CE 319 F

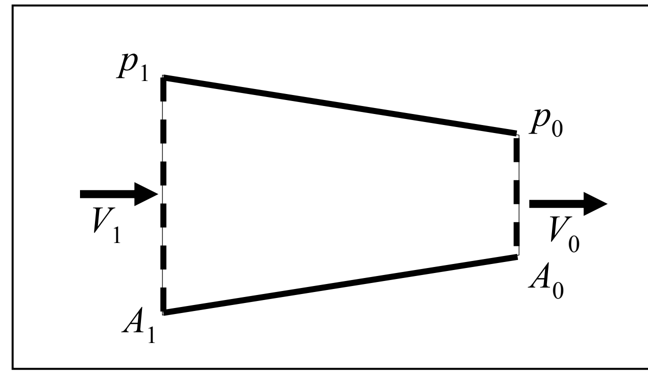
Daene McKinney

Elementary Mechanics of Fluids

Dimensional Analysis

Dimensional Analysis

- Want to study pressure drop as function of velocity (V_1) and diameter (d_o)
- Carry out numerous experiments with different values of V_1 and d_o and plot the data



$$p_1 + \rho \frac{V_1^2}{2} = p_0 + \rho \frac{V_0^2}{2}$$

$$\Delta p = \frac{\rho}{2}\left(V_0^2 - V_1^2\right)$$

$$\Delta p = \frac{\rho}{2} V_1^2 \left(\left(\frac{d_1}{d_0}\right)^4 - 1 \right)$$

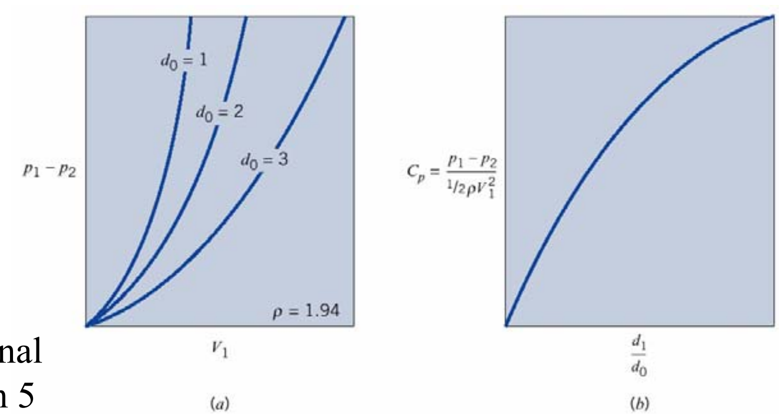
5 parameters:
Δp, ρ, V_1, d_1, d_o

$$\frac{\Delta p}{\frac{\rho}{2} V_1^2} = \left(\frac{d_1}{d_0}\right)^4 - 1$$

2 dimensionless parameters:
$\Delta p/(\rho V^2/2)$, (d_1/d_o)

$$C_p = f\left(\frac{d_1}{d_0}\right)$$

Much easier to establish functional relations with 2 parameters, than 5



Exponent Method

Force F on a body immersed in a flowing fluid depends on: L, V, ρ, and μ

$$F = f(L, V, \rho, \mu)$$

$n = 5$ No. of dimensional parameters
$j = 3$ No. of dimensions
$k = n - j = 2$ No. of dimensionless parameters

F	L	V	ρ	μ
MLT^{-2}	L	LT^{-1}	ML^{-3}	$ML^{-1}T^{-1}$

Select "repeating" variables: L, V, and ρ
Combine these with the rest of the variables: F & μ

