$$

Assuming that we have a guess for $x_{1,I}$ and $x_{2,I}$ that does not satisfy Equation (1), we want to find Δx_1 and Δx_2 such that

$$x_{1,i} + \Delta x_1$$
$$x_{2,i} + \Delta x_2$$

do satisfy Equation (1), or at least they provide a better approximation.

If we sett the right-hand-side of Equation (2) equal to zero produces a set of linear, simultaneous equations in the Δx's.

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{1,i},x_{2,i})} \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{1,i},x_{2,i})} \Delta x_2 = -f_1(x_{1,i},x_{2,i})$$
$$\left.\frac{\partial f_2}{\partial x_1}\right|_{(x_{1,i},x_{2,i})} \Delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_{1,i},x_{2,i})} \Delta x_2 = -f_2(x_{1,i},x_{2,i})$$

Example:

$$\frac{\partial f_1}{\partial x_1} = 2x_1 + x_2 \qquad \frac{\partial f_1}{\partial x_2} = x_1$$
$$\frac{\partial f_2}{\partial x_2} = 3x_2^2 \qquad \frac{\partial f_2}{\partial x_2} = 1 + 6x_1 x_2$$

so the system of equations to solve is:

$$\begin{bmatrix} 2x_1 + x_2 & x_1 \\ 3x_2^2 & 1 + 6x_1x_2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -(x_1^2 + x_1x_2 - 10) \\ -(x_2 + 3x_1x_2^2 - 57) \end{bmatrix}$$

Guess $x_1 = 1.5$, $x_2 = 3.5$, then

$$\begin{bmatrix} 6.5 & 1.5 \\ 36.75 & 32.5 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -1.625 \end{bmatrix}$$

Solving for the unknowns, we have

$$\Delta x_1 = 0.536 = x_{1,i+1} - x_{1,i}$$
$$\Delta x_2 = -0.6561 = x_{2,i+1} - x_{2,i}$$

so that the next iterates of the unknown values are

$$x_{1,i+1} = 2.036$$
$$x_{2,i+1} = 2.84388$$

If one continues this process, the solution will converge to the values

$$x_1 = 2$$
$$x_2 = 3$$

Exercises

1. Given the following nonlinear equation:

$$0.9x^2 - 1.7x - 2.5 = 0$$

(a) Write the equation for first iteration of Fixed point iteration

(b) Write the equation for first iteration of Newton’s method

2. Use fixed-point iteration to locate the root of:

$$f(x) = \sin(\sqrt{x}) - x$$

Use an initial guess of x_0=0.5 and iterate until the approximate error is less than 0.01%.

3. Solve the following nonlinear equation using the fixed point iteration method:

$$0.9x^2 - 1.7x - 2.5 = 0$$

4. Solve the following nonlinear equation using Newton's method

$$0.9x^2 - 1.7x - 2.5 = 0$$

5. Determine the smallest real root of

$$f(x) = -11 - 22x + 17x^2 - 2.5x^3$$

graphically and (b) using the bisection method using a stopping criterion of 0.05%. (Root is between –1.0 and 0.0).

6. Determine the real roots of

$$f(x) = -2.0 + 6x - 4x^2 + 0.5x^3$$

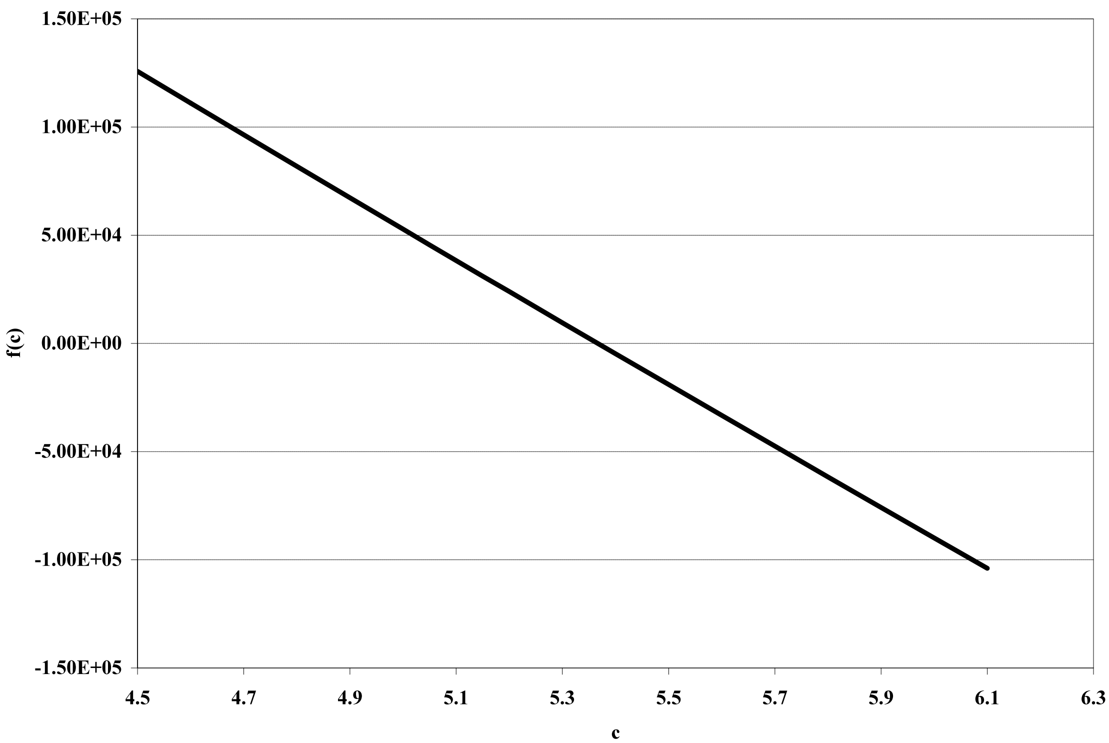
to within 0.01% using Newton's method using initial guesses of (a) 0.5, (b) 1.5, and (c) 6.0.