$$\pi_1 = \mu(L^a V^b \rho^c)$$

$$M^0 L^0 T^0 = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c$$

$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$

$$L: \quad 0 = -1 + a + b - 3c \quad \Rightarrow \quad a = -1$$

$$T: \quad 0 = -1 - b \quad \Rightarrow \quad b = -1$$

$$\pi_1 = \frac{\mu}{LV\rho} \quad or \quad \pi_1 = \Re = \frac{\rho VL}{\mu}$$

Reynolds number

Exponent Method

F	L	V	ρ	μ
MLT^{-2}	L	LT^{-1}	ML^{-3}	$ML^{-1}T^{-1}$

$$\pi_2 = F(L^a V^b \rho^c)$$

$$M^0 L^0 T^0 = (MLT^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c$$

$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$

$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -2$$

$$T: \quad 0 = -2 - b \quad \Rightarrow \quad b = -2$$

$$\pi_2 = \frac{F}{L^2 V^2 \rho} \qquad and \qquad \pi_2 = f(\pi_1)$$

$$\frac{F}{\rho V^2 L^2} = f(\Re)$$

Dimensionless force is a function of the Reynolds number

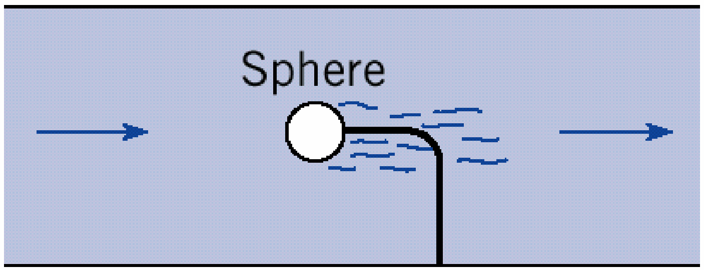
Exponent Method

1. List all n variables involved in the problem
   - Typically: all variables required to describe the problem geometry (D) or define fluid properties (ρ, μ) and to indicate external effects (dp/dx)
2. Express each variables in terms of MLT dimensions (j)
3. Determine the required number of dimensionless parameters ($n - j$)
4. Select a number of repeating variables = number of dimensions
   - All reference dimensions must be included in this set and each must be dimensionalls independent of the others
5. Form a dimensionless parameter by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an unknown exponent
6. Repeat for each nonrepeating variable
7. Express result as a relationship among the dimensionless parameters

Example (8.7)

- **Find:** Drag force on rough sphere is function of D, ρ, μ, V and k. Express in form:

$$\pi_3 = f(\pi_1, \pi_2)$$



F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

$n = 6$ No. of dimensional parameters
$j = 3$ No. of dimensions
$k = n - j = 3$ No. of dimensionless parameters

Select "repeating" variables: D, V, and ρ
Combine these with nonrepeating variables: F, μ & k

$$\pi_1 = \mu(D^a V^b \rho^c)$$

$$M^0 L^0 T^0 = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c$$

$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$

$$L: \quad 0 = -1 + a + b - 3c \quad \Rightarrow \quad a = -1$$

$$T: \quad 0 = -1 - b \quad \Rightarrow \quad b = -1$$

$$\pi_1 = \frac{\mu}{DV\rho} \quad or \quad \pi_1 = \Re = \frac{\rho VD}{\mu}$$

Example (8.7)

F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

Select "repeating" variables: *D*, *V*, and ρ
Combine these with nonrepeating variables: *F*, μ & *k*

$$\pi_2 = k(D^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (L)(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = c \quad \Rightarrow \quad c = 0$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -1$$
$$T: \quad 0 = -b \quad \Rightarrow \quad b = 0$$

$$\pi_2 = \frac{k}{D}$$

$$\pi_3 = F_D(D^a V^b \rho^c)$$
$$M^0 L^0 T^0 = (MLT^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -2$$
$$T: \quad 0 = -2 - b \quad \Rightarrow \quad b = -2$$

$$\pi_3 = \frac{F_D}{\rho V^2 D^2}$$

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}, \frac{k}{D}\right)$$

Common Dimensionless No's.

- Reynolds Number (inertial to viscous forces)
  - Important in all fluid flow problems

$$\Re = \frac{\rho V d}{\mu}$$

- Froude Number (inertial to gravitational forces)
  - Important in problems with a free surface

$$F = \frac{V}{\sqrt{gh}}$$

- Euler Number (pressure to inertial forces)
  - Important in problems with pressure differences

$$C_p = \frac{\Delta p}{\rho V^2}$$

- Mach Number (inertial to elastic forces)
  - Important in problems with compressibility effects

$$M = \frac{V}{\sqrt{E / \rho}} = \frac{V}{c}$$

- Weber Number (inertial to surface tension forces)
  - Important in problems with surface tension effects

$$W = \frac{\rho L V^2}{\sigma}$$

Similitude

- Similitude
  - Predict prototype behavior from model results
  - Models resemble prototype, but are
    - Different size (usually smaller) and may operate in
    - Different fluid and under
    - Different conditions
  - Problem described in terms of dimensionless parameters which may apply to the model or the prototype