7. A mass balance for a pollutant in a well-mixed lake can be written as:

$$V\frac{dc}{dt} = W - Qc - kV\sqrt{c}$$

where c is the concentration of the pollutant in g/m^3 the parameter values are $V = 1\text{x}10^6$ m^3, $Q = 1\text{x}10^5$ m^3/yr, W=1x10^6 g/yr, and $k = 0.2$ m$^{0.5}$/g$^{0.5}$/yr. The steady-state concentration can be found from the equation:

$$W - Qc - kV\sqrt{c} = 0$$



The root can be found using Fixed-Point Iteration using

$$c_{i+1} = \left(\frac{W - Qc_i}{kV}\right)^2 \qquad \text{or} \qquad c_{i+1} = \frac{W - kV\sqrt{c_i}}{Q}$$

Only one of these equations will work all the time for initial guesses for $c > 1$. Select the correct one and demonstrate that it will work for an initial guess of c_0=2.0 g/m^3.

8. Use Newton's method to find the smallest value of x satisfying the following equation:

$$0.7 = \frac{1}{2}(1 + \sin x \cdot e^{-\frac{x}{2\pi}})$$

Begin with the guess $x = 0.0$. Show the results of the computations for 2 iterations.

9. Starting from an initial guess $x_0 = 2.5$, take 2 iterations of Newton's method to find the **minimum value** of the function:

$$f(x) = 2\sin x - \frac{x^2}{8}$$

10. Newton's method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, applied to the function $f(x) = x^3 - 4x + 2$ with $x_1 = 1.0$ gives x_2 as

a) 0.0 _____; b) 2.0 ______; c) 1.0 ______

11. The values of x and the corresponding values of $f(x) = x^3 - 4x + 2$ in the bisection method are

x	*f*(*x*)
0.0	2.0
1.0	-1.0
0.5	0.125
0.75	-0.578125

The next value of x is

a) 0.625 _____; b) 0.875 ______; c) 1.250 ______

12. A flat plate of mass *m* falling freely in air with a velocity *V* is subject to a downward gravitational force and an upward frictional drag force due to air. The drag force F_D is given by the expression

$$F_D = \frac{0.3V^2}{500 + (\ln V)^3} - 0.02V$$

Terminal velocity is reached when the drag force equals the gravitational force

$$F = F_D - mg = 0$$

Find the terminal velocity using the Bisection Method if m = 1 kg and g = 9.8 m/s^2. Use an initial interval of V = 0 to 200 m/s. Show your work for computing the first 2 iterations of the Bisection Method.

13. In the turbulent flow of fluid in a smooth pipe, the frictional force on the fluid is represented in terms of a friction factor *f*, which is positive and less than 0.1. The equation for *f* is

$$\frac{1}{\sqrt{f}} = 2\log_{10}\left(\mathrm{Re}\sqrt{f}\right) - 0.8$$

where *Re* is a constant, called the Reynolds number, which varies with the fluid properties, flow rate, and the pipe diameter. Use the Newton-Raphson method to obtain an approximate value for the friction factor *(f)* if the $Re = 10^4$. Show your computations for the first 2 iterations of Newton's method. Be sure to show all equations for the method.

14. The n-th root of a number *A* can be found by solving the equation

$$x^n - A = 0$$

(a) For this equation, show that Newton's method gives

$$x_{i+1} = \frac{1}{n}\left[(n-1)x_i + \frac{A}{x_i^{n-1}}\right]$$

(b) Use the above result to solve for the cube root of A=161, use a starting value $x_0 = 6.0$. Perform three Newton iterations, show all steps in the calculations.

15. Infiltration is the process of water penetrating from the ground surface into soil. Depending on the amount of infiltration and the physical properties of the soil, water may penetrate a few centimeters to several meters into a soil. The cumulative infiltration is the accumulated depth of water infiltrated during a given time period.

Using approximations of the governing equations of mass and momentum conservation for soil water, hydrologists have developed equations to estimate infiltration. One of the most commonly used infiltration equations is the Green-Ampt equation, which assumes that there is a sharp boundary dividing dry soil from saturated soil. The Green-Ampt equation for the cumulative infiltration of water into soil after a period of time is

$$F = Kt + \psi\Delta\theta \ln\left(1 + \frac{F}{\psi\Delta\theta}\right)$$

where:

F	=	the cumulative infiltration (cm)
K	=	the hydraulic conductivity of the soil (cm/hr)
t	=	time (hr)
ψ	=	the suction head (cm)
$\Delta\theta$	=	the change in soil moisture content from the dry soil to the saturated soil
ln	=	natural logarithm

Compute the cumulative infiltration after one hour of infiltration into a soil that has the characteristics:

K	=	0.65 cm/hr
ψ	=	16.7 cm
$\Delta\theta$	=	0.34

Use Newton's Method and start at the value $F = Kt$. Be sure to show details of:

a. the nonlinear function you solve,
b. the equation you use for Newton's method
c. the results for 3 Newton iterations.

16. Given the equation:

$$r = \frac{1}{2}(1 + e^{-x/2\pi} \sin x)$$

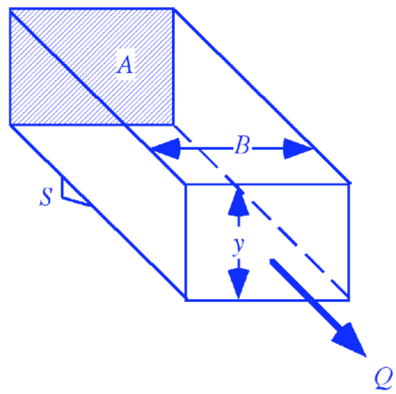
Suppose that you need to find the smallest value of x such that r = 0.7. Beginning with the guess $x = 0.0$, use the Newton method to determine the appropriate root.

a) Write the complete expression that you are using in the Newton method to find the root.
b) Show the results of the bisection computations for 3 iterations.

17. Solve the system of simultaneous nonlinear equations using Newton's method for systems:

$$f_1(x,y) = x^2 - y^2 - 1 = 0$$
$$f_2(x,y) = x^3y^2 - 1 = 0$$

18. The following figure shows a rectangular open channel of constant dimensions.



Under uniform flow conditions the flow in the channel is given by Manning's equation

$$Q = \frac{1}{n}\sqrt{S}AR^{2/3} \qquad (1)$$

where S is the slope of the channel, A is the area of the channel, R is the hydraulic radius of the channel (the radius of a circle with equal area), and n is Manning's roughness coefficient. For a rectangular channel, the following relationships hold

$$A = By \qquad \text{and} \qquad R = \frac{By}{B+2y} \qquad (2)$$

where y is the depth of the water in the channel.

(a) Substitute the relationships from Equation (2) into Equation (1) and determine the nonlinear equation to be solved to find the depth of water y in the channel.

(b) Draw a simple flow chart illustrating how you would use the bisection method to solve the equation you developed in part (a) for the depth of water in the channel. Be sure to include a mechanism in your method to stop the calculations when the desired accuracy has been achieved.

(c) Use the data:

$Q = 14.15 \text{ m}^3/\text{s}$

$B = 4.572$ m

$n = 0.017$

$S = 0.0015$

$[1, 2]$ = initial interval for bisection method

to solve for the depth of water in the channel using the bisection method. Take 3 iterations of the bisection method.

19. Newton's method for systems. Consider the system of simultaneous, nonlinear equations

$$\begin{aligned} f_1(x_1, x_2) &= 3x_1^2 - x_2^2 = 0 \\ f_2(x_1, x_2) &= 3x_1 x_2^2 - x_1^3 - 1 = 0 \end{aligned} \qquad (1)$$

(a) In *very* general terms (don't plug any number in yet!), what is the formula for Newton's method for 2 nonlinear equations? That is $\mathbf{x}^{(k+1)} = \cdots$, and how do you get it.

(b) For the two equations given above in (1), show the system of equations which must be solved in order to apply Newton's method (Don't plug in any numbers yet, just write down the equations using the functions from Equation (1) above.)

(c) Start with k=0, and $x_1^{(0)} = x_2^{(0)} = 1$, what is the system of linear equations that you must solve to obtain $x_1^{(1)} = x_2^{(1)}$? What is the solution to that system of linear equations?

(d) Now, knowing $\mathbf{x}^{(1)}$ you can calculate $\mathbf{x}^{(2)}$ for $k = 2$ in a similar manner. What is the system of linear equations that you must solve to obtain $x_1^{(2)} = x_2^{(2)}$? What is the solution to that system of linear equations?

(e) Draw a simple flowchart illustrating your algorithm for solving these equation.

20. Explain how you would use Newton's method to solve the following simultaneous set of nonlinear equations:

$$f(x,y,z) = 4x + 5\sin y + 0.1z - 5 = 0$$

$$g(x,y,z) = x^2 + 2y + \exp(-0.5z) - 5 = 0$$

$$h(x,y,z) = x + y + z^2 - 12 = 0$$