$$\pi_1 = f(\pi_2, \pi_3, \ldots, \pi_n)$$

  - Suppose it describes the prototype

$$\pi_{1p} = f(\pi_{2p}, \pi_{3p}, \ldots, \pi_{np})$$

  - A similar relationship can be written for a model of the prototype

$$\pi_{1m} = f(\pi_{2m}, \pi_{3m}, \ldots, \pi_{nm})$$

Similitude

- If the model is designed & operated under conditions that

$$\left.\begin{aligned} \pi_{2m} &= \pi_{2p} \\ \pi_{3m} &= \pi_{3p} \\ &\cdots \\ \pi_{nm} &= \pi_{np} \end{aligned}\right\}$$

Similarity requirements or modeling laws

- then $\pi_{1m} = \pi_{1p}$ → Dependent variable for prototype will be the same as in the model

Example

- Consider predicting the drag on a thin rectangular plate (w*h) placed normal to the flow.
- Drag is a function of: w, h, μ, ρ, V

$$F_D = f(w, h, \mu, \rho, V)$$

- Dimensional analysis shows:
- And this applies BOTH to a model and a prototype

$$\pi_1 = f(\pi_2, \pi_3)$$

$$\frac{F_D}{w^2 \rho V^2} = f(\frac{w}{h}, \frac{\rho V w}{\mu})$$

- We can design a model to predict the drag on a prototype.
- Model will have:

$$\pi_{1m} = f(\pi_{2m}, \pi_{3m})$$

$$\frac{F_{Dm}}{w_m^2 \rho_m V_m^2} = f(\frac{w_m}{h_m}, \frac{\rho_m V_m w_m}{\mu_m})$$

- And the prototype will have:

$$\pi_{1p} = f(\pi_{2p}, \pi_{3p})$$

$$\frac{F_{Dp}}{w_p^2 \rho_p V_p^2} = f(\frac{w_p}{h_p}, \frac{\rho_p V_p w_p}{\mu_p})$$

Example

- Similarity conditions

Geometric similarity

$$\pi_{2m} = \pi_{2p} \quad \frac{w_m}{h_m} = \frac{w_p}{h_p} \quad \Rightarrow \quad w_m = \frac{h_m}{h_p} w_p$$

Gives us the size of the model

Dynamic similarity

$$\pi_{3m} = \pi_{3p} \quad \frac{\rho_m V_m w_m}{\mu_m} = \frac{\rho_p V_p w_p}{\mu_p} \quad \Rightarrow \quad V_m = \frac{\mu_m}{\mu_p} \frac{\rho_p}{\rho_m} \frac{w_p}{w_m} V_p$$

Gives us the velocity in the model

Then

$$\pi_{1m} = \pi_{1p} \quad \frac{F_{Dm}}{w_m^2 \rho_m V_m^2} = \frac{F_{Dp}}{w_p^2 \rho_p V_p^2} \quad \Rightarrow \quad F_{Dp} = \left(\frac{w_p}{w_m}\right)^2 \frac{\rho_p}{\rho_m} \left(\frac{V_p}{V_m}\right)^2 F_{Dm}$$

Example (8.28)

- Given: Submarine moving below surface in sea water
  (ρ=1015 kg/m³, ν=μ/ρ=1.4x10-6 m²/s).
  Model is 1/20-th scale in fresh water (20°C).
- Find: Speed of water in the testdynamic similarity and the ratio of drag force on model to that on prototype.
- Solution: Reynolds number is significant parameter.

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p}$$

$$V_m = \frac{L_p}{L_m}\frac{\nu_m}{\nu_p} V_p$$

$$= \frac{20}{1}\frac{1}{1.4} 2\, m/s$$

$$V_m = 28.6\, m/s$$

$$\frac{F_m}{\rho_m V_m^2 l_m^2} = \frac{F_p}{\rho_p V_p^2 l_p^2}$$

$$\frac{F_m}{F_p} = \frac{\rho_m V_m^2 l_m^2}{\rho_p V_p^2 l_p^2}$$

$$= \frac{1000}{1015}\left(\frac{28.6}{2}\right)^2\left(\frac{1}{20}\right)^2$$

$$\frac{F_m}{F_p} = 0.504$$